Paper foldings as chaotic dynamical systems *

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Abstract

A paper folding sequence is the sequence of ridges and valleys obtained by unfolding a sheet of paper which has been folded infinitely many times. To study the complexity of such sequences, we consider foldings as a dynamical system obtained by composing very simple systems. This allows to prove the existence of a Cantor invariant set in the space of infinite landscapes, and that folding systems are chaotic on this invariant.

Keywords: paper folding sequence, dynamical system, composition, chaos.

1 Introduction

Although it is well known that no reasonable sheet of paper can be folded more than 7 times, a paper folding sequence is the sequence of ridges and valleys obtained by unfolding a sheet of paper which has been folded infinitely many times.

Paper folding sequences and their complexity have been studied by several authors, using formal power series, continued fractions, language theory and morphisms, measure theory, group theory, etc. [11, 23, 12, 22, 2, 5].

The folding process behind the abstract mathematical terms used to describe these infinite symbolic sequences has been analyzed in another field: hyperbolic dynamical systems [33, 7, 24]. For example, Smale's "stretch-and-squeeze" horseshoe map [28] shows a typical chaotic behavior due to iterative folding of its underlying state space. Inspired by these results, we propose a way to characterize the complexity of paper folding sequences as in the behavior of horseshoe-like maps and other chaotic dynamical systems: we consider paper foldings as dynamical systems. Up and down foldings correspond to very simple dynamical systems defined on the space of infinite sequences of valleys and ridges. Mixing up and down foldings is shown the be equivalent to composing the corresponding systems in an adequate way. The composed system has

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an invariant which is the set of all possible sequences. Using composition, we prove that this invariant is a Cantor set, on which the system behaves in a chaotic way.

Composition is used here as a tool to explore complexity of systems [26, 27, 15, 16]. The approach allows to treat complexity has a structural property of some systems, which avoids long technical developments usually found in classical references.

Studying formal systems as dynamical systems is not frequent [10, 9], even though formal systems are often used to characterize dynamical properties of systems (e.g. see [21, 3]). This simple but promising example illustrates how apparently disjoint fields can enrich each other. Many formal systems could be analyzed by composition of dynamical systems: formal grammars, L-systems, dynamic proofs, etc.

This paper is organized as follows: in §2, we present the formal definition of paper foldings; in §3 we recall the necessary background in dynamical systems; in §4 we analyze foldings as dynamical systems; finally, in §5, we draw some conclusions.

Conventions. Since we deal with strings of symbols, we use classical conventions: Σ is a finite alphabet, Σ^n is the set of words of length n, Σ^* is the set of finite words on Σ including the empty word ε, Σ^ω is the set of infinite words; $|s|$ is the length of word s; juxtaposition of symbols stands for concatenation; exponentiation stands for multiple concatenation; for any word w of length at least n, $w|_n$ represents its prefix of length n.

2 Paper folding sequences

A *folding action* can be either up (U) or down (D); an *instruction* is a sequence of actions; the set of instructions is denoted by $\mathcal{J}$. The elementary result of a folding action is a *profile*; it can be either a valley (V) or a ridge (Λ); a *landscape* is a sequence of profiles; the set of landscapes is denoted by $\mathcal{L}$.

More precisely, seeing $\{U, D\}$ and $\{V, \Lambda\}$ as alphabets, we use the following notations: $\mathcal{J}^{n/*/\omega} = \{U, D\}^{n/*/\omega}$ and $\mathcal{L}^{n/*/\omega} = \{V, \Lambda\}^{n/*/\omega}$.

Not all landscapes are "legal" in the sense they should be obtainable by successive folding actions. Let us give the recursive definition of "legal" landscapes, that is, paper folding sequences [5].

Definition 1 (Paper folding sequences)
The sequence $(w_n)_{n\geq 1} \subset \mathcal{L}^\omega$ is a paper folding sequence iff $\forall n \geq 0$

$$\begin{aligned} w_{4n+1} &= V \text{ (resp. } \Lambda) \\ w_{4n+3} &= \Lambda \text{ (resp. } V) \end{aligned}$$

and $(w_{2n})_{n\geq 1}$ is a paper folding sequence, too.

