divides the characteristic polynomial of T.

(c) Let T be a linear operator on a finite-dimensional vector space V, and let x and y be elements of V. If W is the T-cyclic subspace generated by x, W′ is the T-cyclic subspace generated by y, and W = W′, then $x = y$.

(d) If T is a linear operator on a finite-dimensional vector space V, then for any $x \in$ V the T-cyclic subspace generated by x is the same as the T-cyclic subspace generated by T(x).

(e) Let T be a linear operator on an n-dimensional vector space. Then there exists a polynomial $g(t)$ of degree n such that $g(\mathsf{T}) = \mathsf{T}_0$.

(f) Any polynomial of the form

$$(-1)^n(a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n)$$

is the characteristic polynomial of some linear operator.

(g) If T is a linear operator on a finite-dimensional vector space V, and if V is a direct sum of k T-invariant subspaces, then there is a basis β for V such that $[\mathsf{T}]_\beta$ is a direct sum of k matrices.

2. For each of the following linear operators T, determine if the given subspace W is a T-invariant subspace of V.

(a) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f) = f'$, and $\mathsf{W} = \mathsf{P}_2(R)$

(b) $\mathsf{V} = \mathsf{P}(R)$, $\mathsf{T}(f)(x) = xf(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$

(c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}(a, b, c) = (a + b + c, a + b + c, a + b + c)$, and $\mathsf{W} = \{(t, t, t): t \in \mathsf{R}\}$

(d) $\mathsf{V} = C([0, 1])$, $\mathsf{T}(f)(t) = \left[\int_0^1 f(x)\,dx\right]t$, and $\mathsf{W} = \{f \in \mathsf{V}: f(t) = at + b \text{ for some } a \text{ and } b\}$

(e) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $\mathsf{W} = \{A \in \mathsf{V}: A^t = A\}$

3. Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

(a) $\{0\}$ and V

(b) N(T) and R(T)

(c) E_λ, for any eigenvalue λ of T

4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(\mathsf{T})$-invariant for any polynomial $g(t)$.

5. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

6. For each linear operator T on the vector space V find a basis for the T-cyclic subspace generated by the vector z.

(a) $\mathsf{V} = \mathsf{R}^4$, $\mathsf{T}(a, b, c, d) = (a + b, b - c, a + c, a + d)$, and $z = e_1$

(b) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f) = f''$, and $z = x^3$

(c) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(d) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

7. Prove that the restriction of a linear operator T to a T-invariant subspace is a linear operator on that subspace.
8. Let T be a linear operator on a vector space with a T-invariant subspace W. Prove that if x is an eigenvector of T_W with corresponding eigenvalue λ, then the same is true for T.
9. For each linear operator T and cyclic subspace W of Exercise 6, compute the characteristic polynomial of T_W in two ways as in Example 6.
10. Verify that $A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$ in the proof of Theorem 5.26.
11. Let T be a linear operator on a vector space V, let x be a nonzero element of V, and let W be the T-cyclic subspace of V generated by x. Prove:
    (a) W is T-invariant.
    (b) Any T-invariant subspace of V containing x also contains W.
12. For each linear operator of Exercise 6, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of T_W (computed in Exercise 9) divides $f(t)$.
13. Let T be a linear operator on a vector space V, let x be a nonzero element of V, and let W be the T-cyclic subspace of V generated by x. For any $y \in \mathsf{V}$, prove that $y \in \mathsf{W}$ if and only if there exists a polynomial $g(t)$ such that $y = g(\mathsf{T})x$.
14. Prove that the polynomial $g(t)$ of Exercise 13 can always be chosen so that its degree is less than or equal to dim(W).
15. Use the Cayley–Hamilton theorem (Theorem 5.28) to prove its corollary for matrices.
16. Let T be a linear operator on a finite-dimensional vector space V.
    (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.
    (b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.
17. Let A be an $n \times n$ matrix. Prove that

$$\dim(\text{span}(\{I_n, A, A^2, \ldots,\})) \leq n.$$

18. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

    (a) Prove that A is invertible if and only if $a_0 \neq 0$.
    (b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

    (c) Use part (b) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

19. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}.$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Hint: Use mathematical induction on k, expanding the determinant along the first row.

20. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then UT = TU if and only if U = g(T) for some polynomial $g(t)$. *Hint:* Suppose that V is generated by x. Choose $g(t)$ according to Exercise 13 so that $g(\mathsf{T})(x) = \mathsf{U}(x)$.

21. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or T = cI for some scalar c.

22. Let T be a linear operator on a two-dimensional vector space V and suppose that T ≠ cI for any scalar c. Show that if U is any linear operator on V such that UT = TU, then U = g(T) for some polynomial $g(t)$.

23. Let T be a diagonalizable linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $x_1, x_2, \ldots, x_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $x_1 + x_2 + \cdots + x_k$ is in W, then $x_i \in$ W for all i. *Hint:* Use mathematical induction on k.

24. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. *Hint:* Use the result of Exercise 23.

25. (a) Prove a converse of Exercise 16(a) of Section 5.2: If T and U are diagonalizable linear operators on a finite-dimensional vector space V such that UT = TU, then T and U are *simultaneously diagonalizable.* (See the definition in the exercises of Section 5.2.) *Hint:* For any eigenvalue λ of T show that E_λ is U-invariant, and apply Exercise 24 to obtain a basis for E_λ of eigenvectors of T.
(b) State and prove a matrix version of (a).

Exercises 26 through 30 require familiarity with Exercise 29 of Section 1.3 and Exercise 30 of Section 2.1. It is also advisable to review Exercise 26 of Section 1.6 and Exercise 22 of Section 2.4.

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Materia: ÁLGEBRA LINEAL

I) Encuentre todas las formas canónicas de Jordan posibles para aquéllas matrices cuyos polinomio característico $p(t)$ y polinomio mínimo $m(t)$ son:

a) $p(t) = (t-2)^4(t-3)^2,\ m(t) = (t-2)^2(t-3)^2$

b) $p(t) = (t-7)^5,\ m(t) = (t-7)^2$

c) $p(t) = (t-2)^7,\ m(t) = (t-2)^3$

d) $p(t) = (t-3)^4(t-5)^4,\ m(t) = (t-3)^2(t-5)^2$

II) Hallar la forma canónica de Jordan para cada matriz A:

a)

$$A = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

III) El operador derivación sobre el espacio de los polinomios de grado menor o igual a 3 está representado en la base ordenada canónica por la matriz

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

¿Cuál es la forma canónica de Jordan para esta matriz?

IV) Para cada operador lineal T Encuentre la forma canónica de Jordan J de T.

a) $\mathcal{V}$ es el espacio vectorial sobre $\mathbb{R}$ generado por el conjunto $\{1, t, t^2, e^t, te^t\}$ de funciones definidas en $\mathbb{R}$ y T es el operador lineal sobre $\mathcal{V}$ definido por $T(f) = f'$.

b) T es el operador lineal sobre $\mathcal{M}_{2\times 2}(\mathbb{R})$ definido por

$$T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A,$$

para cada $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$.

Puebla, Pue., a 30 de abril de 2017