$W + W' = U + U' + U''$. We claim that, in fact, $W + W' = U \oplus U' \oplus U''$. Indeed, assume that $u + u' + u'' = 0_V$, where $u \in U$, $u' \in U'$, and $u'' \in U''$. Then $u' = -u'' - u \in W' \cap W = U$. But $u' \in U'$ and the intersection of U' and U is $\{0_V\}$. Therefore $u' = 0_V$ and so $u = -u''$. But then this is in $U \cap U'' = \{0_V\}$, which implies that $u = u'' = 0_V$. We have thus shown that the set $\{U, U', U''\}$ is independent and hence have established our claim. Moreover, in this case we have

$$\begin{aligned} dim(W + W') &= dim(U \oplus U' \oplus U'') \\ &= dim(U) + dim(U') + dim(U'') \\ &= dim(W) + dim(U'') \\ &= dim(W) + dim(U'') + dim(U) - dim(U) \\ &= dim(W) + dim(W') - dim(U) \\ &= dim(W) + dim(W') - dim(W \cap W') \end{aligned}$$

as desired □

EXAMPLE

Consider the subspaces

$$W_1 = \mathbb{R}\{[1,0,2],[1,2,2]\}$$

and

$$W_2 = \mathbb{R}\{[1,1,0],[0,1,1]\}$$

of $\mathbb{R}^3$. Each of these subspaces has dimension 2 over $\mathbb{R}$ and so $2 \leq dim(W_1 + W_2) \leq 3$. By Proposition 3.14 we see that this implies that $1 \leq dim(W_1 \cap W_2) \leq 2$. In order to ascertain the exact dimension of $W_1 \cap W_2$ we have to find a basis for it. If $v \in W_1 \cap W_2$ then there exist real numbers a, b, c, d satisfying

$$v = a[1,0,2] + b[1,2,2] = c[1,1,0] + d[0,1,1]$$

and so $a + b = c$, $2b = c + d$, and $2a + 2b = d$. This can happen only when $b = -3a$, $c = -2a$, and $d = -4a$ and so we see that v must be of the form

$$a[-2,-6,4] = (-2a)[1,1,0] + (-4a)[0,1,1].$$

From this we conclude that $\{[-2,-6,-4]\}$ is a basis for $W_1 \cap W_2$, and so the dimension of $W_1 \cap W_2$ is 1.

Problems

1. Let V be a vector space over a field F. Let v_1, v_2, v_3 be elements of V and let c_1, c_2, c_3 be scalars in F. Show that the set of vectors

$$\{c_2v_3 - c_3v_2, c_1v_2 - c_2v_1, c_3v_1 - c_1v_3\}$$

is linearly dependent.

2. Find rational numbers a and b such that the subset $\{[2, a-b, 1], [a, b, 3]\}$ of $\mathbb{Q}^3$ is linearly dependent.

3. Is the subset $\{[1+i, 3+8i, 5+7i], [1-i, 5, 2+i], [1+i, 3+2i, 4-i]\}$ of $V = \mathbb{C}^3$ linearly dependent over $\mathbb{C}$? Is it linearly dependent when we consider V as a vector space over $\mathbb{R}$?

4. For each nonnegative integer n let $f_n: \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n: t \mapsto sin^n(t)$. Is the subset $\{f_n \mid n \geq 0\}$ of $\mathbb{R}^{\mathbb{R}}$ linearly independent?

5. Let V be the vector space over $\mathbb{R}$ consisting of all continuous functions from the interval $[-1, 1]$ to $\mathbb{R}$. Let $f, g \in V$ be the functions defined by $f: x \mapsto x^2$ and $g: x \mapsto x|x|$. Is the set $\{f, g\}$ linearly independent over $\mathbb{R}$?

6. Let V be a vector space over the field $F = \mathbb{Z}/(5)$ and let v_1, v_2, v_3 be vectors in V. Is the set of vectors $\{v_1 + v_2, v_1 - v_2 + v_3, 2v_2 + v_3, v_2 + v_3\}$ linearly independent?

7. Let F be a field of characteristic not equal to 2 and let V be a vector space over F containing a linearly-independent set of vectors $\{v_1, v_2, v_3\}$. Show that the set $\{v_1 + v_3, v_2 + v_3, v_1 + v_2\}$ is also linearly independent.

8. Let $F = \mathbb{Z}/(3)$. Is the subset $\{[1, 1, 2, 2], [1, 2, 1, 2], [1, 1, 1, 2], [0, 2, 2, 0]\}$ of F^4 linearly independent?

9. Let $t \leq n$ be positive integers and for all $1 \leq i \leq t$ let $v_i = [a_{i1}, \ldots, a_{in}]$ be a vector in $\mathbb{R}^n$ satisfying the condition $2|a_{jj}| > \sum_{i=1}^{t} |a_{ij}|$ for all $1 \leq j \leq n$. Show that the set of vectors $\{v_1, \ldots, v_t\}$ is linearly independent.

10. Let $F = \mathbb{Z}/(2)$ and let A be a nonempty set. If $B \neq C$ are nonempty subsets of A, show that $\{\chi_B, \chi_C\}$ is a linearly-independent subset of F^A.

