Searching for Chaos in Cellular Automata: New Tools for Classification

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1 Introduction

Among all discrete-time discrete-space multi-dimensional systems, cellular automata (CA) are very interesting to study because they offer a rich variety of behaviors allowing to model specific physical systems, as well as universal computational devices.

In the theory of CA, classification of behaviors is a central theme. The goal is to impose a structure in the space of CA rules, grouping together CA related to equivalent properties.

Many different behaviors are possible, from very simple (destruction of information) to very complex (propagation of information following complex rules). In general simple behaviors are easy to understand and to characterize a priori, which is not the case when analyzing complex behaviors. Different tools have been introduced, leading to different classification schemes.

The main problem appearing in (almost) every classification scheme is the qualitative definition: several classes are not formally defined. There is thus an ambiguity inside each classification.

The goal of this paper is twofold: to propose a new classification of CA, formally and precisely defined; to investigate the class of complex behaviors (particularly "aperiodic" behaviors).

We propose new tools, i.e. transfinite attraction and shifted hamming distance, giving us a way of defining a new classification of CA in which every class is formally defined. We also analyze three different ways of grouping these classes, which bring new insights in understanding chaotic behaviors. Before concluding,

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we survey the related work, especially concerning classification schemes previously presented, and the study of aperiodicity (i.e. the complex behaviors) in CA.

2 Basic Definitions

2.1 Iterations

Let us consider a space (or set) E. Classically, an iteration scheme is defined as follows: starting from an initial value $x_0 \in E$, one successively takes $f(x_0), f(f(x_0)), \ldots$. More formally,

$$f^0(x_0) = x_0 \text{ and } f^{n+1}(x_0) = f(f^n(x_0)).$$

From this, it is possible to speak about finite-time properties, i.e. where $f^n(x_0)$ is considered for some $n \in \mathbb{N}$, or infinite-time properties, i.e. where the limit of all $f^n(x_0)$ is considered.
In [Geu93], the author shows that strong conditions on function (or relation) f are to be assumed to prove convergence theorems in this classical framework. However, if we relax the hypothesis of finite (and their limit, infinite) iterations, then it is possible to get more general results: the relations involved only need to be monotone. Therefore, we need to introduce transfinite iterations.

2.2 Transfinite iterations

The class of ordinal numbers (denoted $\mathbb{O}$) is well ordered by the classical $\leq$. The expression $\cup_{i\in I}\delta_i$ is used to denote the upper bound of the ordinals family $\{\delta_i | i \in I\}$. A limit ordinal δ is such that $\cup_{\alpha<\delta}\alpha = \delta$. A successor ordinal δ is such that $\cup_{\alpha<\delta}\alpha = \delta - 1$, where the predecessor of δ is denoted by $\delta - 1$.
Let us assume that $L(\subseteq, \emptyset, E, \cup, \cap)$ is a complete lattice, with ordering relation $\subseteq$, infimum $\emptyset$, supremum E, least upper bound operator $\cup$, and greatest lower bound $\cap$. With the same notations, $\mu(L)$ denotes the smallest ordinal number such that $\#\{\delta | \delta \in \mu(L)\} > \#L$.

Definition 2.1 Decreasing iteration
The decreasing iteration starting from E in the complete lattice $L(\subseteq, \emptyset, E, \cup, \cap)$, and defined by a monotonic relation F, is a sequence $(x^\delta)_{\delta\in\mu(L)}$ of elements of L, defined by transfinite recurrence, as follows:

$$\begin{aligned} x^0 &= E \\ x^\delta &= F(x^{\delta-1}) \quad \text{for all successor ordinal } \delta \\ x^\delta &= \cap_{\alpha<\delta} x^\alpha \quad \text{for all limit ordinal } \delta \end{aligned}$$

Definition 2.2 Stationary sequence
The sequence $(x^\delta)_{\delta\in\mu}$ of elements of the complete lattice $L(\subseteq, \emptyset, E, \cup, \cap)$ is stationary iff $\exists \varepsilon \in \mu : (\beta \geq \varepsilon) \Rightarrow (x^\varepsilon = x^\beta)$. The limit of this sequence is x^ε.

Theorem 2.1 Greatest fixed-point ([Cou78] Corollary 2.5.2.0.3)
Let $L(\subseteq, \emptyset, E, \cup, \cap)$ be a complete lattice. A decreasing iteration $(x^\delta)_{\delta \in \mu(L)}$ starting from E, and defined by a monotonic relation F, is a stationary decreasing sequence and its limit is the greatest fixed-point of F.

