(4) (Modular law for subspaces): If $W_1 \subseteq W_3$ then

$$W_3 \cap (W_2 + W_1) = W_1 + (W_2 \cap W_3).$$

Proof: Parts (1) and (2) follow immediately from the definition, while (4) is a special case of (3). We are therefore left to prove (3). Indeed, if v belongs to $W_3 \cap [W_2 + (W_1 \cap W_3)]$, then we can write $v = w_2 + y$, where $w_2 \in W_2$ and $y \in W_1 \cap W_3$. Since $v, y \in W_3$, it follows that $w_2 = v - y \in W_3$, and so $v = y + w_2 \in (W_1 \cap W_3) + (W_2 \cap W_3)$. Thus we see that $W_3 \cap [W_2 + (W_1 \cap W_3)] \subseteq (W_1 \cap W_3) + (W_2 \cap W_3)$. Conversely, assume that $v \in (W_1 \cap W_3) + (W_2 \cap W_3)$. Then, in particular, $v \in W_3$ and we can write $v = w_1 + w_2$, where $w_1 \in W_1 \cap W_3$ and $w_2 \in W_2 \cap W_3$. Thus $v = w_1 + w_2 \in W_3 \cap W_2 + (W_1 \cap W_3)$. This shows that $(W_1 \cap W_3) + (W_2 \cap W_3) \subseteq W_3 \cap [W_2 + (W_1 \cap W_3)]$ and so we have the desired equality. $\square$

Exercises

Exercise 48 *Let* $V = \mathbb{Q}^2$, *with the usual vector addition. If* $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ *and if* $\begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{Q}^2$, *set* $\left(a + b\sqrt{2}\right)\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + 2bd \\ bc + ad \end{bmatrix}$. *Do these operations turn* $\mathbb{Q}^2$ *into a vector space over* $\mathbb{Q}(\sqrt{2})$?

Exercise 49 *Is it possible to define on* $V = \mathbb{Z}/(4)$ *the structure of a vector space over* $GF(2)$ *in such a way that the vector addition is the usual addition in* $\mathbb{Z}/(4)$?

Exercise 50 *Consider the set* $\mathbb{Z}$ *of integers, together with the usual addition. If* $a \in \mathbb{Q}$ *and* $k \in \mathbb{Z}$, *define* $a \cdot k$ *to be* $\lfloor a \rfloor k$, *where* $\lfloor a \rfloor$ *denotes the largest integer less than or equal to* a. *Using this as our definition of "scalar multiplication", have we turned* $\mathbb{Z}$ *into a vector space over* $\mathbb{Q}$?

Exercise 51 *Let* $V = \{0, 1\}$ *and let* $F = GF(2)$. *Define vector addition and scalar multiplication by setting* $v + v' = \max\{v, v'\}$, $0v = 0$, *and* $1v = v$ *for all* $v, v' \in V$. *Does this define on* V *the structure of a vector space over* F?

Exercise 52 *Let* $V = C(0, 1)$. *Define an operation* $\boxplus$ *on* V *by setting* $f \boxplus g : x \mapsto \max\{f(x), g(x)\}$. *Does this operation of vector addition, together with the usual operation of scalar multiplication, define on* V *the structure of a vector space over* $\mathbb{R}$?

Exercise 53 *Let* $V \neq \{0_V\}$ *be a vector space over* $\mathbb{R}$. *For each* $v \in V$ *and each complex number* $a + bi$, *let us define* $(a + bi)v = av$. *Does* V, *together with this new scalar multiplication, form a vector space over* $\mathbb{C}$?

Exercise 54 *Let* $V = \{i \in \mathbb{Z} \mid 0 \leq i < 2^n\}$ *for some given positive integer* n. *Define operations of vector addition and scalar multiplication on* V *in such a way as to turn it into a vector space over the field* $GF(2)$.

Exercise 55 *Let* V *be a vector space over a field* F. *Define a function from* $GF(3) \times V$ *to* V *by setting* $(0, v) \mapsto 0_V$, $(1, v) \mapsto v$, *and* $(2, v) \mapsto -v$ *for all* $v \in V$. *Does this function, together with the vector addition in* V, *define on* V *the structure of a vector space over* $GF(3)$?

