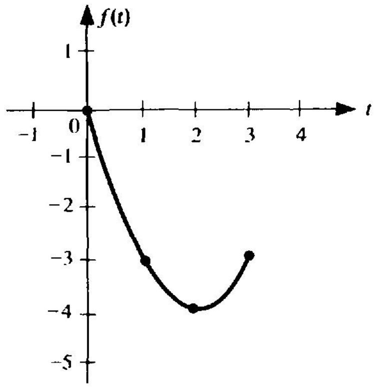

Fig. 6-11

6.59. Prove Theorem 6.25.

If $u \neq v$, then $u - v \neq 0$, and hence $d(u, v) = \| u - v \| > 0$. Also, $d(u, u) = \| u - u \| = \| 0 \| = 0$. Thus $[M_1]$ is satisfied. We also have

$$d(u, v) = \| u - v \| = \| -1(v - u) \| = | -1 | \| v - u \| = \| v - u \| = d(v, u)$$

and

$$d(u, v) = \| u - v \| = \| (u - w) + (w - v) \| \leq \| u - w \| + \| w - v \| = d(u, w) + d(w, v)$$

Thus $[M_2]$ and $[M_3]$ are satisfied.

Supplementary Problems

INNER PRODUCTS

6.60. Verify that the following is an inner product on $\mathbf{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$f(u, v) = x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 5x_2 y_2$$

6.61. Find the values of k so that the following is an inner product on $\mathbf{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$f(u, v) = x_1 y_1 - 3x_1 y_2 - 3x_2 y_1 + kx_2 y_2$$

6.62. Consider the vectors $u = (1, -3)$ and $v = (2, 5)$ in $\mathbf{R}^2$. Find:

(*a*) $\langle u, v \rangle$ with respect to the usual inner product in $\mathbf{R}^2$.

(*b*) $\langle u, v \rangle$ with respect to the inner product in $\mathbf{R}^2$ in Problem 6.60.

(*c*) $\| v \|$ using the usual inner product in $\mathbf{R}^2$.

(*d*) $\| v \|$ using the inner product in $\mathbf{R}^2$ in Problem 6.60.

6.63. Show that each of the following is not an inner product on $\mathbf{R}^3$ where $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$:

(*a*) $\langle u, v \rangle = x_1 y_1 + x_2 y_2$ and (*b*) $\langle u, v \rangle = x_1 y_2 x_3 + y_1 x_2 y_3$

6.64. Let V be the vector space of $m \times n$ matrices over $\mathbf{R}$. Show that $\langle A, B \rangle = \text{tr}\,(B^T A)$ defines an inner product in V.

6.65. Let V be the vector space of polynomials over **R**. Show that $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$ defines an inner product in V.

6.66. Suppose $|\langle u, v \rangle| = \|u\|\,\|v\|$. (That is, the Cauchy–Schwarz inequality reduces to an equality.) Show that u and v are linearly dependent.

6.67. Suppose $f(u, v)$ and $g(u, v)$ are inner products on a vector space V over **R**. Prove:

(*a*) The sum $f + g$ is an inner product on V where $(f + g)(u, v) = f(u, v) + g(u, v)$.

(*b*) The scalar product kf, for $k > 0$, is an inner product on V where $(kf)(u, v) = kf(u, v)$.

ORTHOGONALITY, ORTHOGONAL COMPLEMENTS, ORTHOGONAL SETS

6.68. Let V be the vector space of polynomials over **R** of degree ≤ 2 with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$. Find a basis of the subspace W orthogonal to $h(t) = 2t + 1$.

6.69. Find a basis of the subspace W of $\mathbf{R}^4$ orthogonal to $u_1 = (1, -2, 3, 4)$ and $u_2 = (3, -5, 7, 8)$.

6.70. Find a basis for the subspace W of $\mathbf{R}^5$ orthogonal to the vectors $u_1 = (1, 1, 3, 4, 1)$ and $u_2 = (1, 2, 1, 2, 1)$.

6.71. Let $w = (1, -2, -1, 3)$ be a vector in $\mathbf{R}^4$. Find (*a*) an orthogonal and (*b*) an orthonormal basis for $w^\perp$.

6.72. Let W be the subspace of $\mathbf{R}^4$ orthogonal to $u_1 = (1, 1, 2, 2)$ and $u_2 = (0, 1, 2, -1)$. Find (*a*) an orthogonal and (*b*) an orthonormal basis for W. (Compare with Problem 6.69.)

6.73. Let S consist of the following vectors in $\mathbf{R}^4$:

$$u_1 = (1, 1, 1, 1) \qquad u_2 = (1, 1, -1, -1) \qquad u_3 = (1, -1, 1, -1) \qquad u_4 = (1, -1, -1, 1)$$

(*a*) Show that S is orthogonal and a basis of $\mathbf{R}^4$.