2.3 Predicate transformers

Let us take a relation $f \subseteq E \times E$, and a predicate (or set of states) P included in its domain. We define the **pre-image** of P, $f_-.P$, as the set of states from which a state in P can be reached by an application of f, and the **post-image** of P, $f_+.P$, as the set of states which can be reached from a state in P by an application of f.

Definition 2.3 Pre-image, post-image

$$\begin{aligned} f_-.P &= \{u | (u \in Dom(f)) \wedge \exists v : (v = f(u)) \wedge (P.v)\} \\ f_+.P &= \{v | \exists u : (u \in Dom(f)) \wedge (P.u) \wedge (v = f(u))\} \end{aligned}$$

These operators are called predicate transformers and can be seen as inverse (resp. direct) extensions of relations from point-to-point to set-to-set.
It is easy to see that we have $f_+.E \subseteq E$, $f_-.E \subseteq E$, and monotonicity of these predicate transformers, i.e.

$$\forall X, Y \subseteq E, (X \subseteq Y) \Rightarrow (f_\pm.X \subseteq f_\pm.Y).$$

2.4 Cellular Automata

Finally, let us recall the definition of cellular automata. We consider one-dimensional cellular automata, the cells of which being arranged on a linear bi-infinite lattice. Formally, any linear cellular automaton is a structure

$$C = (G, r, g)$$

where

- $G = \{0, 1, \ldots, k-1\}$ is the set of the states;
- $r \in \mathbb{N}$ is the radius of the neighborhood;
- $g : G^{2r+1} \rightarrow G$ is the local transition function.

A configuration of a CA is a function that specifies a state for each cell

$$x : \mathbb{Z} \rightarrow G$$

and can be represented by a doubly-infinite sequence:

$$x = (\ldots, x_{i-n}, x_{i-n+1}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+n-1}, x_{i+n}, \ldots).$$

So the set of the configurations of the CA is $G^{\mathbb{Z}}$. The neighborhood of a cell $i \in \mathbb{Z}$ is the vector

$$(i-r, i-r+1, \dots, i-1, i, i+1, \dots, i+r-1, i+r) \in G^{2r+1},$$

The global transition function of the CA

$$f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$$

specifies the next state of each cell as the local function applied to the states of the neighborhood ($\forall i \in \mathbb{Z}$)

$$f_i(x) = g(x_{i-r}, x_{i-r+1}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+r-1}, x_{i+r}).$$

In the next sections, we will only use elementary cellular automata, i.e. with $r = 1$ and $k = 2$. We will also use $\mathcal{C}$ instead of $2^{\mathbb{Z}}$ to denote the configuration space.
Let us remark that we use a configuration space $\mathcal{C}$ the cardinality of which is $2^{\aleph_0}$, at least equal to $\aleph_1$. If we consider the power set of this configuration space, i.e. the set of all subsets of $G^{\mathbb{Z}}$, the cardinality is $2^{\aleph_1}$, at least equal to $\aleph_2$. This cardinality is equal to the one of the power set of a one-dimensional real interval. We know that $\aleph_1, \aleph_2, \aleph_3, \dots$ are greater than the first transfinite ordinal number ω and we will see below why transfinite iterations are useful in this framework.

3 New tools

3.1 Transfinite Attraction

We now extend definition 2.2 and split it into three forms, each making use of theorem 2.1 very easily. Working in a space or set E, we consider the set of its subsets, namely $\mathcal{P}E$. As it is well known, this set is a complete lattice. We denote it as above: $L(\subseteq, \emptyset, E, \cup, \cap)$.
Classically, one says that a state x is attracted by p under iteration of f iff $(\exists U : \forall n \in \mathbb{N}, f^n(x) \in U) \wedge (\lim_{n\to\infty} f^n(x) = p)$.
We can extend this definition as follows:

Definition 3.1 Asymptotic attraction
A set P is asymptotically attracted by a set Q iff $lim_{n\to\infty} f_+^n.P \subseteq Q$. In general we consider the smallest such Q.

We can also give a finite version of this notion:

Definition 3.2 Finite attraction
P is finitely attracted by Q iff $\exists n \in \mathbb{N} : f_+^n.P = Q$.

