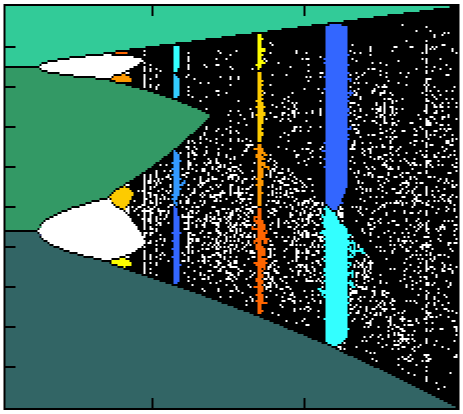


Basic Concepts in Nonlinear Dynamics and Chaos

"Out of confusion comes chaos.

Out of chaos comes confusion and fear.

Then comes lunch."

A **Workshop** presented at the **Society for Chaos Theory in Psychology and the Life Sciences** meeting, July 31,1997 at Marquette University, Miwaukee, Wisconsin.

Table of Contents

Introduction to Dynamic Systems

What is a dynamic system?

A dynamic system is a set of **functions** (rules, equations) that specify how variables change over time.

First example ...
Alice's height diminishes by half every minute...

Second example ...
$\mathbf{x}_{new} = \mathbf{x}_{old} + \mathbf{y}_{old}$
$\mathbf{y}_{new} = \mathbf{x}_{old}$

The second example illustrates a system with two variables, **x** and **y**. Variable **x** is changed by taking its old value and adding the current value of **y**. And **y** is changed by becoming **x**'s old value. Silly system? Perhaps. We're just showing that a dynamic system is any well-specified set of rules.

Here are some important Distinctions:

- **variables** (dimensions) vs. **parameters**
- **discrete** vs. **continuous** variables
- **stochastic** vs. **deterministic** dynamic systems

  How they differ:
- **Variables** change in time, **parameters** do not.
- **Discrete** variables are restricted to integer values, **continuous** variable are not.
- **Stochastic** systems are one-to-many; **deterministic** systems are one-to-one

  This last distinction will be made clearer as we go along ...

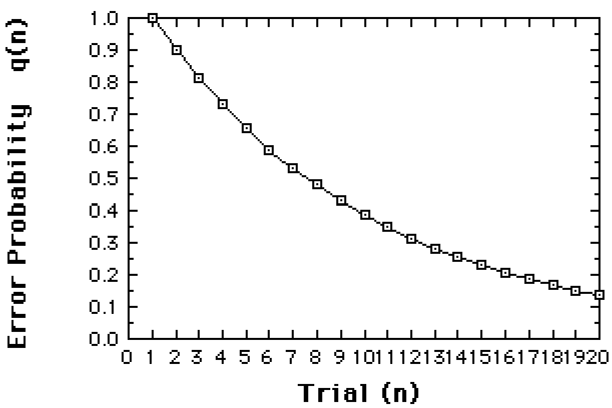
Terms

The current **state** of a dynamic system is specified by the current value of its variables, x, y, z, ...
The process of calculating the new state of a *discrete* system is called **iteration**.

To evaluate how a system behaves, we need the functions, parameter values and **initial conditions** or **starting state.**
To illustrate...Consider a classic learning theory, the *alpha model*, which specifies how q_n, the probability of making an error on trial n, changed from one trial to the next
$\mathbf{q_{n+1} = ß\, q_n}$ The new error probability is diminished by ß (which is less than 1, greater than 0). For example, let the the probability of an error on trial 1 equal to 1, and ß equal .9. Now we can calculate the dynamics by iterating the function,

and plot the results.

$q_1 = 1$
$q_2 = ßq_1 = (.9)(1) = .9$
$q_3 = (.9)q_2 = (.9)(.9) = .81$
etc. ...



Error probabilities for the alpha model, assuming q_1=1, ß =.9. This "learning curve" is referred to as a **time series**.

So far, we have some new ideas, but much is old ...

What's *not* new

Dynamic Systems
Certainly the idea that systems change in time is not new. Nor is the idea that the changes are probabilistic.

What's new

Deterministic nonlinear dynamic systems.

As we will see, these systems give us:

- A new meaning to the term *unpredictable*.
- A different attitude toward the concept of *variability*.
- Some new *tools* for exploring time series data and for modeling such behavior.
- And, some argue, a new *paradigm*.

This last point is not pursued here.

Nonlinear Dynamic Systems

Nonlinear functions

What's a linear function?

Well, gee Mikey, it's one that can be written in the form of a straight line. Remember the formula ...
$y = mx + b$
where m is the slope and b is the y-intercept?

What's a nonlinear function?

What makes a dynamic system *nonlinear* ....
is whether the function specifying the change is nonlinear.
Not whether its behavior is nonlinear.
And y is a nonlinear function of x if x is multiplied by another (non-constant) variable, or multiplied by itself (i. e., raised to some power).
We illustrate *nonlinear* systems using ...

Logistic Difference Equation

... a model often used to introduce chaos. The Logistic Difference Equation, or *Logistic Map*, though simple, displays the major chaotic concepts.

Growth model

We start, generally, with a model of growth.

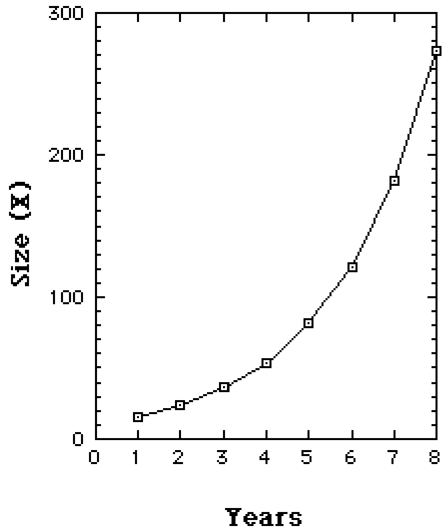
$$\mathbf{x}_{new} = \mathbf{r}\ \mathbf{x}_{old}$$

We prefer to write this in terms of n:

$$\mathbf{x}_{n+1} = \mathbf{r}\ x_n.$$

This says x changes from one time period, n, to the next, n+1, according to r. *If r is larger than one*, x gets larger with successive iterations *If r is less than one*, x diminishes. (In the "Alice" example at the beginning, r is .5).
Let's set r to be larger than one...

We start, year 1 (n=1), with a population of 16 [x_1=16], and since r=1.5, each year x is increased by 50%. So years 2, 3, 4, 5, ... have magnitudes 24, 36, 54, ...
Our population is growing exponentially. By year 25 we have over a quarter million.



Iterations of Growth model with r = 1.5

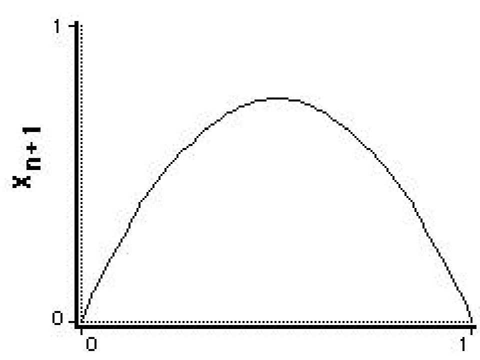
So far, notice, we have a *linear* model that produces unlimited growth.

Limited Growth model - Logistic Map.

The Logistic Map prevents unlimited growth by inhibiting growth whenever it achieves a high level. This is achieved with an additional term, [1 - x_n].
The growth measure (x) is also rescaled so that the maximum value x can achieve is transformed to 1. (So if the maximum size is 25 million, say, x is expressed as a proportion of that maximum.)
Our new model is

$$\mathbf{x_{n+1} = r\, x_n\, [1 - x_n]}$$

[r between 0 and 4.]
The **[1-x_n]** term serves to inhibit growth because as **x** approaches **1**, **[1-x_n]** approaches **0**.

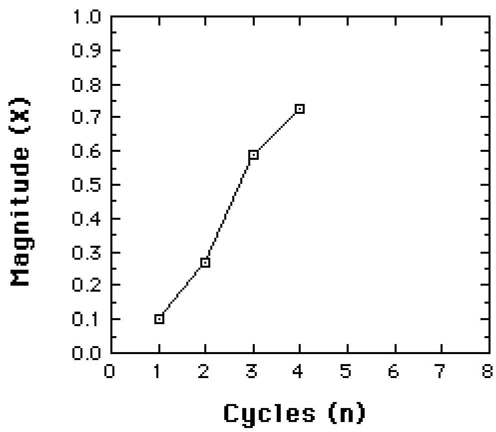


Plotting x_{n+1} vs. x_n, we see we have a nonlinear relation.

Limited growth (Verhulst) model. X_{n+1} vs. x_n, r = 3.

