(e) Let $A \in \mathsf{M}_{n\times n}(F)$ and $\beta = \{x_1, \ldots, x_n\}$ be a basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose ith column is x_i $(i = 1, 2, \ldots, n)$, then $Q^{-1}AQ$ is a diagonal matrix.
(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.
(g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.
(h) If a vector space is the direct sum of subspaces $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$, then $\mathsf{W}_i \cap \mathsf{W}_j = \{0\}$ for $i \neq j$.
(i) If

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \quad \text{and} \quad \mathsf{W}_i \cap \mathsf{W}_j = \{0\} \qquad \text{for } i \neq j,$$

then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

2. For each of the following matrices A in $\mathsf{M}_{n\times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find a matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T, test T for diagonalizability. If T is diagonalizable, find a basis β such that $[\mathsf{T}]_\beta$ is a diagonal matrix.
(a) $\mathsf{T}\colon \mathsf{P}_3(R) \to \mathsf{P}_3(R)$ defined by $\mathsf{T}(f) = f' + f''$, where f' and f'' are the first and second derivatives of f, respectively.
(b) $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$.
(c) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

(d) $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}(f)(x) = f(0) + f(1)(x + x^2)$.
(e) $\mathsf{T}\colon \mathsf{C}^2 \to \mathsf{C}^2$ defined by $\mathsf{T}(z, w) = (z + iw, iz + w)$.
(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ defined by $\mathsf{T}(A) = A^t$.

4. Prove the matrix version of the corollary to Theorem 5.10: If $A \in \mathsf{M}_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.