The previous definitions are not new but we can extend them in a third way, making use of transfinite iterations:

Definition 3.3 Transfinite attraction
P is transfinitely attracted by Q iff there exists an ordinal number $n \in \mathbb{O}$ *such that* $f_+^n.P = Q$.

In general, the symbol ∞ used in the first definitions is equivalent to the first transfinite ordinal ω. We extend the notion to all ordinal numbers (finite, ω, and all other transfinite ones) to simplify our next developments.
To find the whole attractor of configuration space $\mathcal{C}$, we have to compute, for a certain ordinal number $n \in \mathbb{O}$, the negative invariant of the system [Sin92, SG94]:

$$\boxed{f_+^n.\mathcal{C}}.$$

This expression is computable by successive approximations, and leads to the attractor, thanks to monotonicity of f_+ assumed by theorem 2.1.

3.2 Shifted Hamming Distance

We introduce here a notion of distance that is vey close to the very well known "Hamming distance". At first we extend this notion, defined on finite strings of symbols, to bi-infinite strings of symbols. Then, we present the new notion itself.
Let us remember the first notion. We work with a finite alphabet isomorphic to $\Sigma = \{0, 1, ..., n-1\} \subseteq \mathbb{N}$. On this alphabet we define a distance

$$\delta(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

For two strings of symbols $a, b \in \Sigma^m$, the Hamming distance between a and b, $H(a, b)$ is defined as the number of places (or indices) where a and b differ:

$$H(a, b) = \sum_{i=1}^{m} \delta(a_i, b_i).$$

For two bi-infinite sequences a and b of symbols,

$$H(a, b) = \sum_{i \in \mathbb{Z}} \delta(a_i, b_i).$$

Let us now introduce the new notion.

Definition 3.4 Shifted Hamming distance
The shifted Hamming distance between two bi-infinite sequences x and y of $\mathcal{C}$ *is defined by:*

$$H^\sigma(x, y) = \min_{j \in \mathbb{Z}} H(x, \sigma^j(y))$$

where σ *is the classical shift function:* $\forall i \in \mathbb{Z}, \sigma(x)_i = x_{i+1}$.

It is easy to see that $(\mathcal{C}, H^\sigma)$ is a metric space.

4 Classification of simple CA

The following behavioral descriptions can be seen as definitions of the resulting classes. In a next section, we present a characterization of these classes using attraction. This last phenomenon is a property of the systems studied. Here is the classification in brief:

- CA evolving to null configurations (type $\mathcal{N}$)
- CA evolving to fixed points (type $\mathcal{F}$)
- CA with periodic behavior (type $\mathcal{P}$)
- CA with subshift behavior (type $\mathcal{S}$)
- CA with aperiodic behavior (type $\mathcal{A}$)

4.1 Type $\mathcal{N}$ cellular automata

They fastly evolve to homogeneous configurations, i.e. without information (all ones or zeroes), after finite transients.

4.2 Type $\mathcal{F}$ cellular automata

They evolve to fixed-points after finite transients. This class contains the first one, which is a particular case.

4.3 Type $\mathcal{P}$ cellular automata

They evolve to periodic configurations, after finite transients. This class contains the two previous ones.

4.4 Type $\mathcal{S}$ cellular automata

They evolve to configurations where a generalized alternating subshift behavior occurs. Here is a definition of this behavior, generalizing [CFMS93a]:

Definition 4.1 Generalized alternating subshift rule
A CA rule h is a generalized alternating subshift rule if the corresponding global function g is such that there is a closed invariant subset Σ_1 of Σ on which it operates as follows:

$$\forall x \in \Sigma_1, g^n(x) = \sigma^m(x)$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, and $\sigma : \mathcal{C} \to \mathcal{C}$ is the classical shift defined as follows: $\forall i \in \mathbb{Z}, \sigma(x)_i = x_{i+1}$.

It is possible to prove (see Appendix 9.1) that this kind of behavior leads to chaos in the sense of Devaney (topological transitivity, density of periodic points) [Dev89, BBC+92].

4.5 Type $\mathcal{A}$ cellular automata

This is the same class as type "c" of [CFMS93b]. We prefer calling it "aperiodic" because it is not really chaotic in the sense of Devaney's definition. Actually, we could also call this phenomenon "spatio-temporal chaos" or "intermittency". There does not exist any analytical definition for these phenomena. Aperiodicity seems to be the least restrictive definition for the moment: a configuration is aperiodic if it is not eventually periodic (neither periodic nor one of its forward iterations). Phenomenologically, what we observe is often a number of different triangles or other patterns growing, vanishing and moving towards the future. There is a kind of regularity (these forms are far from noise) but also a diversity (different forms). There is no stabilization as such but there is no real disorder. We remark that aperiodicity entails that almost the whole domain is visited through successive iterations. Therefore, we could also call aperiodic systems "total" or "full" systems.