Exercise 56 *Let* $V = \mathbb{R} \cup \{\infty\}$ *and extend the usual addition of real numbers by defining* $v + \infty = \infty + v = \infty$ *for all* $v \in V$. *Is it possible to define an operation of scalar multiplication on* V *in such a manner as to turn it into a vector space over* $\mathbb{R}$?

Exercise 57 *Let* $V = \mathbb{R}^2$. *If* $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix} \in V$ *and* $r \in \mathbb{R}$, *set* $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} a + a' + 1 \\ b + b' \end{bmatrix}$ *and* $r\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra + r - 1 \\ rb \end{bmatrix}$. *Do these operations define on* V *the structure of a vector space over* $\mathbb{R}$? *If so, what is the identity element for vector addition in this space?*

Exercise 58 *Let* $V = \mathbb{R}$ *and let* $\circ$ *be an operation on* R *defined by* $a \circ b = a^3 b$. *Is* V, *together with the usual addition and "scalar multiplication" given by* $\circ$, *a vector space over* $\mathbb{R}$?

Exercise 59 *Show that* $\mathbb{Z}$ *cannot be turned into a vector space over any field.*

Exercise 60 *Let* V *be a vector space over the field* $GF(2)$. *Show that* $v = -v$ *for all* $v \in V$.

Exercise 61 *Give an example of a vector space having exactly 125 elements.*

Exercise 62 *In the definition of a vector space, show that the commutativity of vector addition is a consequence of the other conditions.*

Exercise 63 *Let* W *be the subset of* $\mathbb{R}^5$ *consisting of all vectors an odd number of the entries in which are equal to* 0. *Is* W *a subspace of* $\mathbb{R}^5$?

Exercise 64 *Let* $V = \mathbb{R}^{\mathbb{R}}$ *and let* W *be the subset of* V *containing the constant function* $x \mapsto 0$ *and all of those functions* $f \in V$ *satisfying the condition that* $f(a) = 0$ *for at most finitely-many real numbers* a. *Is* W *a subspace of* V?

Exercise 65 *Let* $V = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix} \middle| \; 0 < a_i \in \mathbb{R} \right\}$. *If* $v = \begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix}$ *and* $w = \begin{bmatrix} b_1 \\ \vdots \\ b_5 \end{bmatrix}$ *belong to* V, *and if* $c \in \mathbb{R}$, *set* $v + w = \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_5 b_5 \end{bmatrix}$ *and* $cv = \begin{bmatrix} a_1^c \\ \vdots \\ a_5^c \end{bmatrix}$. *Do these operations turn* V *into a vector space over* $\mathbb{R}$?

Exercise 66 *Let* $F = GF(3)$. *How many elements are there in the subspace of* F^3 *generated by* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$?

Exercise 67 *A function* $f \in \mathbb{R}^{\mathbb{R}}$ *is* ***piecewise constant*** *if and only if it is a constant function* $x \mapsto c$ *or there exist* $a_1 < a_2 < \ldots < a_n$ *and* $c_0, \ldots, c_n$ *in* $\mathbb{R}$ *such that*

$$f : x \mapsto \begin{cases} c_0 & \text{if } x < a_1 \\ c_i & \text{if } a_i \leq x < a_{i+1} \text{ for } 1 \leq i < n \\ c_n & \text{if } a_n \leq x \end{cases} .$$

Does the set of all piecewise constant functions form a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ *over* $\mathbb{R}$?

Exercise 68 *Let* V *be the vector space of all continuous functions from* $\mathbb{R}$ *to itself and let* W *be the subset of all those functions* $f \in V$ *satisfying the condition that* $|f(x)| \leq 1$ *for all* $-1 \leq x \leq 1$. *Is* W *a subspace of* V?

Exercise 69 *Let* $F = GF(2)$ *and let* W *be the subspace of* F^5 *consisting of all vectors* $\begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix}$ *satisfying* $\sum_{i=1}^{5} a_i = 0$. *Is* W *a subspace of* F^5?