We have to **iterate this function** to see how it will behave ...
Suppose r=3,
and x_1=.1

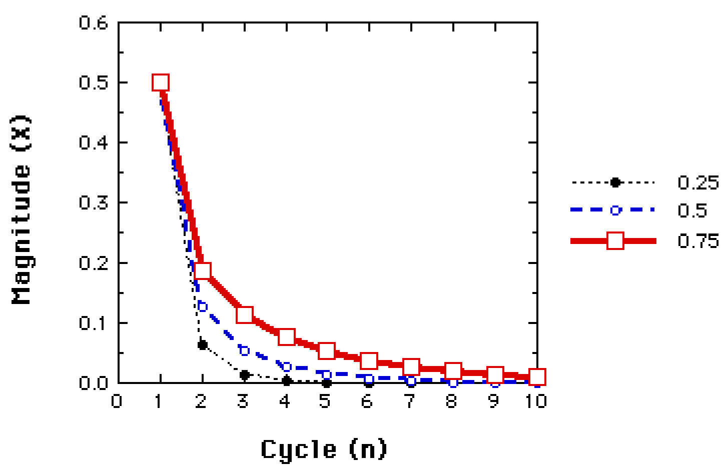
$x_2 = rx_1[1-x_1] =$
$3(.1)(.9) = .27$
$x_3= r\ x_2[1-x_2]=$
$3(.27)(.73) =$
$.591$
$x_4= r\ x_3[1-_3]=$
$3(.591)(.409) =$
$.725$



Behavior of the Logistic map for r = 3, x_1 = .1, iterated to give x_2, x_3, and x_4

It turns out that the logistic map is a very different animal, depending on its control parameter r. To see this, **we next examine the time series produced at different values of r**, starting near 0 and ending at r=4. Along the way we see very different results, revealing and introducing major features of a chaotic system.

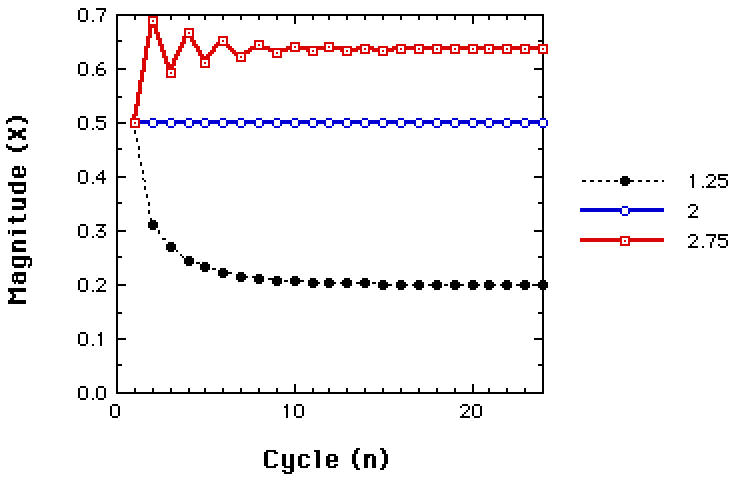
When r is less than 1



Behavior of the Logistic map for r=.25, .50, and .75. In all cases x_1=.5.

The same fates awaits any starting value. So long as r is less than 1, x goes toward 0. This illustrates a **one-point attractor.**

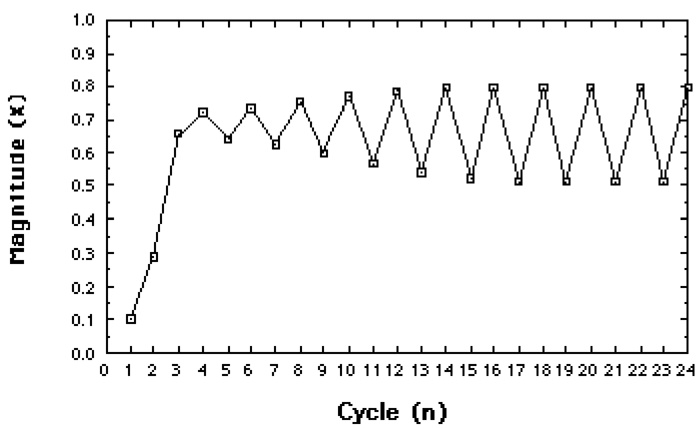
When r is between 1 and 3



Behavior of the Logistic map for r=1.25, 2.00, and 2.75. In all cases x_1*=.5.*

Now, regardless, of the starting value, we have non-zero one-point attractors.

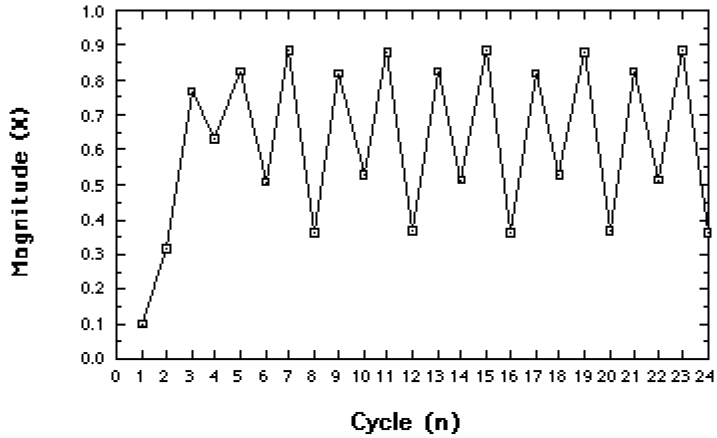
When r is larger than 3



Behavior of the Logistic map for r=3.2.

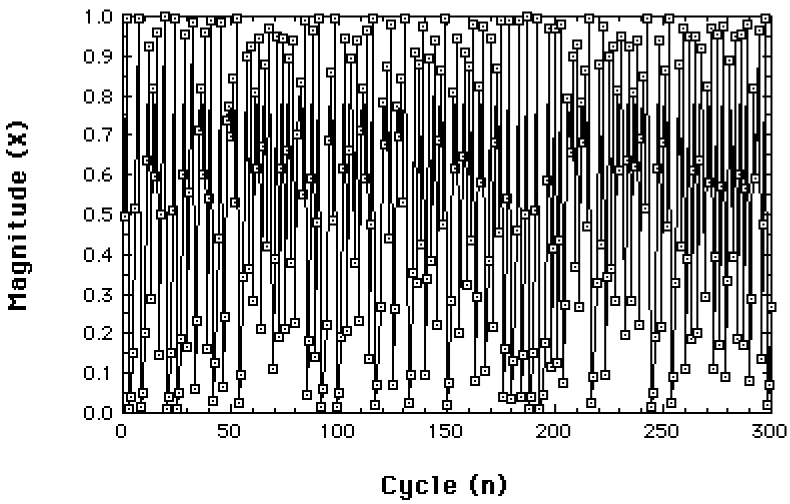
Moving just beyond r=3, the system settles down to alternating between two points. We have a *two-point attractor*. We have illustrated a **bifurcation**, or **period**

doubling,



Behavior of the Logistic map for r= 3.54. Four-point attractor

Another bifurcation. The concept: an **N-point attractor**.



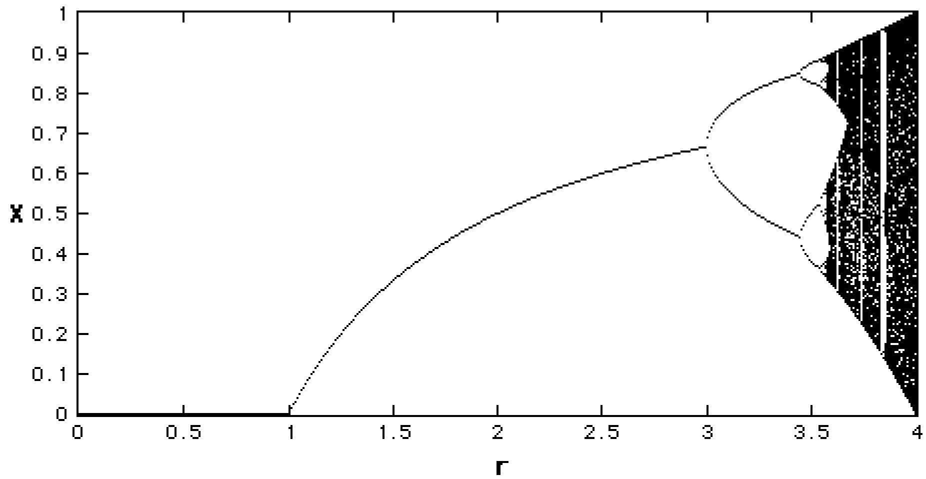
Chaotic *behavior of the Logistic map at r= 3.99.*

So, what is an **attractor**? Whatever the system "settles down to".
Here is a very important concept from nonlinear dynamics: A system eventually "settles down". But what it settles down to, its attractor, need not have 'stability'; it can be very 'strange'.