5 Classification and Attraction

Our motivation to classify the different behaviors with respect to the generalized attraction we have presented above is the following. There seems to exist three kinds of observable behaviors: finitely regular (null, fixed-point, or periodic rules), completely irregular (our type $\mathcal{A}$), and "chaotic" behaviors (subshift rules).
Let us discuss the third class. These rules are chaotic, as proved above, in the sense of Devaney's definition. However, when we observe their long term behavior, what we see is a beautiful regularity. The problem is a bit technical: the proof is based on a metric assigning a weighting to the cells of the cellular automaton, which strongly influences the consideration made, whereas we don't have this weighting in mind when we observe the successive configurations generated. Thus, we have to find a technical aspect closer to our own observation and interpretation. We propose to use attraction as technical tool but, since we work with doubly infinite lattices of cells, we need to consider transfinite iterations and attraction.
Several choices are possible for the study of attraction phenomena in cellular automata. Some authors work with finite configurations in zero backgrounds. We consider bi-infinite configurations in zero backgrounds! [1] We take one

[1]This may seem very strange but since we consider bi-infinite configurations, we have to consider transfinite iterations; then we have to precise what happens at the "borders" of these lattices. Another possible choice is a circular bi-infinite CA, having cells $-\infty$ and ∞

configuration at a time ($\#P = 1$), randomly, and we "observe" its successive iterations, over a certain amount of time (finite, infinite, or transfinite): for a cellular automaton the global function of which is given by f, we observe, $\forall n \in \mathbb{O}, f_+^n.P$. We also get information by studying the same expression with $P = \mathcal{C}$. Finally, it is interesting to compare the results obtained with or without the constraint of finite iterations.
Let us first try to justify transfinite iterations intuitively. We consider a single deterministic automaton with a finite number of possible states. We let the system progress, or iterate, and we look at the orbits generated from different initial states. If we only take a smaller orbit than the total number of states, then different behaviors are observable: fixed-point attraction, periodicity, seemingly random orbits. What is really random is another question [Cha90, Cha94] and that's why we add "seemingly". If we consider more iterations than the number of possible states, since the system is deterministic, then random orbits vanish. Everything becomes eventually fixed or periodic. We consider now bigger and bigger state spaces, until we reach some kind of infinity. For example, we take a state space of positive integers $\mathbb{N}$ of cardinality $\aleph_0$, the first transfinite cardinal, also equal to ω. The same behaviors appear and we have to allow more than ω iterations to see only periodic behaviors. Let us now examine the different classes separately.

5.1 Type $\mathcal{N}$ cellular automata

They fastly evolve to homogeneous configurations, i.e. any configuration is finitely attracted to the same configuration, homogeneously composed of quiescent cell states. The homogeneous state is a function of the rule itself:

$$\exists h \in \mathcal{C} : \begin{cases} & \forall x_0 \in \mathcal{C}, \exists n \in \mathbb{N} : f_+^n.\{x_0\} = \{h\} \\ \wedge & f(h) = h \end{cases}$$

or, more globally,

$$\boxed{f_+^\omega.\mathcal{C} = \{h\}}.$$

This class is called $\mathcal{N}_0$ because another version is possible, with several possible quiescent configurations:

$$\exists H \subset \mathcal{C} : \forall x_0 \in \mathcal{C}, \exists h \in H : \begin{cases} & \exists n \in \mathbb{N} : f_+^n.\{x_0\} = \{h\} \\ \wedge & f(h) = h \end{cases}$$

or, more globally,

$$\boxed{f_+^\omega.\mathcal{C} = H}.$$

We call this subclass $\mathcal{N}_1$.

equivalent.

