7. Prove that the restriction of a linear operator T to a T-invariant subspace is a linear operator on that subspace.
8. Let T be a linear operator on a vector space with a T-invariant subspace W. Prove that if x is an eigenvector of T_W with corresponding eigenvalue λ, then the same is true for T.
• 9. For each linear operator T and cyclic subspace W of Exercise 6, compute the characteristic polynomial of T_W in two ways as in Example 6.
10. Verify that $A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$ in the proof of Theorem 5.26.
• 11. Let T be a linear operator on a vector space V, let x be a nonzero element of V, and let W be the T-cyclic subspace of V generated by x. Prove:
(a) W is T-invariant.
(b) Any T-invariant subspace of V containing x also contains W.
12. For each linear operator of Exercise 6, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of T_W (computed in Exercise 9) divides $f(t)$.
• 13. Let T be a linear operator on a vector space V, let x be a nonzero element of V, and let W be the T-cyclic subspace of V generated by x. For any $y \in V$, prove that $y \in W$ if and only if there exists a polynomial $g(t)$ such that $y = g(T)x$.
• 14. Prove that the polynomial $g(t)$ of Exercise 13 can always be chosen so that its degree is less than or equal to dim(W).
15. Use the Cayley–Hamilton theorem (Theorem 5.28) to prove its corollary for matrices.
16. Let T be a linear operator on a finite-dimensional vector space V.
(a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.
(b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.
17. Let A be an $n \times n$ matrix. Prove that

$$\dim(\mathrm{span}(\{I_n, A, A^2, \ldots,\})) \leq n.$$

18. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that A is invertible if and only if $a_0 \neq 0$.
(b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

(c) Use part (b) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$