Bifurcation Diagram

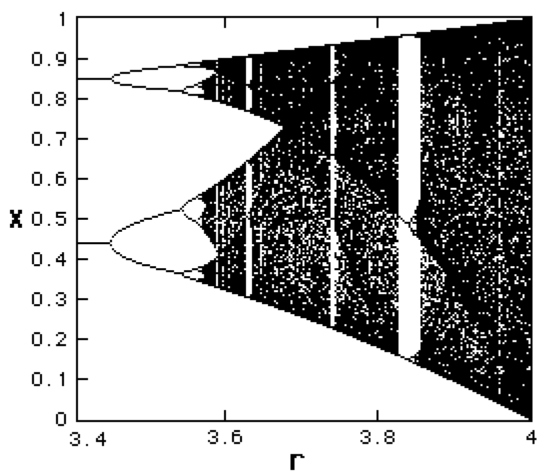
So, again, what is a *bifurcation*? A bifucation is a period-doubling, a change from an N-point attractor to a 2N-point attractor, which occurs when the control parameter is

changed.
A Bifurcation *Diagram* is a visual summary of the succession of period-doubling produced as r increases. The next figure shows the bifurcation diagram of the logistic map, r along the x-axis. For each value of r the system is first allowed to settle down and then the successive values of x are plotted for a few hundred iterations.



Bifurcation Diagram r between 0 and 4

We see that for r less than one, all the points are plotted at zero. Zero is the one point attractor for r less than one. For r between 1 and 3, we still have one-point attractors, but the 'attracted' value of x increases as r increases, at least to r=3. Bifurcations occur at r=3, r=3.45, 3.54, 3.564, 3.569 (approximately), etc., until just beyond 3.57, where the system is chaotic.
However, the system is not chaotic for all values of r greater than 3.57.
Let's zoom in a bit.

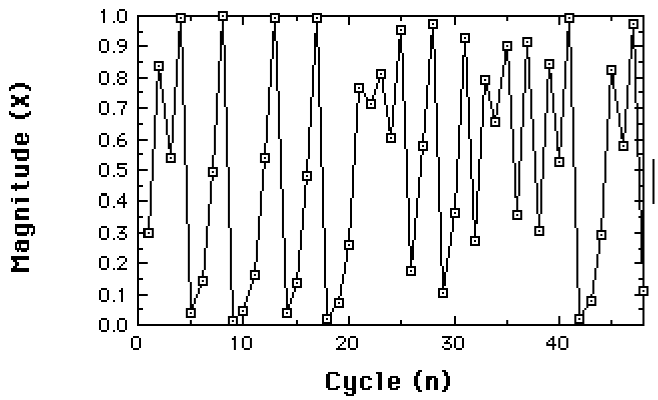


Bifurcation Diagram r between 3.4 and 4

Notice that at several values of r, greater than 3.57, a small number of x=values are visited. These regions produce the 'white space' in the diagram. Look closely at r=3.83 and you will see a three-point attractor.
In fact, between 3.57 and 4 there is a rich interleaving of chaos and order. A small change in r can make a stable system chaotic, and vice versa.

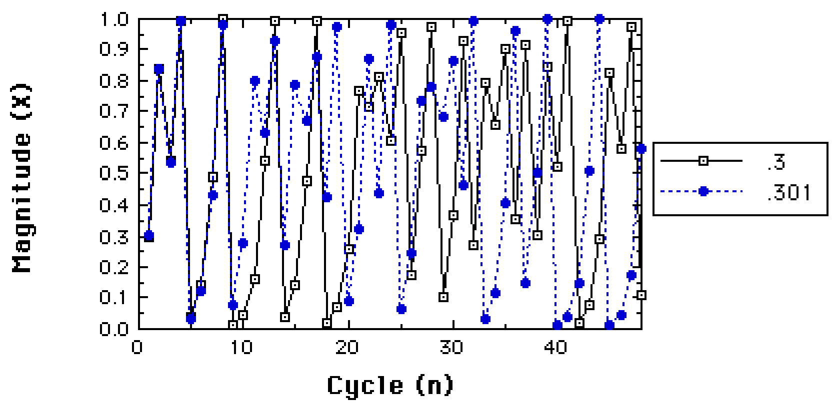
Sensitivity to initial conditions

Another important feature emerges in the chaotic region ... To see it, we set r=3.99 and begin at x_1=.3. The next graph shows the time series for 48 iterations of the logistic map.



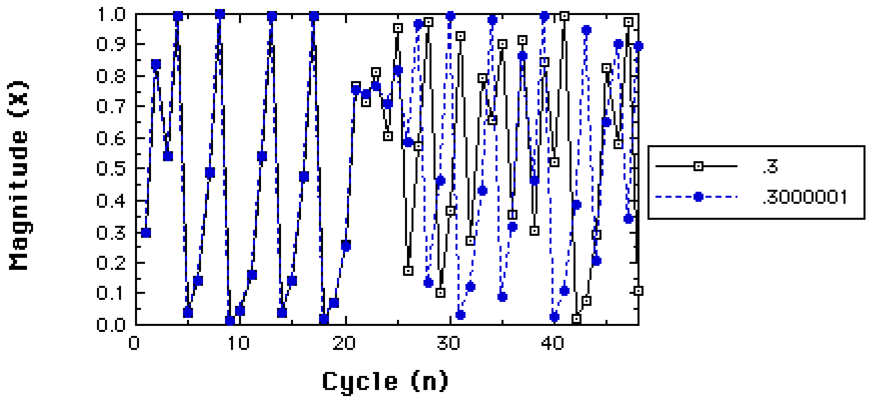
Time series for Logistic map r=3.99, x_1=.3, 48 iterations.

Now, suppose we alter the starting point a bit. The next figure compares the time series for x_1=.3 (in black) with that for x_1=.301 (in blue).



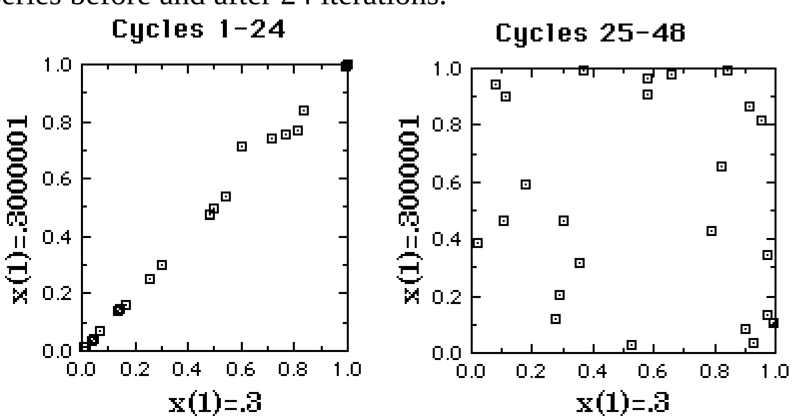
Two time series for r=3.99, x_1=.3 compared to x_1=.301

The two time series stay close together for about 10 iterations. But after that, they are pretty much on their own. Let's try starting closer together. We next compare starting at .3 with starting at .3000001...



Two time series for r=3.99, x_1*=.3 compared to* x_1*=.3000001*

This time they stay close for a longer time, but after 24 iterations they diverge. To see just how independent they become, the next figure provides scatterplots for the two series before and after 24 iterations.



Scatterplots of series starting at .3 vs. series starting at .3000001.
The first 24 cycles on the left, next 24 on the right.

The correlation after 24 iterations (right side), is essentially zero. Unreliability has replaced reliability.

We have illustrated here one of the symptoms of chaos. A chaotic system is one for which *the distance between two trajectories from nearby points in its state space diverge over time*. The magnitude of the divergence increases *exponentially* in a chaotic system.

So what? Well, it means that a chaotic system, even one determined by a simple rule, is in principle unpredictable. Say what? It is unpredictable, "in principle" because in order to predict its behavior into the future we must know its currrent value *precisely*. We have here an example where a slight difference, in the sixth decimal place, resulted in prediction failure after 24 iterations. And six decimal places far exceeds the kind of measuring accuracy we typically achieve with natural biological systems.

Symptoms of Chaos

We are beginning to sharpen our definition of a chaotic system. First of all, it is a *deterministic* system. If we observe behavior that we suspect to be the product of a chaotic system, it will also be

- difficult to distinguish from random behavior
- sensitive to initial conditions

Note well: Neither of these symptoms, on their own, are *sufficient* to identify chaos.

Note on technical vs. metaphorical uses of terms:

Students of chaotic systems have begun to use the (originally mathematical) terms in a "metaphorical" way. For example, 'bifurcation', defined here as a period doubling has come to be used to refer to any qualititave change. Even the term 'chaos', has become synomous, for some, with 'overwhelming anxiety'.

Metaphors enrich our understanding, and have helped extend nonlinear thinking into new areas. On the other hand, it is important that we are aware of the technical/metaphorical difference.

Two- and Three-Dimension Systems

First we practice the distinction between variables (dimensions) and parameters

Consider again the Logistic map

$$x_{n+1} = r\, x_n[1- x_n]$$

Multiply the right side out

$$x_{n+1}= r\, x_n - r\, x_n^2,$$

and replace the two r's with separate parameters, a and b,

$$x_{n+1}= a\, x_n - b\, x_n^2.$$

Now, separate **parameters**, a and b, govern growth and suppression, but we still have only one **variable**, x.