5.2 Type $\mathcal{F}$ cellular automata

They evolve to fixed configurations after finite transients. The final fixed configuration is in general dependent on the initial one. We have here a finite attraction, too:

$$\forall x_0 \in \mathcal{C}, \exists s(x_0) \in \mathcal{C} : \begin{cases} & \exists n \in \mathbb{N} : f_+^n.\{x_0\} = \{s\} \\ \wedge & f(s) = s \end{cases}$$

More globally, we have:

$$\boxed{f_+^\omega.\mathcal{C} = \cup_{x_0 \in \mathcal{C}} \{s(x_0)\}}.$$

5.3 Type $\mathcal{P}$ cellular automata

They evolve to cycles of configurations after finite transients. The limit cycle is dependent on the initial condition. We have a finite attraction to a set of points rather than to a single fixed-point:

$$\forall x_0 \in \mathcal{C}, \exists C(x_0) \subseteq \mathcal{C}, y \in C, m \in \mathbb{N}, n \in \mathbb{N} : \begin{cases} & f_+^n.\{x_0\} = y \\ \wedge & f^m(y) = y \\ \wedge & \forall y' \in C, f^m(y') = y' \end{cases}$$

More globally, we have:

$$\boxed{f_+^\omega.\mathcal{C} = \cup_{x_0 \in \mathcal{C}} C(x_0)}.$$

5.4 Type $\mathcal{S}$ cellular automata

They behave like generalized alternating subshifts. When observing a specific cellular automaton starting from a random initial configuration, what we see is the initial configuration progressively escaping (or shifting) to the right or to the left, like sliding along the linear lattice of cells, together with a kind of periodic behavior. If we take an initial finite configuration in a zero background, for example, we will see our configuration escaping the finite observation domain, unless this domain can indefinitely grow. Let us imagine we could iterate more times than the total amount of cells composing the lattice of the automaton, even if this lattice possesses a bi-infinite number of cells. Then, starting from any initial configuration, we could observe an attraction to a homogeneous configuration, exactly as type $\mathcal{N}$ cellular automata behave. From this point of view, the behavior of type $\mathcal{S}$ cellular automaton becomes more regular and simple than chaotic, if we accept transfinite iterations:

$$\exists h \in \mathcal{C} : \begin{cases} & \forall x_0 \in \mathcal{C}, \exists n \in \mathbb{O} \backslash \mathbb{N} : f_+^n.\{x_0\} = \{h\} \\ \wedge & f(h) = h \end{cases}$$

or, more generally,

$$\exists H \subseteq \mathcal{C} : \forall x_0 \in \mathcal{C}, \exists m \in \mathbb{O}, n \in \mathbb{O}\backslash\mathbb{N} : \left\{ \begin{array}{cl} & \exists y \in H : f_{+}^{n}.\{x_0\} = \{y\} \\ \wedge & \forall y' \in H, f^{m}(y') = y' \end{array} \right.$$

where H is a cycle of homogeneous configurations. It is also possible to write:

$$\boxed{\exists n \in \mathbb{O}\backslash\mathbb{N} : f_{+}^{n}.\mathcal{C} = H}.$$

Here, we see a difference regarding the use of finite/transfinite iterations. Finite iterations lead to a typical shift behavior which can be seen as chaotic (classical definition). Transfinite iterations show a simple behavior of attraction to homogeneous configurations.

5.5 Type $\mathcal{A}$ cellular automata

They have an aperiodic behavior which is responsible for the observable (spatio-temporal) chaos. Since no simple definition actually exists to characterize this kind of behavior or to analyze it more deeply, we prefer calling it "aperiodic". A part of our future work is to find a subclassification of this phenomenon with more appropriate and refined definitions. Back to attraction, we have here an "attraction" to a huge cycle containing (almost) the whole configuration space:

$$\forall x_0 \in \mathcal{C}, \exists C' \overset{\subseteq}{\approx} \mathcal{C}, m \in \mathbb{N}, n \in \mathbb{O}\backslash\mathbb{N} : \left\{ \begin{array}{cl} & \exists y \in C' : f_{+}^{n}.\{x_0\} = y \\ \wedge & \forall y' \in C', f^{m}(y') = y' \end{array} \right.$$

where the symbol "$\overset{\subseteq}{\approx}$" means "dense subset of" but is still to be precised more formally. [2] It is important because it makes the difference with type $\mathcal{S}$ cellular automata. It is also possible to write:

$$\boxed{\exists n \in \mathbb{O}\backslash\mathbb{N} : f_{+}^{n}.\mathcal{C} \overset{\subseteq}{\approx} \mathcal{C}}.$$

Here also, we have a difference between finite and transfinite iterations. Finite iterations show irregular behaviors, spatio-temporal chaotic patterns, aperiodic evolutions. The problem is that it is difficult to give an explicit characterization of this kind of behavior. On the other hand, transfinite iterations allow us to give a very simple definition, saying that the system involved is periodic with a huge period very close to the cardinality of the configuration space itself.