Paper folding sequences can be seen as Toeplitz sequences (see [4] for a survey on Toeplitz sequences), which provides another way to generate them.

We now turn to the iterative construction of legal lanndscapes. From an empty landscape ε, i.e. a clean paper, folding up or down leads to a folded paper, nothing else. We must unfold this paper in the reverse order to get a new landscape. Thus, the first point to make precise is what we call an "action" does not really correspond to the folding alone, nor to the unfolding alone, but to both folding then unfolding in the reverse order.

Let us first apply U or D to the finite landscape obtained after a finite instruction. Intuitively, a folding action consists in inserting between each profile another profile, since all existing ones lie at the borders of the folded paper and the folding takes place in the middle of the folded paper. Of course, the extreme borders do not represent anything in the landscape. Formally, we have the following definition.

Definition 2 (Paper folding construction – 1)
Let $w = w_1 w_2 \cdots w_n$ be in $\mathcal{L}^*$, i.e. $\forall i, w_i \in \mathcal{L}$; then

$$\begin{aligned} U(w) &= V && \text{if } w = \varepsilon \\ &= V w_1 \Lambda w_2 V \cdots V w_n \Lambda && \text{if } n \text{ is odd} \\ &= V w_1 \Lambda w_2 V \cdots \Lambda w_n V && \text{if } n \text{ is even;} \\ D(w) &= \Lambda && \text{if } w = \varepsilon \\ &= \Lambda w_1 V w_2 \Lambda \cdots \Lambda w_n V && \text{if } n \text{ is odd} \\ &= \Lambda w_1 V w_2 \Lambda \cdots V w_n \Lambda && \text{if } n \text{ is even.} \end{aligned}$$

Secondly, we extend these definitions by composition, i.e. finite instructions applied to finite landscapes.

Definition 3 (Paper folding construction – 2)
For $a \in \{U, D\}, W \in \mathcal{J}^*, w \in \mathcal{L}^*$, we have:

$$\begin{aligned} \varepsilon(w) &= w \\ aW(w) &= W(a(w)) \\ Wa(w) &= a(W(w)). \end{aligned}$$

Finally, extending instructions and landscapes to infinity is straightforward, using the classical continuous limit of finite embedded sequences of increasing length.

Definition 4 (Paper folding construction – 3)
Let us fix $w \in \mathcal{L}^\omega$; then

$$\begin{aligned} U(w) &= \sqcup_n U(w|_{2n+1}) \\ D(w) &= \sqcup_n D(w|_{2n+1}) \end{aligned}$$

where $\sqcup$ expresses the least upper bound defined by the prefix ordering on sequences.

Before characterizing $\mathcal{J}^\omega(\mathcal{L}^\omega)$, it is important to remark that an infinite landscape can only appear after an infinite instruction. Thus, writing $U(w)$, where w is an infinite landscape, is equivalent to $U(W(\varepsilon))$, where W is an infinite instruction leading to the constructible landscape w from the empty landscape, i.e. the clean paper. The last expression can be rewritten as $WU(\varepsilon)$ and it justifies to introduce a right-juxtaposition to infinite instructions. Moreover, thanks to the definitions of U and D given hereabove, we intuitively see that if we want to find the first letters (that is, the leftmost ones) of an infinite landscape w appearing after an infinite instruction W, it is more useful to know the rightmost part of W than its leftmost part. For any landscape $w \in \mathcal{L}^\omega$, n folding actions shift w of $2^n - 1$ positions to the right. To know $W(w)$ with a finite precision of $2^n - 1$ profiles, it thus suffices to know the n last actions of W, independently of w. Actually, for $W \in \mathcal{J}^\omega$ and $w \in \mathcal{L}^\omega \cup \mathcal{L}^*$, $W(w) = W(\varepsilon)$ since $2^n - 1$ tends to ω as n does.