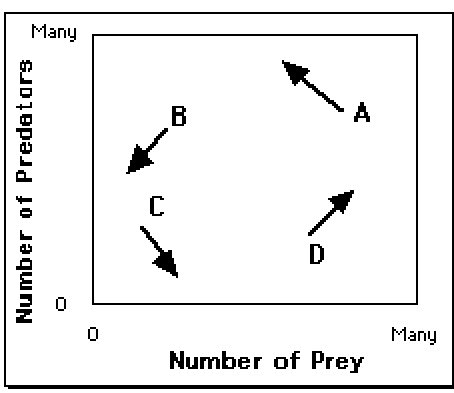
When we have a system with two or more variables,

- its current **state** is the current values of its variables, and is
- treated as a **point** in **phase (state) space**, and
- we refer to its **trajectory** or **orbit** in time.

Predator-prey system

This is a two-dimensional dynamic system in which two variables grow, but one grows at the expense of the other. The number of **predators** is represented by y, the number of **prey** by x.

We plot next the **phase space** of the system, which is a two-dimension plot of the possible states of the system.

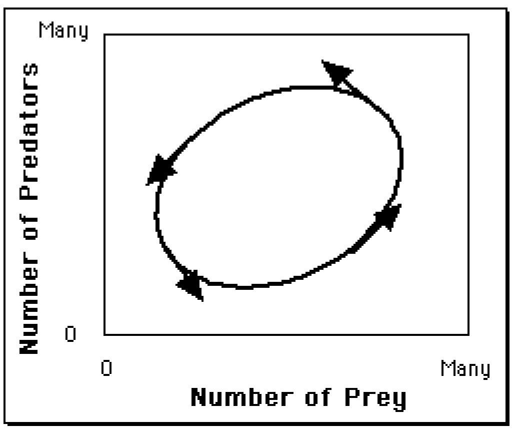


A = Too many predators.
B = Too few prey.
C = Few predator and prey; prey can grow.
D= Few predators, ample prey.

The phase-space of the predator-prey system.

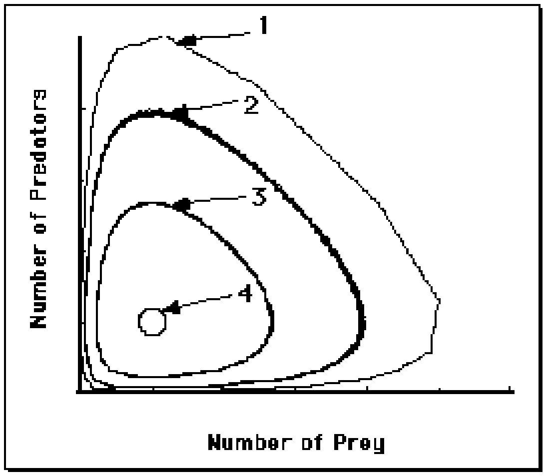
Four states are shown. At **Point A** there are a large number of predators and a large number of prey. Drawn from point A is an arrow, or **vector**, showing how the system would change from that point. Many prey would be eaten, to the benefit of the predator. The arrow from point A, therefore, points in the direction of a smaller value of x and a larger value of y.

At **Point B** there are many predators but few prey. The vector shows that both decrease; the predators because there are too few prey, the prey because the number of predators is still to the prey's disadvantage. At **Point C**, since there are a small number of predators the number of prey can increase, but there are still too few prey to sustain the predator population. Finally, at **point D**, having many prey is advantageous to the predators, but the number of prey is still too small to inhibit prey growth, so their numbers increase. The full **trajectory** (somewhat idealized) is shown next.



The phase-space of the predator-prey system.

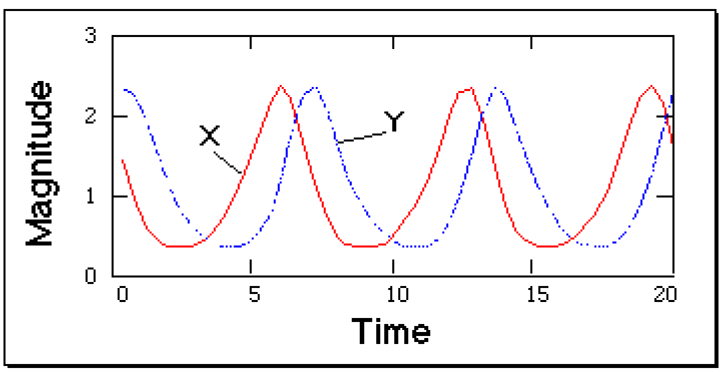
An attractor that forms a loop like this is called a **limit cycle**. However, in this case the system doeasn't start outside the loop and move into it as a final attractor. In this system any starting state is already in the final loop. This is shown in the next figure, which shows loops from four different starting states.



Phase-portrait *of the predator-prey system, showing the influence of starting state.*

Points 1-4 start with about the same number of prey but with different numbers of predators.

Let's look at this system over time, that is, as two **time series**.



The time series of the predator-prey system.

This figures shows how the two variables oscillate, out of phase.

Continuous Functions and Differential Equations

- Changes in ***discrete*** variables are expressed with **difference equations**, such as the logistic map.
- Changes in ***continuous*** variables are expressed with **differential equations**

For example, the Predator-prey system is typically presented as a set of two differential equations:

$$dx/dt = (a-by)x$$
$$dy/dt = (cx-d)y$$

Types of two-dimensional interactions

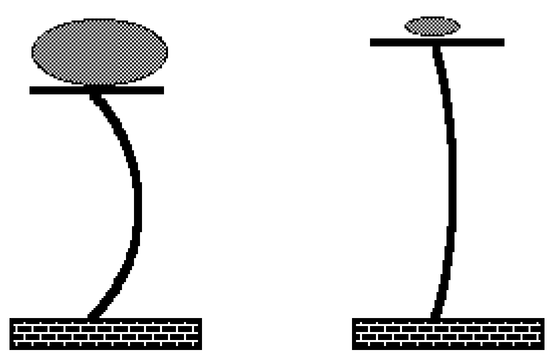
Other types of two-dimensional interactions are possible, as nicely categorized by van Geert (1991).

- mutually supportive - the larger one gets, the faster the other grows
- mutually competitive - each negatively affects the other
- supportive-competitive - as in Predator-prey

The Buckling column system

Abraham, Abraham, & Shaw (1990) used the Buckling Column system to discuss psychological phenomena that exhibit oscillations (for example, mood swings, states of consciousness, attitude changes). The model is a single, flexible, column that supports a mass within a horizontally constrained space. If the mass of the object is sufficiently heavy, the column will "give", or buckle. There are two dimensions, **x** representing the sideways displacement of the column, and **y** the velocity of its movement.

Shown next are two situations, differing in the magnitude of the mass.



The buckling column model (Abraham, Abraham, & Shaw, 1990).

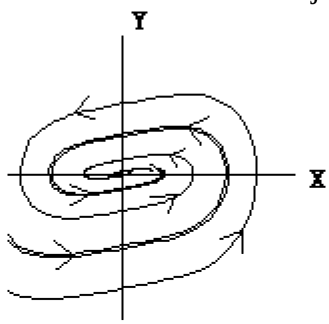
The mass on the left is larger than the mass on the right.

What are the dynamics? The column is elastic, so an initial give is followed by a springy return and bouncing (oscillations). If there is resistance (friction), the bouncing will diminish and the mass will come to rest. The equations are given for completeness only:

$$\frac{dx}{dt} = y$$

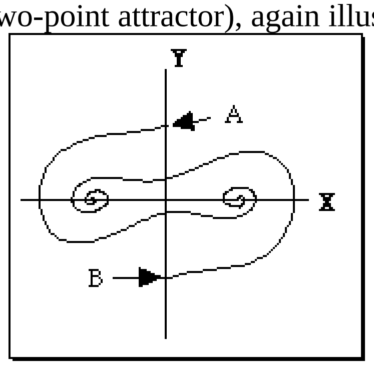
$$\frac{dy}{dt} = (1 - m)(ax^3 + b + cy)$$

The parameters m and c represent mass and friction respectively. If there is friction (c>0), and mass is small, the column eventually returns to the upright position (x=0, y=0), illustrated next with two trajectories.



Phase portrait of the buckling column model.

With a heavy mass, the column comes to rest in one of two positions (two-point attractor), again illustrated with two trajectories.

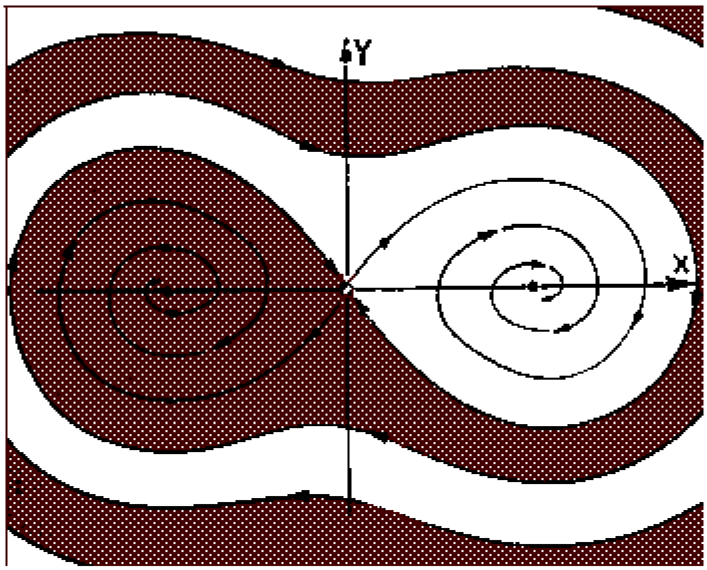


Phase portrait of the buckling column model.