5.6 Discussion

For the sake of simplicity, we will include $\mathcal{N}_1$ in $\mathcal{F}$ and keep $\mathcal{N}$ equal to $\mathcal{N}_0$.

[2] Chaitin'theorem saying that most strings, or real numbers, are not computable (reachable by an algorithm), this inclusion is always strict [Cas92, Cha94].

We see here that transfinite attraction gives us a new way of defining the behavior of differents classes of CA, from very simple classes to the most complex ones.
If we restrict our attention to basis CA (i.e. the rules of which are 0, 1, 2, 4, 8, 16, 32, 64, or 128), our classification is of course decidable. Our goal is to extend the notions and our classification to more complex CA, constructed from bais CA with composition operators [FG94].

6 Classes Organization

6.1 Linear Periodicity Hierarchy

Though we have the following inclusions:

$$\mathcal{N} \subset \mathcal{F} \subset \mathcal{P},$$

it is difficult to compare the first classes with type $\mathcal{S}$ and type $\mathcal{A}$. However, if we try to see all classes with respect to transfinite iterations, we can see a hierarchy of periodic systems. From type $\mathcal{N}$ to type $\mathcal{A}$, the period grows from one to an ordinal "close" to the cardinality of our configuration space [3] , and the resulting attractor grows from a homogeneous fixed configuration to almost the whole configuration space. Hence, we have a linear hierarchy:

$$\boxed{\mathcal{N} \ll \mathcal{F} \ll \mathcal{P} \ll \mathcal{S} \ll \mathcal{A}}$$

where we volontary do not give a precise definition to "$\ll$".

6.2 Periodicity Clusterization

In this second organization, we introduce two criteria:

- (in)dependence to initial conditions
- (trans)finite iterations.

They permit to build a classification table of our different sources of behaviors:

Periodicity	Dep. to I.C.	Indep. to I.C.
Finite	$\mathcal{F} \subset \mathcal{P}$	$\mathcal{N}$
Transfinite	$\mathcal{A}$	$\mathcal{S}$

[3] We quote close because, for example, we allow to say that $\frac{\aleph_1}{2}$ is closer to $\aleph_1$ than to $\aleph_0$.

6.3 Organization wrt Shifted Hamming Distance

With the help of the tool previously introduced, we can classify our different behaviors in a very simple way. Thanks to shifted Hamming distance we are able to show that subshifts behaviors are simple. The intuition is the following. Let us consider a very simple shift behavior, the classical left shift for example. This shift can be proved chaotic under some assumptions and with a "center-first" metric. However, the patterns observed when the shift evolves let us think that the system is simple: there is just a shift to the left at each step. If we take the sequence obtained after a few steps, it very resembles the initial one: we just have to shift it to the right to let it be equivalent to the former again. This is the idea behind the shifted Hamming distance: a Hamming distance forgetting the shifting motions. An elementary CA with neighborhood fixed by the parametre $r = 1$ has the following property: each cell has an effect on $2rn + 1$ cells after n iteration steps.
If the system is n-periodic or if it has a generalized subshift behavior including a n-periodicity, they both appear very simple through the eye of our shifted Hamming distance: for any x in the orbit of an initial condition, after the transient,

$$H^{\sigma}(x, f^n(x)) = 0.$$

The dual behavior is aperiodic. This gives us another characterization for aperiodicity: there is no x in the configuration space $\mathcal{C}$ for which the previous condition applies and thus, for all state x, and all n, we have:

$$H^{\sigma}(x, f^n(x)) \neq 0.$$

We summarize this last organization as follows:

Null SHD	Positive SHD
$\mathcal{N} \cup \mathcal{F} \cup \mathcal{P} \cup \mathcal{S}$	$\mathcal{A}$

Under "center-first" metrics, subshifts can be considered as chaotic (Devaney's definition [Dev89]). Under SHD, subshifts are very simple, just as periodic behaviors.

7 Related Work

In this related work, we do not want to treat everything in detail but we only would like to cite some other works and compare them with our approach. Two themes are important: classification and aperiodicity.

7.1 Classification: State-of-the-Art

Although we call this part "state-of-the-art", we mainly present classification schemes close to Wolfram's one. Other schemes are mentioned in the papers we give here as references.