Notation 5
From now on, when speaking about infinite instructions, we shall consider infinite words on $\mathcal{J}^\omega$ whose rightmost end is known, and for any $W = \cdots W_2 W_1 \in \mathcal{J}^\omega$, $W|_n = W_n \cdots W_1$.

We have $\mathcal{J}^\omega(\mathcal{L}^\omega) = \mathcal{J}^\omega(\varepsilon)$ and we can define the last expressions in the following way.

Definition 6 (Paper folding construction – 4)
Let $W = \cdots W_2 W_1$ be in $\mathcal{J}^\omega$, then

$$W(\varepsilon) = \sqcup_n (W|_n)(\varepsilon).$$

This expression is well defined since for each n, $(W|_n)(\varepsilon)$ is a strict prefix of $(W|_{n+1})(\varepsilon)$.

3 Compositional analysis of dynamical systems

In this section, we briefly present the notions in dynamical systems theory we need in the sequel. Further details can be found in [26, 27, 16].

Whatt is compositionality all about in the context of dynamical systems? The aim is to characterize some dynamical property of a system S, noted as $\mathcal{D}(S)$. We first decompose S into simpler components S_i such that $S = \star_i S_i$. Then, after individual analysis of these components, we want to combine the results to get a global analysis, using an operator on properties $\diamond$ such that $\mathcal{D}(\star_i S_i) = \diamond_i \mathcal{D}(S_i)$. In other words, we want to find homomorphisms between systems and their dynamical properties.

Definition 7 (Dynamical system [1])
A dynamical system *is a relation $f \subseteq E \times E$. Its* iterations *are defined as follows: $\forall P \subseteq E$,*

$$f(P) \quad = \quad \{y | \exists x : (x \in P) \wedge ((x, y) \in f)\}$$

$$\begin{aligned} f^0(P) &= P \\ f^{n+1}(P) &= f(f^n(P)), \forall n \geq 0. \end{aligned}$$

To define backwards iterations, *it suffices to consider*

$$\begin{aligned} f^{-1} &= \{(y,x)|(x,y) \in f\}, \\ f^{-n} &= (f^{-1})^n, \forall n > 0. \end{aligned}$$

Remark 8
Each *non-deterministic execution* of a system chooses one possible path among all available ones, that is, one possible image at each step in an arbitrary way. The dynamics of a system can thus be defined as the set of all possible state sequences it is able to produce.

From now on, we restrict our attention to continuous injective functions, constant functions (basic systems), and their union (composition operator). Other operators are proposed in [26, 27, 16]; they are omitted here for the sake of clarity, since paper foldings can be treated as simple unions of systems.

Definition 9 (Union)
The union of two systems f and g is defined by their set-union:

$$f \cup g = \{(x,y)|(x,y) \in f \vee (x,y) \in g\}.$$

From this definition, we derive $(f \cup g)^{-1} = f^{-1} \cup g^{-1}$.

As well known, it is interesting to study dynamical systems through symbolic dynamics [17, 13, 33]. To this end, we decompose a given system S into a finite set of subsystems S_i, such that S can be obtained as the union of these subsystems. Then, we attribute a different symbol i to each independent component S_i, which gives an alphabet Σ of symbols. For example, if S is the union of three relations, i.e. $S = f \cup g \cup h$, one can consider $S_1 = f \cup g$ and $S_2 = h$, whose corresponding symbols are 1 and 2. Remark that the choice of decomposition can have a strong influence on the results of symbolic dynamics.

Definition 10 (Trace-parametrized relations)
Let $i \in \Sigma, \sigma \in \Sigma^\omega \cup \Sigma^*$; then

$$\begin{aligned} f_\varepsilon(P) &= P \\ f_{i\sigma}(P) &= f_i(f_\sigma(P)). \end{aligned}$$

We define the invariant of a system as a set of points in which the system *can stay* when iterated forward *or* backwards indefinitely. When we speak about the invariant of a system, we mean its greatest invariant J, i.e. for any other invariant A we have $A \subseteq J$.