Starting at point A, the system comes to rest buckled slightly to the right, starting at B ends up buckled to the left. Now we can introduce another major concept...

Basins of attraction

With sufficient mass, the buckling column can end up in one of two states, buckled to the left or to the right. What determines which is its fate? For a given set of parameter values, the fate is determined entirely by where it starts, the initial values of x and y. In fact, each point in phase space can be classified according to its attractor. The set of points associated with a given attractor is called that attractors' **basin of attraction**. For the two-point attractor illustrated here, there are two basins of attraction. These are shown in the next figure, which has the phase space shaded according to attractor.



The basins of attraction for the buckling column system. Reproduced from Abraham et al (1990).

The basin of attraction for the positive attractor (the one on the right) are shaded. The basin of attraction for the other attractor is unshaded in the figure. The term **seperatrix** is used to refer to the boundary between basins of attraction.

Questions to ponder

Is the buckling column system a chaotic system? Why (not)?

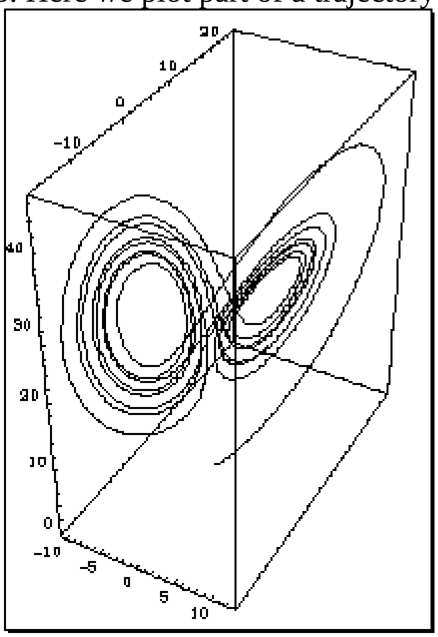
Three-dimensional Dynamic Systems

The Lorenz System

Lorenz's model of atmospheric dynamics is a classic in the chaos literature. The model nicely illustrates a three-dimensional system.

dx/dt = a(y-x)
dy/dt = x(b-z) - y
dz/dt = xy-cz

There are three variables reflecting temperature differences and air movement, but the details are irrelevant to us. We are interested in the trajectories of the system in its phase space for a=10, b=28, c=8/3. Here we plot part of a trajectory starting from (5,5,5).



The Lorenz system. Only a portion of one trajectory is shown.

Although the figure suggests that a trajectory may intersect with earlier passes, in fact it never does. Although not demonstrated here, the Lorenz system shows sensitivity to initial conditions. This is chaos, the first strange attractor, and it has become the icon for chaos.

Beasts in Phase space - Limit Points

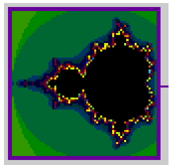
There are three kinds of limit points.

- **Attractors** - where the system 'settles down' to.
- **Repellors** - a point the system moves away from.
- **Saddle points** - attractor from some regions, repellor to others.

Examples

- **Attractors** - we've seen many
- **Repellors** - the value 0 in the Logistic Map
- **Saddle points** - the point (0,0) in the Buckling Column

Fractals and the Fractal Dimension



Mandelbrot and Nature

> "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."(Mandelbrot, 1983).

The Concept of Dimension

So far we have used "dimension" in two senses:

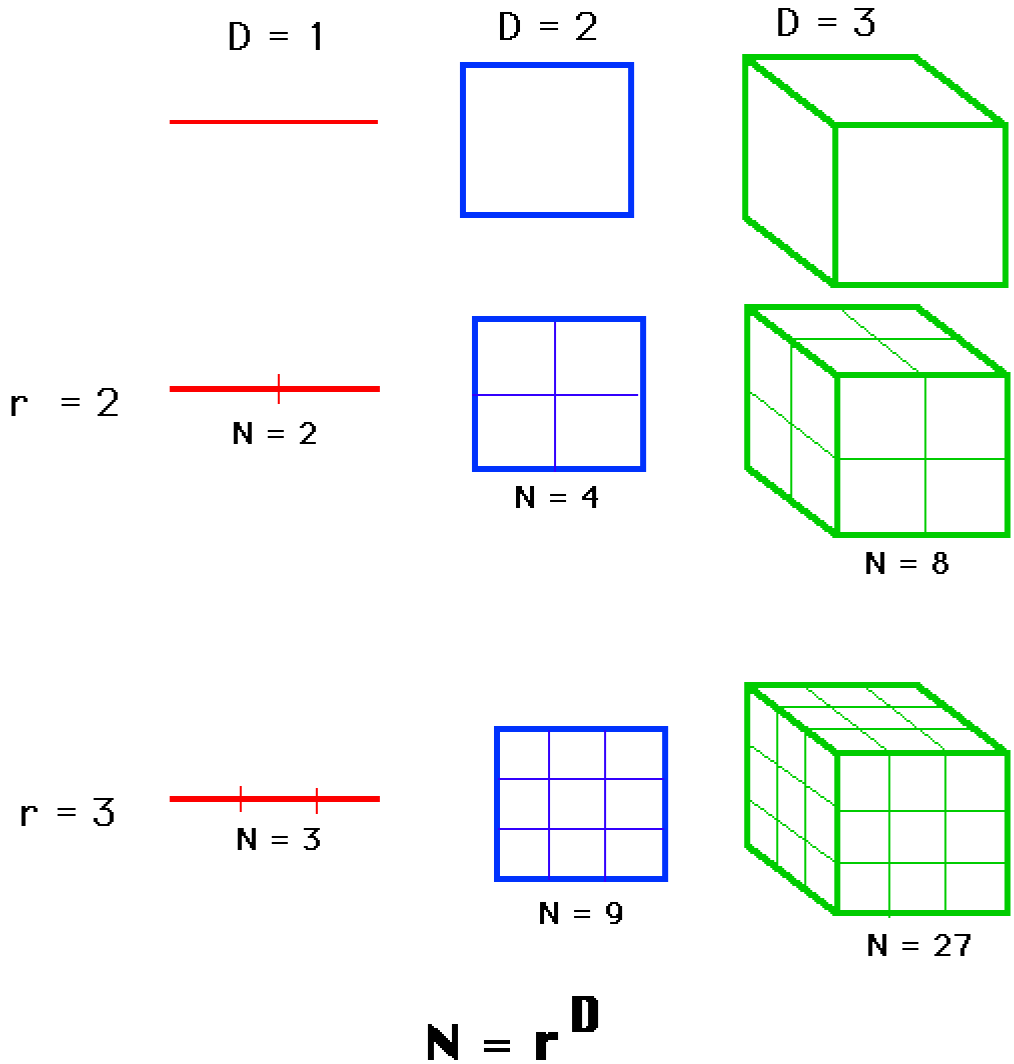
- The three dimensions of Euclidean space (D=1,2,3)
- The number of variables in a dynamic system

Fractals, which are irregular geometric objects, require a third meaning:

The Hausdorff Dimension

If we take an object residing in Euclidean dimension D and reduce its linear size by 1/r in each spatial direction, its measure (length, area, or volume) would increase to $N=r^D$

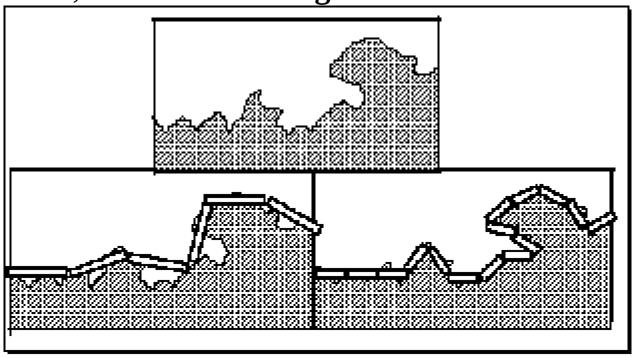
times the original. This is pictured in the next figure.



We consider $N=r^D$, take the log of both sides, and get log(N) = D log(r). If we solve for D. D = log(N)/log(r) The point: examined this way, D need not be an integer, as it is in Euclidean geometry. It could be a fraction, as it is in fractal geometry. This generalized treatment of dimension is named after the German mathematician, Felix Hausdorff. It has proved useful for describing natural objects and for evaluating trajectories of dynamic systems.

The length of a coastline

Mandelbrot began his treatise on fractal geometry by considering the question: "How long is the coast of Britain?" The coastline is irregular, so a measure with a straight ruler, as in the next figure, provides an estimate. The estimated length, L, equals the length of the ruler, s, multiplied by the N, the number of such rulers needed to cover the measured object. In the next figure we measure a part of the coastline twice, the ruler on the **right** is **half** that used on the **left**.