Classification is one of the central themes in the theory of CA. A structure is imposed on the space of CA, grouping together CA with related properties. Several authors have proposed different classifications, starting with Wolfram in 1983.

- In [Wol86], the first classification of CA appears, grouping together systems having the same long-term behavior:
  - class *I*: evolution to homogeneous state;
  - class *II*: evolution to separated simple states or periodic structures;
  - class *III*: chaotic patterns;
  - class *IV*: complex localized structure, sometimes long-lived.

  The main problem of this classification is that it is only qualitatively defined. The fourth class is related to universal computational devices. Wolfram considers only symmetric rules. When asymmetry appears, we get subshift behaviors.

- In [Kan86], the author relates the previous classification to two properties (number of attractors, period size) and gives thus a more precise description of each class.

	small number of attractors	big number of attractors
short periods	no information creation small storage class *I*	no information creation large storage class *II*
long periods	creation of information small storage class *III*	creation of information large storage class *IV*

- In [CHY90] the authors give a hierarchy of CA starting with finite configurations:
  - class *I*: CA converging to fixed-points in finite time;
  - class *II*: CA converging to periodic configurations in finite time;
  - class *III*: CA for which it is decidable to know whether a configuration occurs in the orbit of another;
  - class *IV*: all CA.

  Each class contains the previous one(s). The authors put an accent on (un)decidability results

- In [Gut90], the author studies the action of CA on n-step Markow measure. This approach is really different from ours, because we do not make use of probabilities.

- In [LP90], the authors give a classification looking like Wolfram's one but not completely comparable with it:
  - class 1: null rules, leading to honogeneous configurations (equivalent to Wolfram's class I);
  - class 2: evolution to fixed-point configurations (class II, partly);
  - class 3: evolution to periodic configurations (class II);
  - class 4: locally chaotic rules (class II);
  - class 5: globally chaotic rules (class III);
  - class 6: complex behaviors (class IV).

  They also study inter, and intra-class probabilities and give a mean-field description of their classification.

- In [Sut90], the author concentrates on circular CA and uses the same definitions as in [CHY90].

- In [CFMS93b], a variation of Wolfram's classification is presented, adding classes behaving like subshifts, due to the consideration of asymmetric rules. Here, the authors take bi-infinite configurations without any restriction:
  - class n: evolution to quiescent configurations (Wolfram's class I);
  - class f: evolution to fixed-point configurations (class II, partly);
  - class p: evolution to periodic configurations (class II);
  - class s: simple subshift behaviors (class II);
  - class s': complex subshift behaviors (class II);
  - class c: "chaotic" behavior (classes III and IV);

  Classes s and s' are considered separately from simple rules because, when studied on bi-infinite configurations, they generate chaos (as proved in Appendix 9.1), whereas they are seen as very simple rules when evolving from finite configurations. This classification is our starting point. We will come back to this below.

- In [BCFV94], the authors propose a new classification of CA, based on the observation of finite initial conditions in bi-infinite configurations. The tool presented is a measure of pattern growth. The classification goes as follows:
  - class C_1: patterns vanish;
  - class C_2: pattern length stays finite (fixed or periodic finite size);
  - class C_3: pettern length grows to infinity.

We have the following relation between classes: $C_1 \subset C_2 = \overline{C_3}$. An important advantage of this classification over other ones is that it is decidable. For each rule, it is possible to determine a priori wheter it belongs to class C_1, C_2, or C_3.

Let us try to summarize what are the problems in general and compare our approach with these previous ones.
Two problems appear when classification is studied:

- it is difficult to give a formal definition of each class of CA (in particular, spatio-temporal chaos is not precisely defined in this context);
- these definitions are often based on undecidable properties.

Our classification is strongly influenced by [CFMS93b]'s. We put together classes s and s' and give a generalized version of subshift behaviors. We also generalize class c into a class of aperiodic behaviors (see [Jen90] below). We present new tools allowing us to give a precise formal definition of each class. These tools can be related to the characterization appearing in [Kan86]. Our classification is not decidable for all CA rules but only for simple basis ones. We are trying to extend these results to more complex rules, with the help of composition operators [FG94].