Exercise 70 *Let* $V = \mathbb{R}^{\mathbb{R}}$ *and let* W *be the subset of* V *consisting of all monotonically-increasing or monotonically-decreasing functions. Is* W *a subspace of* V?

Exercise 71 *Let* $V = \mathbb{R}^{\mathbb{R}}$ *and let* W *be the subset of* V *consisting of the constant function* $a \mapsto 0$, *and all epic functions. Is* W *a subspace of* V?

Exercise 72 *Let* $V = \mathbb{R}^{\mathbb{R}}$ *and let* W *be the subset of* V *containing the constant function* $a \mapsto 0$ *and all of those functions* $f \in V$ *satisfying the condition that* $f(\pi) > f(-\pi)$. *Is* W *a subspace of* V?

Exercise 73 *Let* $V = \mathbb{R}^{\mathbb{R}}$ *and let* W *be the subset of* V *consisting of all functions* f *satisfying the condition that there exists a real number* c *(which depends on* f*) such that* $|f(a)| \leq c|a|$ *for all* $a \in \mathbb{R}$. *Is* W *a subspace of* V?

Exercise 74 *Let* $V = \mathbb{R}^{\mathbb{R}}$ *and let* W *be the subset of* V *consisting of all functions* f *satisfying the condition that there exist real numbers* a *and* b *such that* $|f(x)| \leq a|\sin(x)| + b|\cos(x)|$ *for all* $x \geq 0$. *Is* W *a subspace of* V?

Exercise 75 *Let* F *be a field and let* $V = F^F$, *which is a vector space over* F. *Let* W *be the set of all functions* $f \in V$ *satisfying* $f(1) = f(-1)$. *Is* W *a subspace of* V?

Exercise 76 *For any real number* $0 < t \leq 1$, *let* V_t *be the set of all functions* $f \in \mathbb{R}^{\mathbb{R}}$ *satisfying the condition that if* $a < b$ *in* $\mathbb{R}$ *then there exists a real number* $u(a,b)$ *satisfying* $|f(x) - f(y)| \leq u(a,b)\,|x-y|^t$ *for all* $a \leq x, y \leq b$. *For which values of* t *is* V_t *a subspace of* $\mathbb{R}^{\mathbb{R}}$*?*

Exercise 77 *Let* U *be a nonempty subset of a vector space* V. *Show that* U *is a subspace of* V *if and only if* $au + u' \in U$ *for all* $u, u' \in U$ *and* $a \in F$.

Exercise 78 *Let* V *be a vector space over a field* F *and let* v *and* w *be distinct vectors in* V. *Set* $U = \{(1-t)v + tw \mid t \in F\}$. *Show that there exists a vector* $y \in V$ *such that* $\{u + y \mid u \in U\}$ *is a subspace of* V.

Exercise 79 *Let* V *be a vector space over a field* F *and let* W *and* Y *be subspaces of* V^2. *Let* U *be the set of all vectors* $\begin{bmatrix} v \\ v' \end{bmatrix} \in V^2$ *satisfying the condition that there exists a vector* $v'' \in V$ *such that* $\begin{bmatrix} v \\ v'' \end{bmatrix} \in W$ *and* $\begin{bmatrix} v'' \\ v' \end{bmatrix} \in Y$. *Is* U *a subspace of* V^2?

Exercise 80 *Consider* $\mathbb{R}$ *as a vector space over* $\mathbb{Q}$. *Given a nonempty subset* W *of* $\mathbb{R}$, *let* $\overline{W}$ *be the set of all real numbers* b *for which there exists a sequence* $a_1, a_2, \ldots$ *of elements of* W *satisfying* $\lim_{i \to \infty} a_i = b$. *Show that* $\overline{W}$ *is a subspace of* $\mathbb{R}$ *whenever* W *is.*

Exercise 81 *Let* V *be a vector space over a field* F *and let* P *be the collection of all subsets of* V, *which we know is a vector space over* $GF(2)$. *Is the collection of all subspaces of* V *a subspace of* P?