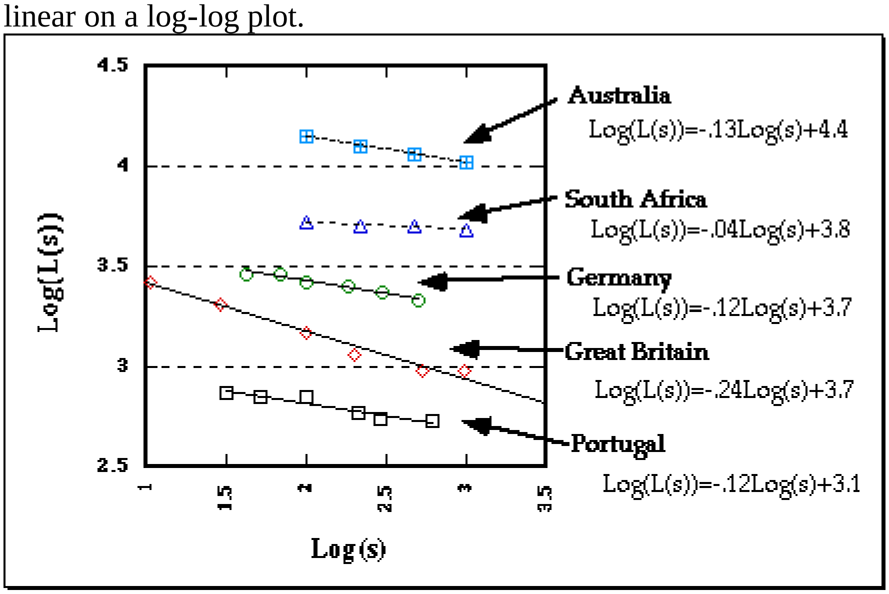


Measuring the length of a coastline using rulers of varying lengths.

But the estimate on the right is longer. If the the scale on the left is one, we have six units, but halving the unit gives us 15 rulers (L=7.5), not 12 (L=6). If we halved the scale again, we would get a similar result, a longer estimate of L. In general, as the ruler gets diminishingly small, the length gets infinitely large. The **concept of length, begins to make little sense.**

The "Richardson Effect"

Lewis Fry Richardson first noted the regularity between the length of national boundaries and scale size. As shown next, the relation between length estimate and length of scale is linear on a log-log plot.



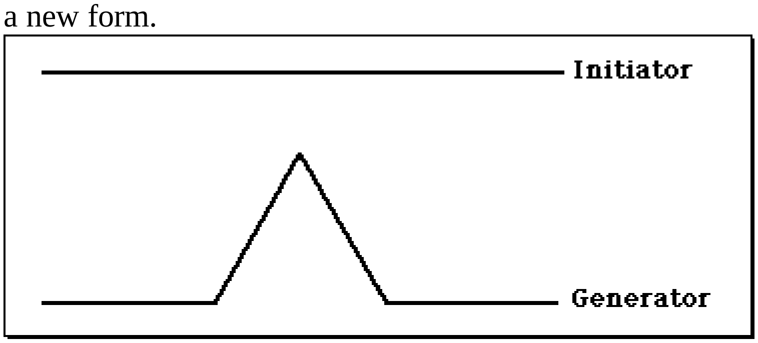
The Richardson Effect.

Mandelbrot assigned the term (1-D) to the slope, so the functions are:
log[L(s)] = (1-D)log(s) + b where D is the Fractal Dimension.
For Great Britain, 1 - D = -.24, approximately. D = 1-(-.24) = 1.24, a fractional value.The coastline of South Africa is very smooth, virtually an arc of a circle. The slope estimated above is very near zero. D = 1-0 = 1. This makes sense because the coastline is very nearly a regular Euclidean object, a line, which has dimensionality of one. In general, the "rougher' the line, the steeper the slope, the larger the fractal dimension.

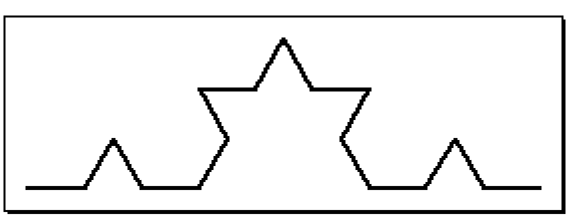
Examples of geometric objects with non-integer dimensions

Koch Curve

We begin with a straight line of length 1, called the **initiator**. We then remove the middle third of the line, and replace it with two lines that each have the same length (1/3) as the remaining lines on each side. This new form is called the **generator**, because it specifies a rule that is used to generate a new form.

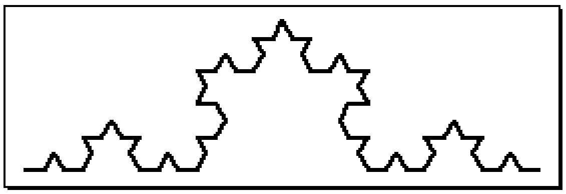


The Initiator and Generator for constructing the Koch Curve.



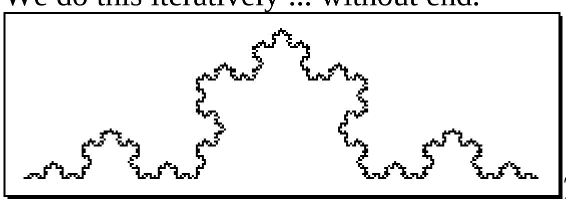
The rule says to take each line and replace it with four lines, each one-third the length of the original.

Level 2 in the construction of the Koch Curve.



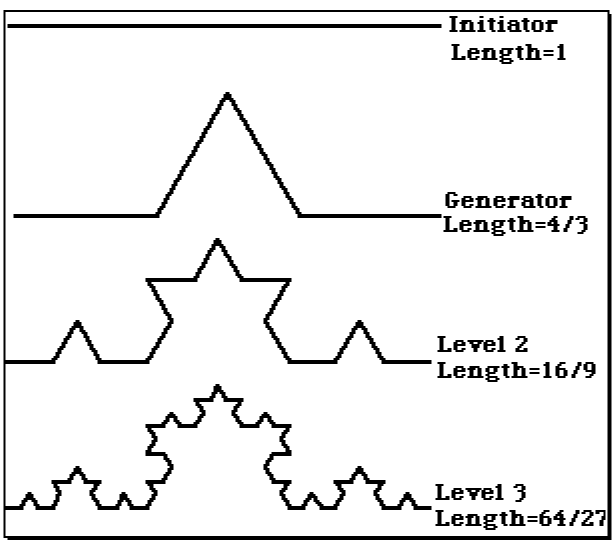
Level 3 in the construction of the Koch Curve.

We do this iteratively ... without end.

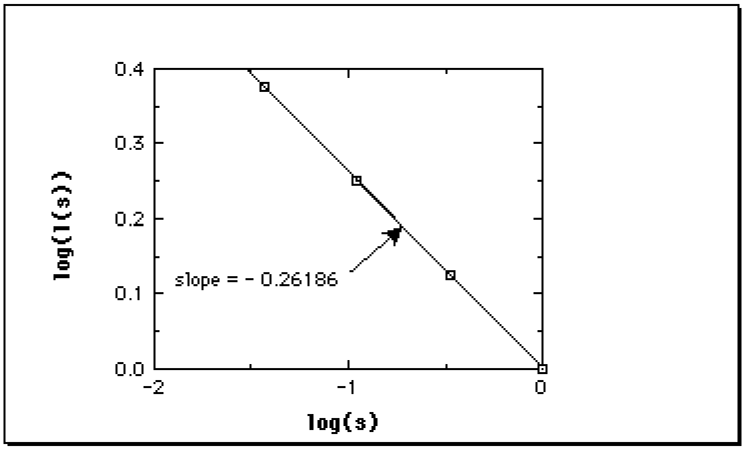


The Koch Curve.

What is the **length** of the Koch curve?



The length of the curve increases with each iteration. It has infinite length. But if we treat the Koch curve as we did the coastline, ...

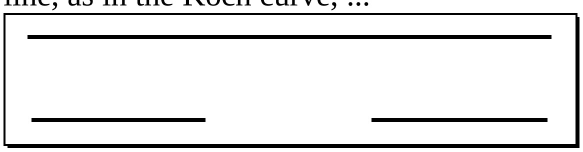


The relation between log(L(s)) and log(s) for the Koch curve ...

we find its fractal dimension to be 1.26. The same result obtained from D = log(N)/log(r) D = log(4)/log(3) = 1.26.

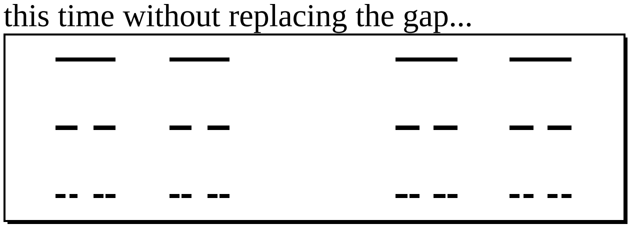
Cantor Dust

Iteratively removing the middle third of an initiating straight line, as in the Koch curve, ...



Initiator and Generator for constructing Cantor Dust. ...

this time without replacing the gap...