(*b*) Write $v = (1, 3, -5, 6)$ as a linear combination of u_1, u_2, u_3, u_4.

(*c*) Find the coordinates of an arbitrary vector $v = (a, b, c, d)$ in $\mathbf{R}^4$ relative to the basis S.

(*d*) Normalize S to obtain an orthonormal basis of $\mathbf{R}^4$.

6.74. Let V be the vector space of 2×2 matrices over **R** with inner product $\langle A, B \rangle = \operatorname{tr}(B^T A)$. Show that the following is an orthonormal basis of V:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

6.75. Let V be the vector space of 2×2 matrices over **R** with inner product $\langle A, B \rangle = \operatorname{tr}(B^T A)$. Find an orthogonal basis for the orthogonal complement of (*a*) the diagonal and (*b*) the symmetric matrices.

6.76. Suppose $\{u_1, u_2, \ldots, u_r\}$ is an orthogonal set of vectors. Show that $\{k_1 u_1, k_2 u_2, \ldots, k_r u_r\}$ is orthogonal for any scalars $k_1, k_2, \ldots, k_r$.

6.77. Let U and W be subspaces of a finite-dimensional inner product space V. Show that: (*a*) $(U + W)^\perp = U^\perp \cap W^\perp$; and (*b*) $(U \cap W)^\perp = U^\perp + W^\perp$.

PROJECTIONS, GRAM–SCHMIDT ALGORITHM, APPLICATIONS

6.78. Find an orthogonal and an orthonormal basis for the subspace U of $\mathbf{R}^4$ spanned by the vectors $v_1 = (1, 1, 1, 1)$, $v_2 = (1, -1, 2, 2)$, $v_3 = (1, 2, -3, -4)$.

6.79. Let V be the vector space of polynomials $f(t)$ with inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)\,dt$. Apply the Gram–Schmidt algorithm to the set $\{1, t, t^2\}$ to obtain an orthogonal set $\{f_0, f_1, f_2\}$ with integer coefficients.

6.80. Suppose $v = (1, 2, 3, 4, 6)$. Find the projection of v onto W (or find $w \in W$ which minimizes $\|v - w\|$) where W is the subspace of $\mathbf{R}^5$ spanned by:

(a) $u_1 = (1, 2, 1, 2, 1)$ and $u_2 = (1, -1, 2, -1, 1)$; (b) $v_1 = (1, 2, 1, 2, 1)$ and $v_2 = (1, 0, 1, 5, -1)$

6.81. Let $V = C[-1, 1]$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Let W be the subspace of V of polynomials of degree ≤ 3. Find the projection of $f(t) = t^5$ onto W. [*Hint:* Use the (Legendre) polynomials $1, t, 3t^2 - 1, 5t^3 - 3t$ in Example 6.13.]

6.82. Let $V = C[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Let W be the subspace of V of polynomials of degree ≤ 2. Find the projection of $f(t) = t^3$ onto W. (*Hint:* Use the polynomials $1, 2t - 1, 6t^2 - 6t + 1$ in Problem 6.27.)

6.83. Let U be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$v_1 = (1, 1, 1, 1) \qquad v_2 = (1, -1, 2, 2) \qquad v_3 = (1, 2, -3, -4)$$

Find the projection of $v = (1, 2, -3, 4)$ onto U. (*Hint:* Use Problem 6.78.)

INNER PRODUCTS AND POSITIVE DEFINITE MATRICES, ORTHOGONAL MATRICES

6.84. Find the matrix A which represents the usual inner product on $\mathbf{R}^2$ relative to each of the following bases of $\mathbf{R}^2$: (a) $\{v_1 = (1, 4), v_2 = (2, -3)\}$ and (b) $\{w_1 = (1, -3), w_2 = (6, 2)\}$.

6.85. Consider the following inner product on $\mathbf{R}^2$:

$$f(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2 \qquad \text{where} \quad u = (x_1, x_2) \quad \text{and} \quad v = (y_1, y_2)$$

Find the matrix B which represents this inner product on $\mathbf{R}^2$ relative to each basis in Problem 6.84.

6.86. Find the matrix C which represents the usual basis on $\mathbf{R}^3$ relative to the basis S of $\mathbf{R}^3$ consisting of the vectors: $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 1)$, $u_3 = (1, -1, 3)$.

6.87. Consider the vector space V of polynomials $f(t)$ of degree ≤ 2 with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

(a) Find $\langle f, g\rangle$ where $f(t) = t + 2$ and $g(t) = t^2 - 3t + 4$.

(b) Find the matrix A of the inner product with respect to the basis $\{1, t, t^2\}$ of V.