11. Let A be a subset of $\mathbb{R}$ having at least three elements. Let f_1, f_2, and f_3 be the elements of $\mathbb{R}^A$ defined by $f_1: t \mapsto 2^{t-1}$, $f_2: t \mapsto t2^{t-1}$, and $f_3: t \mapsto t^2 2^{t-1}$. Is the set $\{f_1, f_2, f_3\}$ linearly independent?

12. Let $F = \mathbb{Z}/(5)$ and let $V = F^F$, which is a vector space over F. Let $f: x \mapsto x^2$ and $g: x \mapsto x^3$ be elements of V. Find another element h of V such that the set $\{f, g, h\}$ is linearly independent.

13. Find a basis for the subspace of $\mathbb{R}^4$ generated by

$$\{[4, 2, 6, -2], [1, -1, 3, -1], [1, 2, 0, 0], [1, 5, -3, 1]\}.$$

14. For each real number a let $f_a \in \mathbb{R}^{\mathbb{R}}$ be the function defined by

$$f_a(r) = \begin{cases} 1, & \text{when } r = a, \\ 0, & \text{otherwise.} \end{cases}$$

Is $\{f_a \mid a \in \mathbb{R}\}$ a basis for $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$?

15. Let $F = \mathbb{Z}/(2)$ and let A be a nonempty finite set. Is $\{\chi_{\{a\}} \mid a \in A\}$ a basis for the vector space for F^A over F?

16. Let a, b, c be elements of a field F. Show that

$$\{[1_F, a, b], [0_F, 1_F, c], [0_F, 0_F, 1_F]\}$$

is a basis for F^3 over F.

17. Let $F = \mathbb{Z}/(p)$, where p is a prime integer, and let V be a vector space of dimension n over F. How many distinct bases are there for V over F?

18. Let F be a field and let V be the subspace of $F[X]$ consisting of all polynomials of degree no greater than 9. Is the set $\{1_F, X - 1_F, \ldots, (X - 1_F)^9\}$ a basis for V over F?

19. Let $\{v_1, v_2, v_3\}$ be a given basis for a vector space V over a field F. Is the set $\{v_1 + v_2, v_2 + v_3, v_1 - v_3\}$ also a basis for V over F?

20. Let V be a vector space finite dimensional over $\mathbb{C}$ having a basis $D = \{v_1, \ldots, v_n\}$. Show that $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis for V as a vector space over $\mathbb{R}$.

21. Let V be the subspace of $\mathbb{Q}[X]$ consisting of all polynomials having dimension at most 5. Extend the set $\{X^5 + X^4, X^5 - 7X^3, X^5 - 4X^2, X^5 + 3X\}$ to a basis for V over $\mathbb{Q}$.

22. Let $W = \mathbb{R}\{[-1, 1, 1, 1], [1, 2, 1, 0]\}$ and $Y = \mathbb{R}\{[2, -1, 0, 1], [-5, 6, 0]\}$ be subspaces of $\mathbb{R}^4$. Calculate the dimensions of $W \cap Y$ and $W + Y$.

23. Let F be a subfield of a field K satisfying the condition that the dimension of K as a vector space over F is finite and equal to $r > 0$. Let V be a vector space over K having finite dimension $n > 0$. What is the dimension of V as a vector space over F?

24. Let V be a vector space over a field F and let n be a positive integer. A matrix $A = [a_{ij}] \in \mathcal{M}_{n \times n}(V)$ is called a *Toeplitz matrix* if and only if the entries along each diagonal parallel to the principal diagonal are equal. Thus, for example, the matrix

$$A = \begin{bmatrix} u & v & w \\ x & u & v \\ y & x & u \end{bmatrix}$$

is a Toeplitz matrix of size 3×3. Show that the set of all Toeplitz matrices of size $n \times n$ is a subspace of $\mathcal{M}_{n \times n}(V)$. Find the dimension of this subspace for the case $V = F$.

25. Let V be a vector space of dimension 6 over a field F and let W and Y be distinct subspaces of V of dimension 4. What are the possible dimensions of $W \cap Y$?

26. Let V be a vector space which is not finite dimensional over a field F and let W be a proper subspace of V. Show that there exists an infinite set $\{Y_1, Y_2, \ldots\}$ of subspaces of V satisfying the condition that $\bigcap_{i=1}^{n} Y_i \not\subseteq W$ for each $n \geq 1$ but $\bigcap_{i=1}^{\infty} Y_i \subseteq W$.

27. Let V be a vector space which is not finite dimensional over a field F. Show that there exists a countable set of proper subspaces of V the union of which equals V.

28. Let V be the space of all continuous functions from $\mathbb{R}$ to itself and consider the subspaces Y and W of V defined by

$$Y = \{f \in V \mid f(a) = f(-a) \text{ for all } a \in \mathbb{R}\}$$

and

$$W = \{f \in V \mid -f(a) = f(-a) \text{ for all } a \in \mathbb{R}\}.$$

Show that $V = W \oplus Y$.

29. Let F be a field and let W be the subspace of $F[X]$ consisting of all polynomials of degree no greater than 4. Find a complement for W in $F[X]$.

30. Let A be a nonempty set and let B be a subset of A. Let F be a field and let W be the subspace of F^A consisting of all those functions $f: A \to F$ satisfying $f(b) = 0_F$ for all $b \in B$. Find a complement for W in F^A.

31. Let W a be a subspace of a vector space V over a field F and let W' be a complement of W in V. Use W' to construct a complement for W^2 in V^2.