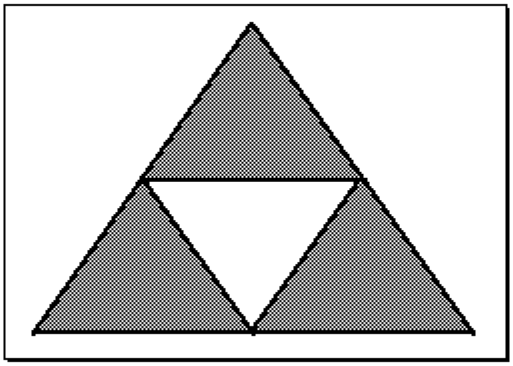


Levels 2, 3, and 4 in the construction of Cantor Dust.

Calculating the dimension ... D = log(N)/log(r) D = log(2)/log(3) = .63 We have an object with dimensionality less than one, between a point (dimensionality of zero and a line (dimensionality 1).

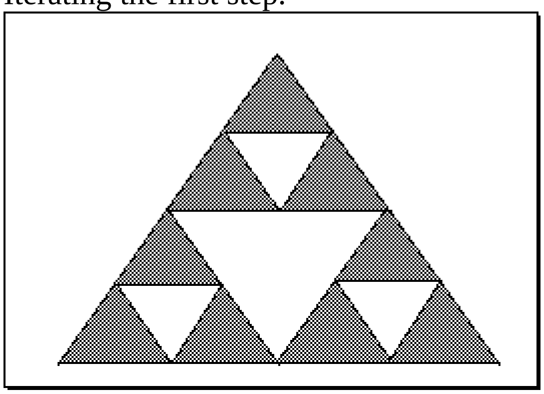
Sierpinski Triangle

We start with an equilateral triangle, connect the mid-points of the three sides and remove the resulting inner triangle.

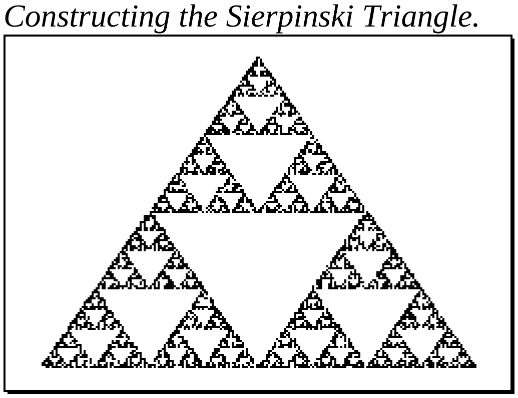


Constructing the Sierpinski Triangle.

Iterating the first step.



Constructing the Sierpinski Triangle.



The Sierpinski Triangle.

Calculating the dimension... D = log(N)/log(r) = log(3)/log(2) = 1.585. This time we get a value between 1 and 2.

The dimensionality of a strange attractor

1. The trajectory of a strange attractor cannot intersect with itself. (Why?)
2. Nearby trajectories diverge exponentially. (Why?)
3. But the attractor is bounded to the phase space.
4. The trajectory does not fill the phase space.

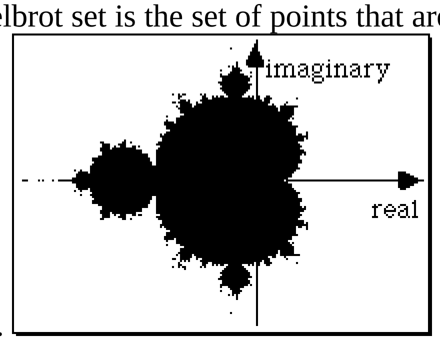
A strange attractor is a fractal, and its fractal dimension is less than the dimensions of its phase space.

Self-similarity

An important (defining) property of a fractal is **self-similarity**, which refers to an infinite nesting of structure on all scales. Strict self- similarity refers to a characteristic of a form exhibited when a substructure resembles a superstructure in the same form.

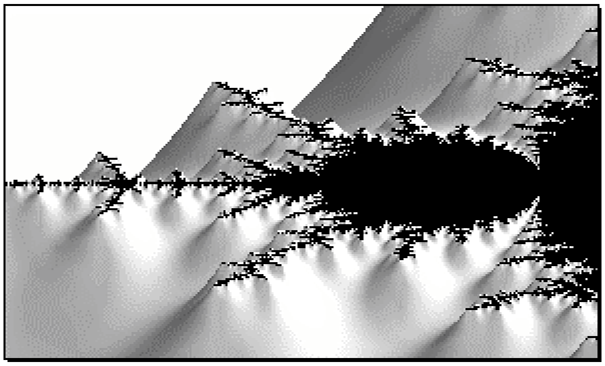
Mandelbrot Set

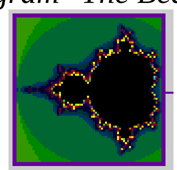
Found by iterating
$z_{n+1} = z_n^2 + c$.
where z is a complex number. $z_0=0$.
For different values of c, the trajectories either: stay near the origin, or "escape".
The Mandelbrot set is the set of points that are not in the Escape Set.



The Mandelbrot set. The points in the set are painted black.

The Escape Set differs in rate of escape, graphically depicted with different colors or altitudes ...



Constructed using the computer program "The Beauty of Fractal Lab", by Thomas Eberhardt. 

So, what is a fractal?

An irregular geometric object with an infinite nesting of structure at all scales.

Why do we care about fractals?

- Natural objects are fractals.
- Chaotic trajectories (strange attractors) are fractals.
- Assessing the fractal properties of an observed time series is informative.

Nonlinear Statistical Tools

A number of statistical techniques have been introduced to try to evaluate time series data. Their purposes include 1) attempting to distinguish chaotic time series from random data ("noise"), 2) assessing the feasibility that the data are the product of a deterministic system, and 3) assessing the dimensionality of the data. Here we introduce some concepts basic to these efforts.

Return Maps

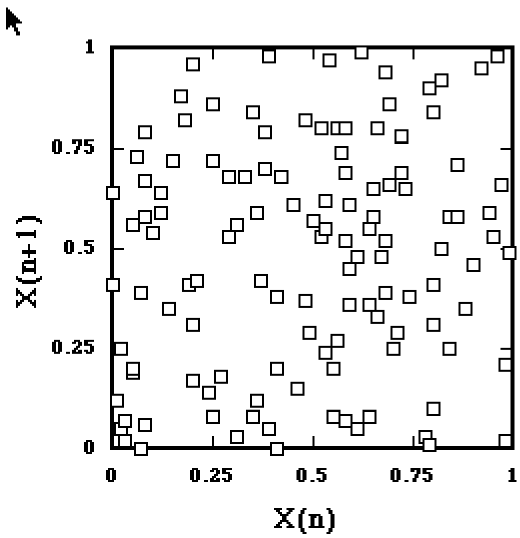
What is a return map?

A plot of x_t against $x_{delta\ t}$

Why is it plotted?

To evaluate the structure of the measured trajectory.

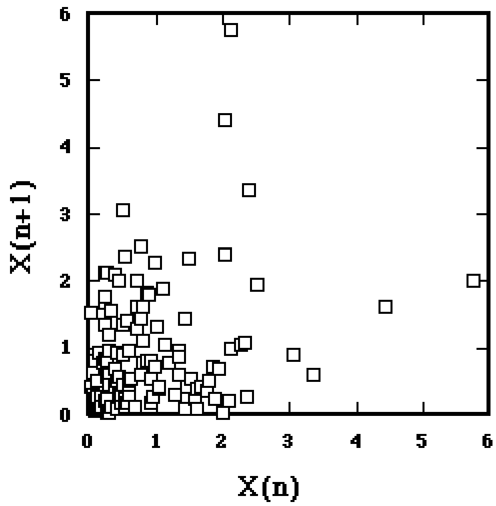
To illustrate, we start with a time series that was generated by randomly sampling from (0,1) interval. If we plot x_n against x_{n+1} we get ...



Return Map of time series from random Uniform distribution.

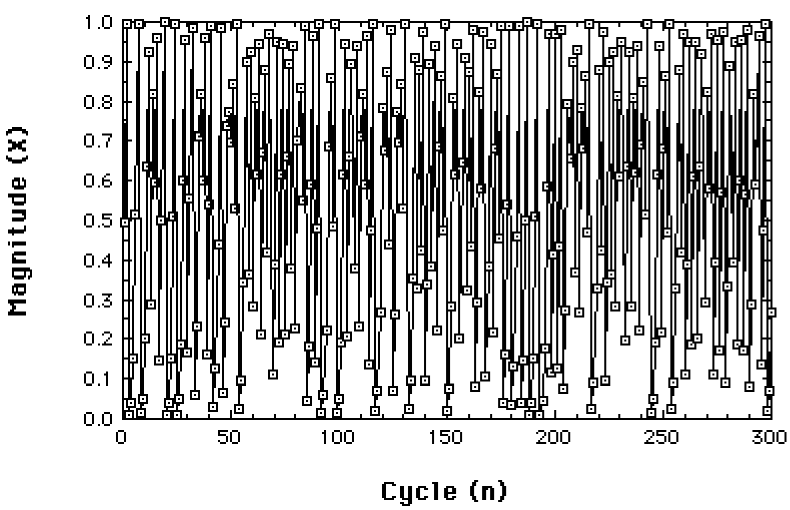
As expected, the points scatter.