Definition 11 (Invariant)
The invariant J of a system f is the maximal solution of the fixed-point equation:

$$X \subseteq f^{-1}(X) \cap f(X).$$

Proposition 12
The invariant J of a system f can be computed by successive approximations:

$$(\cap_n f^n(E)) \cap (\cap_n f^{-n}(E)).$$

PROOF. Using Tarski's fixpoint theorem [29].
□

Following our idea of working with symbolic dynamics, the components of an invariant can be parametrized by traces of symbols, too.

Definition 13 (Trace-based invariant)
The trace-based invariant $J_{\sigma,\tau}$ parametrized by the traces σ, representing the past, and τ, representing the future, is given by the following expression:

$$\forall \sigma, \tau \in \Sigma^\omega, J_{\sigma,\tau} = f_\sigma(E) \cap f_\tau^{-1}(E).$$

Proposition 14
The global invariant is equivalent to the union of all possible trace-based invariants:

$$J = \cup_{\sigma,\tau \in \Sigma^\omega} J_{\sigma,\tau}.$$

PROOF. By induction on trace-parametrized invariants.
□

The dynamical richness of a system seems to strongly depend on the structure of its invariants [13, 33, 25, 30, 31], which motivates the following definition. In terms of these trace-based invariants, it is possible to characterize the structure of the global invariant J of a system f.

Definition 15 (Packed invariance)
The invariant J of f is packed iff each bi-infinite symbolic orbit (trace) determines exactly one state:

$$\forall \sigma, \tau \in \Sigma^\omega, \# J_{\sigma,\tau} = 1.$$

Finally, we have a theorem to characterize the invariant of the union of two systems.

Theorem 16 (Union invariant)
The union of two injective systems having compatible dynamics and different packed fixpoint invariants, has a packed invariant with a Cantor set structure (closed, totally disconnected, perfect set).

By *compatible dynamics*, we mean: f and g have the same dimension; in each dimension, f and g are both contracting or expanding; the union is globally contracting in the future ($\gamma(f) + \gamma(g) < 1$) or in the past ($\gamma(f^{-1}) + \gamma(g^{-1}) < 1$). The Lipschitz constant $\gamma(f)$ is defined as follows:

$$\gamma(f) = \sup_{x \neq y} \frac{d_H(f(x), f(y))}{d(x, y)}$$

where d (d_H) is a classical (Hausdorff) metric. Let us sketch the proof of this result, the full proof of which can be found in [16, Chap. 6, Cor. 6.32].

PROOF. Functions f and g are one-to-one and contracting, $\gamma(f) + \gamma(g) < 1$, and $J^f \neq J^g$. By [34, Theorem D] this implies $\forall \sigma, \tau \in \Sigma^\omega, \# J_{\sigma,\tau} = 1$. Thus, J is packed.

Functions f and g are compatible (e.g., both contracting). This implies that the union $f \cup g$ is contracting, using [18, Lemma 2.3]. By [18, Theorem 3.1], there exists a unique compact set K such that $K = (f \cup g)(K)$, and $\forall Q \subseteq E, \lim_{n \to \omega} (f \cup g)^n(Q) = K$.

Both f and g are contracting, and $\gamma(f) + \gamma(g) < 1$. This implies by [19, §3.1(9)] that J is totally disconnected.

Functions f and g are contracting with different fixpoint invariants $J^f \neq J^g$. This implies, using [18, Theorem 4.3], that J is perfect.

Since J is compact, totally disconnected, and perfect, it is a Cantor set [14].

□

4 Dynamical complexity of paper foldings

In this section, we consider paper foldings as dynamical systems on symbol sequences, and we characterize the invariants and dynamics of $U \cup D$, and their inverses, using Theorem 16.