(c) Verify Theorem 6.14 that $\langle f, g\rangle = [f]^T A[g]$ with respect to the basis $\{1, t, t^2\}$.

6.88. Determine which of the following matrices are positive definite:

(a) $\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 4 \\ 4 & 7 \end{pmatrix}$, (c) $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 6 & -7 \\ -7 & 9 \end{pmatrix}$

6.89. Determine whether or not A is positive definite where

(a) $A = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 2 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 6 \\ 2 & 6 & 9 \end{pmatrix}$

6.90. Suppose A and B are positive definite matrices. Show that: (a) $A + B$ is positive definite; (b) kA is positive definite for $k > 0$.

6.91. Suppose B is a real nonsingular matrix. Show that: (a) B^TB is symmetric, and (b) B^TB is positive definite.

6.92. Find the number and exhibit all 2×2 orthogonal matrices of the form $\begin{pmatrix} \frac{1}{3} & x \\ y & z \end{pmatrix}$.

6.93. Find a 3×3 orthogonal matrix P whose first two rows are multiples of $u = (1, 1, 1)$ and $v = (1, -2, 3)$, respectively.

6.94. Find a symmetric orthogonal matrix P whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. (Compare with Problem 6.43.)

6.95. Real matrices A and B are said to be *orthogonally equivalent* if there exists an orthogonal matrix P such that $B = P^T AP$. Show that this relation is an equivalence relation.

COMPLEX INNER PRODUCT SPACES

6.96. Verify that

$$\langle a_1 u_1 + a_2 u_2, b_1 v_1 + b_2 v_2 \rangle = a_1 \bar{b}_1 \langle u_1, v_1 \rangle + a_1 \bar{b}_2 \langle u_1, v_2 \rangle + a_2 \bar{b}_1 \langle u_2, v_1 \rangle + a_2 \bar{b}_2 \langle u_2, v_2 \rangle$$

More generally, prove that $\left\langle \sum_{i=1}^{m} a_i u_i, \sum_{j=1}^{n} b_j v_j \right\rangle = \sum_{i,j} a_i \bar{b}_j \langle u_i, v_i \rangle$.

6.97. Consider $u = (1 + i, 3, 4 - i)$ and $v = (3 - 4i, 1 + i, 2i)$ in $\mathbf{C}^3$. Find:

(*a*) $\langle u, v \rangle$, (*b*) $\langle v, u \rangle$, (*c*) $\| u \|$, (*d*) $\| v \|$, (*e*) $d(u, v)$.

6.98. Find the Fourier coefficient c and the projection cw of

(*a*) $u = (3 + i, 5 - 2i)$ along $w = (5 + i, 1 + i)$ in $\mathbf{C}^2$;

(*b*) $u = (1 - i, 3i, 1 + i)$ along $w = (1, 2 - i, 3 + 2i)$ in $\mathbf{C}^3$.

6.99. Let $u = (z_1, z_2)$ and $v = (w_1, w_2)$ belong to $\mathbf{C}^2$. Verify that the following is an inner product on $\mathbf{C}^2$:

$$f(u, v) = z_1 \bar{w}_1 + (1 + i) z_1 \bar{w}_2 + (1 - i) z_2 \bar{w}_1 + 3 z_2 \bar{w}_2$$

6.100. Find an orthogonal basis and an orthonormal basis for the subspace W of $\mathbf{C}^3$ spanned by $u_1 = (1, i, 1)$ and $u_2 = (1 + i, 0, 2)$.

6.101. Let $u = (z_1, z_2)$ and $v = (w_1, w_2)$ belong to $\mathbf{C}^2$. For what values of $a, b, c, d \in \mathbf{C}$ is the following an inner product on $\mathbf{C}^2$?

$$f(u, v) = a z_1 \bar{w}_1 + b z_1 \bar{w}_2 + c z_2 \bar{w}_1 + d z_2 \bar{w}_2$$

6.102. Prove the following polar form for an inner product in a complex space V:

$$\langle u, v \rangle = \tfrac{1}{4} \| u + v \|^2 - \tfrac{1}{4} \| u - v \|^2 + \tfrac{1}{4} \| u + iv \|^2 - \tfrac{1}{4} \| u - iv \|^2$$

[Compare with Problem 6.9(*b*).]

6.103. Let V be a real inner product space. Show that:

(i) $\| u \| = \| v \|$ if and only if $\langle u + v, u - v \rangle = 0$;

(ii) $\| u + v \|^2 = \| u \|^2 + \| v \|^2$ if and only if $\langle u, v \rangle = 0$.

Show by counterexamples that the above statements are not true for, say, $\mathbf{C}^2$.

6.104. Find the matrix P which represents the usual inner product on $\mathbf{C}^3$ relative to the basis $\{1, 1 + i, 1 - 2i\}$.