Here's a return map from another random time series. This one sampled from an exponential (positively skewed) distribution.

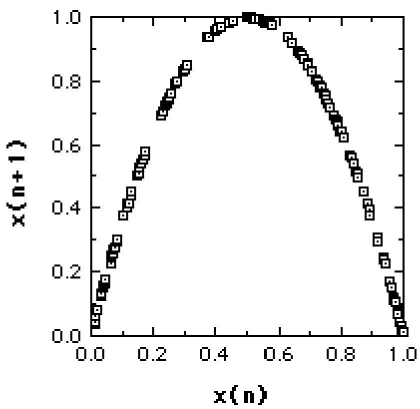


Return Map of time series from exponential distribution.
Here we do not scatter all over. What's the point? You may have heard that a symptom of chaos is when the return map is confined to a region of the map. This illustrates how such a collection can occur, but from a random system.

Now, remember this time series?



A nonrandom Time Series
It's from the Logistic Map in the chaotic region, r=3.99.
What does its Return Map look like?



Return Map from Logistic Map, r=3.99

The structure of the generating function is entirely captured.

So, a return map can be very handy, provided the data are from a one-dimensional system. If the system has more than two-dimensions, the return map has limited utility.

Embedding dimension

Okay. One more meaning of the term 'dimension'.
It comes from extending the concept of a return map.
Successive n- tuples of data are treated as points in n-space.
The Return Map is an embedding dimension of 2.

Suppose, for example, that the first six data values were
4, 2, 6, 1, 5, 3,
then for an embedding dimension of 3.
P(1)= (4,2,6)
P(2)= (2,6,1)
P(3)= (6,1,5), and so forth.

What's the point?

Contemporary statistical analyses examine the geometric structure of obtained time series embedded with differing dimensions.

Types of 'Noise'

An older, linear, tool, for examining time series, is Fourier analysis, specifically, **FFT** (Fast Fourier Transform). FFT transforms the time domain into a frequency domain, and examines the series for periodicity. The analysis produces a **power spectrum**, the degree to which each frequency contributes to the series. If the series is periodic, then the resulting power spectrum reveals peak power at the driving frequency.
Plotting log power versus log frequency,

- **White noise** (and many chaotic systems) have zero slope.
- **Brown noise** has slope equal to -2.
- **1/f (Pink) noise** has a slope of -1.

1/f noise is interesting because it is ubiquitous in nature, and it is a sort of **temporal fractal**. In the way a fractal has self-similarity in space, 1/f noise has self-similarity in time. Pink noise is also a major player in the area of **complexity**, our next topic.

Glossary

Definitions of several terms are a matter of some dispute. For a more technical treatment of some of these terms, see the faq sheet of the sci.nonlinear newsgroup.
attractor The status that a dynamic system eventually "settles down to". An attractor is a set of values in the phase space to which a system migrates over time, or iterations. An attractor can be a single fixed point, a collection of points regularly visited, a loop, a complex orbit, or an infinite number of points. It need not be one- or two-dimensional. Attractors can have as many dimensions as the number of variables that influence its system.
basin of attraction A region in phase space associated with a given attractor. The basin of attraction of an attractor is the set of all (initial) points that go to that attractor.
bifurcation A qualitative change in the behavior (attractor) of a dynamic system associated with a change in control parameter.
bifurcation diagram Visual summary of the succession of period-doubling produced as a control parameter is changed.
chaos Behavior of a dynamic system that has (a) a very large (possibly infinite) number of attractors and (b) is sensitive to initial conditions.
complexity While, chaos is the study of how simple systems

can generate complicated behavior, complexity is the study of how complicated systems can generate simple behavior. An example of complexity is the synchronization of biological systems ranging from fireflies to neurons. (From the FAQ sheet of the sci.nonlinear newsgroup).

complex system Spatially and/or temporally extended nonlinear systems characterized by collective properties associated with the system as a whole--and that are different from the characteristic behaviors of the constituent parts.(From the FAQ sheet of the sci.nonlinear newsgroup)

control parameter A parameter in the equations of a dynamic system. If control parameters are allowed to change, the dynamic system would also change. Changes beyond certain values can lead to bifurcations. .

difference equation A function specifying the change in a variable from one discrete point in time to another.

differential equation A function that specifies the rate of change in a continuous variable over changes in another variable (time, in this book).

dimension See embedding dimension, box-counting dimension, correlation dimension, information dimension, dimension of a system.

dimensions of a system The set of variables of a system.

dynamic system A set of equations specifying how certain variables change over time. The equations specify how to determine (compute) the new values as a function of their current values and control parameters. The functions, when explicit, are either difference equations or differential equations. Dynamic systems may be stochastic or deterministic. In a stochastic system, new values come from a probability distribution. In a deterministic system, a single new value is associated with any current value.

embedding dimension Successive N-tuples of points in a time series are treated as points in N dimensional space. The points are said to reside in embedding dimensions of size N, for N = 1, 2, 3, 4, ... etc.

fractal An irregular shape with self-similarity. It has infinite detail, and cannot be differentiated. "Wherever chaos, turbulence, and disorder are found, fractal geometry is at play" (Briggs and Peat, 1989).

fractal dimension A measure of a geometric object that can take on fractional values. At first used as a synonym to Hausdorff dimension, fractal dimension is currently used as a more general term for a measure of how fast length, area, or volume increases with decrease in scale. (Peitgen, Jurgens, & Saupe, 1992a).

Hausdorff dimension A measure of a geometric object that can take on fractional values. (see fractal dimension).
initial condition the starting point of a dynamic system.
iteration the repeated application of a function, using its output from one application as its input for the next.
iterative function a function used to calculate the new state of a dynamic system.
iterative system A system in which one or more functions are iterated to define the system.
limit cycle An attractor that is periodic in time, that is, that cycles periodically through an ordered sequence of states.
limit points Points in phase space. There are three kinds: attractors, repellors, and saddle points. A system moves away from repellors and towards attractors. A saddle point is both an attractor and a repellor, it attracts a system in certain regions, and repels the system to other regions.
linear function The equation of a straight line. A linear equation is of the form $y=mx+b$, in which y varies "linearly" with x. In this equation, m determines the slope of the line and b reflects the y-intercept, the value y obtains when x equals zero.
logistic difference equation see logistic map
logistic map $x(n+1)= rx(n)[1- x(n)]$. A concave-down parabolic function that (with $0<r$
Lorenz attractor A butterfly-shaped strange attractor. It came from a meteorological model developed by Edward Lorenz with three equations and three variables. It was one of the first strange attractors studied.
Lyapunov Number (Liapunov number) The value of an exponent, a coefficient of time, that reflects the rate of departure of dynamic orbits. It is a measure of sensitivity to initial conditions.
nonlinear function One that's not linear! y would be a nonlinear function of x if x were multiplied by another variable (non-constant) or by itself (that is, raised to some power.
nonlinear dynamics The study of dynamic systems whose functions specifying change are not linear.
orbit (trajectory) A sequence of positions (path) of a system in its phase space.
period-doubling The change in dynamics in which a N-point attractor is replaced by a 2N-point attractor.
phase portrait The collection of all trajectories from all possible starting points in the phase space of a dynamic system.
phase space (state space) An abstract space used to

represent the behavior of a system. Its dimensions are the variables of the system. Thus a point in the phase space defines a potential state of the system. The points actually achieved by a system depend on its iterative function and initial condition (starting point).

recursive process For our purposes, "recursive" and "iterative" are synonyms. Thus recursive processes are iterative processes, and recursive functions are iterative functions.

repellors One type of limit point. A point in phase space that a system moves away from.

return map Plot of a time series values n vs. n+1.

saddle point A point, usually in three-space, that both an attracts and a repels, attracting in one dimension and repelling to another.

self-similarity An infinite nesting of structure on all scales. Strict self- similarity refers to a characteristic of a form exhibited when a substructure resembles a superstructure in the same form.

sensitivity to initial conditions A property of chaotic systems. A dynamic system has sensitivity to initial conditions when very small differences in starting values result in very different behavior. If the orbits of nearby starting points diverge, the system has sensitivity to initial conditions.

starting state see initial condition

state A point in state space designating the current location (status) of a dynamic system.

state space (phase space) An abstract space used to represent the behavior of a system. Its dimensions are the variables of the system. Thus a point in the phase space defines a potential state of the system.

strange attractor N-point attractor in which N equals infinity. Usually (perhaps always) self-similar in form.

time series A set of measures of behavior over time.

Torus An attractor consisting of N independent oscillations. Plotted in phase space, a 2-oscillation torus resembles a donut.

trajectory (orbit) A sequence of positions (path) of a system in its phase space. The path from its starting point (initial condition) to and within its attractor.

vector A two-valued measure associated with a point in the phase space of a dynamic system. Its 1) direction shows where the system is headed from the current point, and its 2) length indicates velocity.

vector field The set of all vectors in the phase space of a dynamic system. For a given continuous system, the vector field is specified by its set of differential equations.