The functions U and D we use are defined in §2 (see Def(s) 2–6). Their domain is the set of infinite instructions $\mathcal{L}^\omega$.

4.1 Metric properties of foldings

Let us fix a metric in $\mathcal{L}^\omega$ (it also holds $\mathcal{J}^\omega$).

Definition 17 (Metric)
Let $x, y \in \mathcal{L}^\omega$, then

$$d(x, y) = c^{\inf\{i | x_i \neq y_i\}}.$$

Remark 18
Altough $c < 1$ is sufficient to keep this distance bounded, we will see in the sequel (see proof of Theorem 23) that we may need a stronger condition to prove that foldings are chaotic. Hence, we will consider $c < \frac{1}{2}$.

Using this metric, it is easy to show that U and D are continuous and contracting: $\forall a \in \{U, D\}$,

$$\forall w, w', d(a(w), a(w')) \leq c \cdot d(w, w').$$

Since we know that $W(w)$ does not depend on w when $W \in \mathcal{J}^\omega$, let us consider the well-defined application

$$\chi : \mathcal{J}^\omega \to \mathcal{L}^\omega : W \to \chi(W) = W(\varepsilon).$$

Proposition 19
The application χ is both continuous and injective.

Proof. To prove continuity, we have to show that $\forall W, \varepsilon, \exists \delta, \forall W'$,

$$d(W, W') \leq \delta \Rightarrow d(\chi(W), \chi(W')) \leq \varepsilon.$$

If W and W' are equal up to position n, i.e. $d(W, W') \leq c^{n+1}$, $\chi(W)$ and $\chi(W')$ are the same up to position $2^n - 1$, i.e. $d(\chi(W), \chi(W')) \leq c^{2^n}$. Thus, ε being fixed, it suffices to take $\delta = c^{1+\log_2 \log_c \varepsilon}$.

Injectivity is easy to prove. Let us suppose that $W, W' \in \mathcal{J}^\omega$ and they differ from position k, i.e. $\forall i < k, W_i = W'_i$ and $W_k \neq W'_k$. In this case, $\forall i < k, (W|_i)(\varepsilon) = (W'|_i)(\varepsilon)$ but $(W|_k)(\varepsilon) \neq (W'|_k)(\varepsilon)$ from position 2^{k-1}. Thus, we have $W \neq W' \Rightarrow \chi(W) \neq \chi(W')$.
□

4.2 Foldings as dynamical systems

Let us summarize the properties of the folding functions U and D.

- Foldings are continuous contracting functions.
- Sequences of landscapes are Cauchy sequences and converge to a limit.
- Moreover, by the contraction mapping theorem, the successive iterations of these functions, starting from any word, converge to a unique limit which is a fixed-point. If we consider uniform instructions like $UUU\cdots$ (resp. $DDD\cdots$), then, by continuity of foldings, the limits are fixed-points of U (resp. D). Remark that these fixed-points are reachable from any initial landscape.

4.3 Cantor structure of paper foldings

The main result of this paper follows: we prove that the union of up and down paper foldings has a packed invariant which has a Cantor-set structure. It is in itself not surprising but the way we prove it is interesting because we use dynamical systems notions in the context of paper foldings, that is, formal systems. Before proving the theorem, let us state three lemmas, ang give their proof.

Lemma 20
The functions defined in Def(s) 2–6 are injective.

PROOF. The two functions U and D are clearly injective: $\forall w, w' \in \mathcal{L}^\omega$, if $\exists k : w_k \neq w'_k$, $U(w)_{2k+1} \neq U(w')_{2k+1}$. Their inverses are also injective when restricted to $V \cdot \Lambda \cdot V \cdot \Lambda \cdots$ and $\Lambda \cdot V \cdot \Lambda \cdot V \cdots$ respectively.
□

Lemma 21
The dynamics of the systems defined by Def(s). 2–6 are compatible.