7.2 Aperiodicity in Cellular Automata

The notion of "chaos" is still not well defined in the context of discrete-time discrete-space multi-dimensional dynamical systems such as, for example, cellular automata. Several authors propose ways of defining complex behaviors in CA. This is one of the goals of classification. We have already presented several classification schemes. Other ones put an accent on transition phenomena in the space of CA rules, allowing new classifications, too. In these latter ones, statistical measures are often used, together with information theory-like measures (entropy, activity, sensitivity to rule change, etc.). This leads to definitions of complex behaviors, based on certain values of parameters. Among others, we refer the interested reader to [Gra84, Ped90, LPL90, WL90, CM90, Bin93].
In [Svo90], the author presents a classification of chaotic behaviors, based on notions of randomness, complexity measures, computability of initial conditions, and (non)determinism of rules.
Finally, in [Jen90], the author studied aperiodicity of some CA analytically. We take this point of view in our classification scheme because it is easier to define than complex or chaotic rules. However, we do not make use of linearity and injectivity notions presented by Jen. This point of view is interesting because aperiodicity includes complex and chaotic behaviors, in some sense.
Before concluding, we apologize for any forgotten reference, and we please the reader to inform us for such a mistake.

8 Conclusion

Our goal was to find a classification of elementary cellular automata in which each class is defined by a mathematical expression. In particular, we wanted to characterize the most "chaotic" classes.
We have refined a given classification and we have added new tools to go deeper: predicate transformers, transfinite attraction, and shifted hamming distance. With these tools we gave an explicit characterization of each class and we have seen that spatio-temporal or aperiodic systems are in fact periodic systems with huge periods.
In a future work, we will use this explicit definition of CA classes to find equivalent classes of more general dynamical systems. In parallel, we will also investigate different composition operators and analyze how they behave in this framework, regarding properties of invariance, attraction, etc.
Another important aspect to investigate is whether there is a link between our definition of aperiodic systems, and classical definitions of chaotic systems [Dev89, Wig90]. Are they equivalent, opposite, or complementary ?

9 Appendix

9.1 Chaos in Generalized Subshifts

The following theorem is inspired from [BCFM93].

Theorem 9.1 *If Σ_1 is a closed invariant subset of Σ, and if a CA rule h is a generalized alternating subshift on Σ_1 with irreducible transition matrix, then the corresponding global function g is chaotic on Σ_1.*

Proof. We have to prove that:

1. there exists a dense orbit for g (leading to topological transitivity),

2. the periodic points of g are dense.

Transitivity.
It is easy to construct a dense sequence for the full shift. This sequence can be expressed as a sequence of all sequences of length 1, all sequences of length 2, etc. All these sequences can be ordered. We note the resulting sequence $...s_5s_3s_1s_0s_2s_4s_6....$ It is easy to deduce that σ^m has a dense orbit, too. The resulting sequence is $...s_5^ms_3^ms_1^ms_0^ms_2^ms_4^m....$ Thus g^n has a dense orbit. Hence g has a dense orbit (the same as for g^n). All these sequences can be adapted to the case of a subshift with irreducible transition matrix: the final resulting sequence is $...t_3(s_1s_1')^{m-1}s_1t_1(s_0s_0')^{m-1}s_0t_0(s_2s_2')^{m-1}s_2t_2...$, where t_i is an admissible sequence connecting s_i and s_{i+2} for every even i, s_i and s_{i-2} for every odd i, t_1 connects s_1 and s_0, and s_i' connects s_i with itself. To simplify the

proof of density of this sequence, let us assume that every connecting sequence has the same length.
Regularity.
We have to prove $\forall x \in \Sigma_1, \forall \varepsilon > 0, \exists y \in B_\varepsilon(x) \cap Per(g)$. We work with the metric $d(x,y) = \sum_{i=-\infty}^{\infty} \frac{1}{4^{|i|}} |x_i - y_i|$ defined on Σ. Let us take a bi-infinite sequence x of Σ. We have to construct a y belonging to $B_\varepsilon(x)$ and to $Per(g)$. It is sufficient that y matches x on a central part (around index 0) of length $2l+1$ to guarantee $y \in B_\varepsilon(x)$. Now, $y \in Per(g) \Leftrightarrow \exists q : g^q(y) = y$. If y is the bi-infinite repetition of $x_{-l}...x_{-1}x_0x_1...x_l$, it is $(2l+1)$-periodic but also $m(2l+1)$-periodic. Thus, we have $y = \sigma^{m(2l+1)}(y) = g^{n(2l+1)}(y)$. Hence y is periodic for g. To treat the subshift case, just add a sequence $c_1...c_K$ in y, where K is less than the smallest exponent k such that the transition matrix of the subshift is irreducible.
□

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