PROOF. The dynamics of these functions are compatible: they are both contracting in the future, and their union is globally contracting in the future, too. The last argument is more technical: $\gamma(U) = \gamma(D)$, thus we have to show that $\gamma(U) \leq \gamma < \frac{1}{2}$ (which is sufficient to guarantee that $\gamma(U) + \gamma(D) < 1$);

$$\begin{aligned} & \gamma(U) \leq \gamma \\ \Leftarrow \quad & \sup_{x \neq y} \frac{d(U(x), U(y))}{d(x,y)} \leq \gamma \\ \Leftarrow \quad & \sup_{x \neq y} \frac{c^{\inf\{i | U(x)_i \neq U(y)_i\}}}{c^{\inf\{i | x_i \neq y_i\}}} \leq \gamma \\ \Leftarrow \quad & \sup_{x \neq y} \frac{c^{1+\inf\{i | x_i \neq x_i\}}}{c^{\inf\{i | x_i \neq y_i\}}} \leq \gamma \\ \Leftarrow \quad & c \leq \gamma < \frac{1}{2}. \end{aligned}$$

Thus, we have to choose a specific c in order to guarantee this last condition.
□

Lemma 22
The individual invariants of the systems defined by Def(s). 2–6 are different packed fixpoints.

PROOF. The invariants of U and D are different: each application of U (resp. D) inserts a V (resp. Λ) at the left end of the word; at infinity, the fixed points cannot be the same. Since they are attractive fixed-points (by global contraction), they are trivially packed. In this case, there is only one possible trace for each system, and this trace determines exactly one point, the unique attractive fixed-point.
□

Theorem 23 (Cantor-set invariant)
The union of paper folding systems defined by Def(s). 2–6 generates a packed invariant having a Cantor-set structure.

PROOF. To prove the theorem, we apply Theorem 16. Thus, we have to verify a few assumptions:

- the functions are injective: Lemma 20;
- the dynamics are compatible: Lemma 21;
- the individual invariants are different packed fixpoints: Lemma 22.

In conclusion, we deduce from Theorem 16 that the invariant of the union of these two systems $U \cup D$ is a packed Cantor set. This union is interpreted as the set of all possible infinite landscapes resulting from infinite instructions.

□

The same result holds for the union of the inverse systems, $(U \cup D)^{-1}$, since our definitions of invariance and related properties are symmetric in time.

4.4 Cantor structure: the classical way

There is a classical way to recover the previous result. Let us investigate it and compare it with the compositional approach used in the proof of Theorem 23. First, we need the following lemma.

Lemma 24
The invariant of the union of U and D, J, is equivalent to $\chi(\mathcal{J}^\omega)$.

PROOF. The invariant is

$$J^{U\cup D} \;=\; \cap_{n\in\mathbb{Z}}(U \cup D)^n(\mathcal{L}^\omega).$$

Since $U^{-1}(\mathcal{L}^\omega) = \mathcal{L}^\omega$ and $D^{-1}(\mathcal{L}^\omega) = \mathcal{L}^\omega$, we have $(U \cup D)^{-1}(\mathcal{L}^\omega) = (U^{-1} \cup D^{-1})(\mathcal{L}^\omega) = U^{-1}(\mathcal{L}^\omega) \cup D^{-1}(\mathcal{L}^\omega) = \mathcal{L}^\omega$. Thus, the invariant can be simplified:

$$J^{U\cup D} \;=\; \cap_{n\in\mathbb{N}}(U \cup D)^n(\mathcal{L}^\omega).$$

The union $U \cup D$ is monotonic:

$$X \subseteq Y \;\Rightarrow\; (U \cup D)(X) \subseteq (U \cup D)(Y).$$

Moreover, $(U \cup D)(\mathcal{L}^\omega) \subseteq \mathcal{L}^\omega$. Hence, we rewrite the invariant as follows:

$$J^{U\cup D} \;=\; \lim_{n\to\omega}(U \cup D)^n(\mathcal{L}^\omega).$$

Finally, $(U \cup D)^n(\mathcal{L}^\omega) = \cup_{w\in\{U,D\}^n} w(\mathcal{L}^\omega)$, and

$$\begin{aligned} J^{U\cup D} &= \lim_{n\to\omega} \cup_{w\in\{U,D\}^n} w(\mathcal{L}^\omega) \\ &= \cup_{w\in\{U,D\}^\omega} w(\mathcal{L}^\omega) \\ &= \chi(\mathcal{J}^\omega). \end{aligned}$$

□

Let us now give another proof of Theorem 23.

PROOF. Since $\mathcal{J}^\omega$ is a Cantor set, and χ is an injective continuous function from $\mathcal{J}^\omega$ to $\mathcal{L}^\omega$, $\chi(\mathcal{J}^\omega)$ is a Cantor set. Thus, J is a Cantor set, too.
□

The classical way involves a quite technical lemma and a proof treating a global system. Our compositional approach states the problem differently: once the system is decomposed into simple subsystems, some easy assumptions have to be verified, and the global result follows automatically. Of course, technically speaking, we have to compare Lemma 24 with Theorem 16, but the "end-user" can consider the proof of Theorem 16 as a black box. This is the general advantage of any compositional (i.e. modular) approach.

4.5 Chaos in paper foldings

In addition to the result of Theorem 23, it is also possible to show that the paper folding dynamical system is chaotic on its Cantor-set invariant.

Classically, chaos is based on the following properties. Let f be a system and J be its invariant.

Topological transitivity (TT): $\exists x \in J : \forall y \in J, \varepsilon, \exists n : d(f^n(x), y) < \varepsilon$.

Density of periodic points (DPP): $\forall x \in J, \forall \varepsilon > 0, \exists y \in J, m \in \mathbb{N} : y$ is periodic $\wedge$ $d(x, y) < \varepsilon$.

Sensitivity to initial conditions (SIC): $\exists \delta > 0 : \forall x, y \in J, \varepsilon, \exists n \in \mathbb{N} : d(x, y) < \varepsilon \wedge d(f^n(x), f^n(y)) > \delta$.

On one hand, Devaney's well-known definition of chaos relies on the conjunction of these three properties [13]. However, it has been shown that they are redundant: (TT) and (DPP) imply (SIC) [6] and, when restricted to intervals, (TT) implies both (DPP) and (SIC) [32]. On the other hand, Knudsen has proposed a more general definition based on the conjunction of (TT) and (SIC) without (DPP) [20].

Definition 25 (Knudsen chaos)
A system f is (Knudsen) chaotic on its invariant J if (TT) and (SIC) hold.

Since packed invariance implies both (TT) and (SIC), we can prove that paper foldings are chaotic on their invariant. First, let us refine packed invariance into two properties, namely fullness and atomicity.

Definition 26 (Full/atomic invariance)
The invariant J of f is full (resp. atomic) iff each bi-infinite symbolic orbit determines exactly at least (resp. at most) one state: $\forall \sigma, \tau \in \Sigma^\omega, \#J_{\sigma,\tau} \geq 1$ (resp. $\#J_{\sigma,\tau} \leq 1$).

Fullness assures that sufficiently many traces are observable, which limits the precision level. Atomicity rather guarantees the observation to be fine enough in order to get useful information on the dynamics.

Proposition 27 (Topological transitivity)
If f_1 and f_2 are injective, then fullness implies that any part of the invariant J of $f_1 \cup f_2$ can be reached from any other part in a finite number of iterations. Let $\Sigma = \{1, 2\}$,

$$\forall \sigma_1, \sigma_2, \tau_1, \tau_2 \in \Sigma^*, f_{\tau_1\sigma_2}(J_{\sigma_1,\tau_1}) \cap J_{\sigma_2,\tau_2} \neq \emptyset$$

Fullness and atomicity entail topological transitivity.

PROOF. Fullness implies that $\forall \sigma_1, \sigma_2, \tau_1, \tau_2 \in \Sigma^*, \exists \sigma, \tau \in \Sigma^\omega, J_{\sigma\sigma_1,\tau_1\sigma_2\tau_2\tau} \neq \emptyset$. Moreover, as $\forall i, f_i$ is injective, we have $f_{\tau_1\sigma_2}(J_{\sigma\sigma_1,\tau_1\sigma_2\tau_2\tau}) = f_{\tau_1\sigma_2}(J_{\sigma\sigma_1,}) \cap f_{\tau_1\sigma_2}(J_{,\tau_1\sigma_2\tau_2\tau})$. Thus,

$$\begin{array}{ll} & J_{\sigma\sigma_1,\tau_1\sigma_2\tau_2\tau} \subseteq J_{\sigma_1,\tau_1} \subseteq J \\ & \quad \because \text{ monotonicity and injectivity} \\ \Rightarrow & J_{\sigma\sigma_1\tau_1\sigma_2,\tau_2\tau} \subseteq f_{\tau_1\sigma_2}(J_{\sigma_1,\tau_1}) \\ & \quad \because \ J_{\sigma\sigma_1\tau_1\sigma_2,\tau_2\tau} \subseteq J_{\sigma_2,\tau_2} \\ \Rightarrow & (f_{\tau_1\sigma_2}(J_{\sigma_1,\tau_1})) \cap J_{\sigma_2,\tau_2} \neq \emptyset. \end{array}$$

By atomicity, these parts of J can be as small as desired.
□

Proposition 28 (Sensitivity to initial conditions)
Fullness and atomicity entail sensitive dependence on initial conditions.

PROOF. Take two distinct states x and y in the invariant J. Atomicity implies a kind of contraction: there is at most one point in every bi-infinite invariant. Fullness guarantees that these sub-invariants are never empty. Since $x \neq y$, there exist $\sigma, \sigma', \tau, \tau'$ such that $x = J_{\sigma,\tau}$ and $y = J_{\sigma',\tau'}$. These bi-infinite traces are different and the first place where they differ gives the n we need to make the iterations diverge.
□

Remark 29
The symmetry of the systems we work with allows us to consider a sensitive dependence on final conditions, too.

Corollary 30 (Chaotic paper foldings)
The dynamical system defined by Def(s) 2–6 and its inverse are both chaotic on their invariant set.

PROOF. By Theorem 23, the invariant is packed and, thus, full and atomic. By Prop(s) 27 and 28, the system is both topologically transitive and sensitive to initial consitions. By Def. 25, the system is Knudsen chaotic.
□

5 Conclusions

Many papers have been written on paper foldings up to now. Our approach to this surprising and interesting part of mathematics is orthogonal to these classicals.

Indeed, it is instructive to study these paper foldings as dynamical systems because the approach itself is new, it brings new results, such as the chaotic aspect of the systems involved, and it allows to recover old results in a clear way, like the presence of a Cantor invariant set.

As illustration of the union-invariant theorem, we have seen that composing two dynamically compatible systems with different fixpoints can lead to a complex behavior sustained by a structurally rich (i.e. Cantor-set structure) invariant set.

We have here a typical example of rich behavior, dynamically complex, resulting from the evolution of a system with a simple structure, the union composition of simple systems.

In fact, the same phenomenon has been observed in different kinds of systems like classical dynamical systems (e.g. Smale Horseshoe Map [26], Cantor relation and logistic map [16]) or cellular automata (disjunctive composition of two shifts [8, 15]): complexity arises from the composition of compatible systems attracting the space to different regions in the future or in the past.

A kind of hyperbolic behavior sustains all these rich dynamics in complex systems composed from simpler ones having simple, attracting, symmetric dynamics. We are investigating this open and interesting question.

Let us conclude on the result we have presented here: we have embedded paper foldings in the context of dynamical systems and we have shown that these systems are chaotic on a Cantor invariant set, using a straightforward decomposition of a global system into subsystems respectively corresponding to up and down foldings.

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