Preface

This book is designed to help the university student make the difficult transition from calculus to university-level pure mathematics. A quick glance at the contents page will indicate to any instructor what the book sets out to do.

I first starting teaching this material in the late 1970s at the University of Lancaster in England. I wrote the first version of this book, published in 1981, to accompany a 6-week course I developed there. Teaching the same kind of material a decade later in the U.S., I looked at the original version, found it wanting, and completely rewrote it. I changed almost everything: I reordered the topics, changed the treatment of each topic, and increased the number of exercises. That second edition was published in 1992.

One decision I faced when I wrote the second edition was the size of the book. The original edition was deliberately written as a "little book" that would be both unintimidating and cheap. At the time, there were no competing books on the market (at least in the U.K.), and as a result the book did quite well. By the time I came to write the second edition, there was a whole slew of books (in the U.S.) targeted at the same students. All of these books were far more substantial than mine and, in keeping with the trend in textbook publishing, getting larger all the time in a furious struggle to be all things to all students and to include everything that appeared in any competing book. I looked at those books and decided firmly — I had to be firm because my publisher had other ideas — to stick with my original plan: Despite the many changes that I felt were necessary, the second edition would remain a "little book".

Now another decade has passed and the time has come to rewrite the book once again. Not because the core material has changed; that is the same now as it was when I myself was a student in the 1960s. What has changed is the background that the beginning student brings to his or her study and the environment in which the study is carried out. This new version of the book tries to take those changes into account. But I still believe that for this material, small is better.

Students Start Here

If you are a mathematics instructor, this book will tell you absolutely nothing you do not already know. Assuming you have read the Preface (which was written for instructors, under the assumption that students never read prefaces but instructors usually do), you really do not need to read any further in this book than this paragraph.

Those two sentences are the last ones in the entire book directed at instructors. Everything else (including this paragraph) is written for the beginning student of post-calculus university-level mathematics ("you", from now on).

You have completed courses on differential and integral calculus. Maybe you aced those courses; or maybe you got through only after a struggle. Either way, now you are trying to make the transition to what comes next. Chances are, regardless of your performance in mathematics up until now, you are going to find the next step unfamiliar and challenging. This book won't make it easy. No book can do that. Anyone who claims otherwise is trying to sell you a book — probably a thick, expensive one at that.[1] But after guiding many generations of young people through precisely this difficult transition,[2] I think I can help.

The main distinction between most high-school mathematics and post-calculus university math lies in the degree of rigor and abstraction required at the university level. In general, you (a student embarking on post-calculus university mathematics) will have had little or no prior experience of wholly rigorous definitions and proofs. The result is that, although you may be competent to handle quite difficult problems in calculus, you are likely to find yourself totally lost when presented with a rigorous definition of limits and derivatives — the fundamental mathematical ideas that lie behind calculus.

In effect, what you need in order to progress further in mathematics is to acquire mastery of what is virtually an entire new language ("the language of mathematics") and to adopt an entirely new mode of thinking ("mathematical thinking"). Throughout my many years teaching university mathematics,[3] I have met very few students who came through this process without a great deal of difficulty. (I certainly didn't when I was a student.) This book is intended to assist you in making this transition.

Chances are that almost everything you find in this book will be new to you and will probably seem very strange. Indeed, you may feel that it is not "mathematics" at all. Be patient. Given time and a fair amount of hard work, this stage will pass.

[1] See the Preface.

[2] See the Preface again.

[3] See the Preface yet again.

Do not try to rush through any part of the book, even if at first glance a particular section looks easy. This entire book consists of *basic* material required elsewhere (indeed practically *everywhere*) in post-calculus mathematics. Everything you will find in this book is included because it generally causes problems for the beginner. (Trust me on this. Hey, I got you to look at the Preface, didn't I, and how often have you done *that* with a math textbook?)

Part of the reason you are likely to find this material difficult is that it will seem unmotivated. It is bound to: the sole motivation is to provide you with the foundation on which to build the mathematics that comes later — mathematics that you do not yet know about!

So take it steadily, and try to *understand* the new concepts as you meet them. There is little in the way of new facts to learn, but a great deal to comprehend! (The actual facts contained in this book could be listed on three or four pages of notes.) And try the exercises — as many of them as possible. They are included for a purpose: to aid your understanding.

Discuss any difficulties that arise with your colleagues and with your instructor. Do not give up. Students all around the world managed it last year. Likewise the previous year, and the year before that. So did I. So will you!

Keith Devlin
Stanford University

Chapter 1

What Is Mathematics and What Does It Do for Us?

You might think these are odd questions to begin with. After all, won't you, my intended reader, already know what mathematics is? And doesn't everyone know that it's important to be able to "do math" — which presumably implies that they think it is useful stuff?

Maybe yes. But in my experience, given the way mathematics is often taught in schools — and in some universities, come to that — many students are never given a "big picture" view of the subject. I've long lost count of the number of adults I've met, many years after they have graduated with degrees in such mathematically rich subjects as engineering, physics, computer science, or even mathematics itself, who have told me that they went through their entire education without ever gaining a good overview of what constitutes modern mathematics. Only later in life do they start to catch a glimpse of the true nature of mathematics and the extent of its pervasive influence in modern-day society.

It's not hard to understand why this is the case. Most of the mathematics that underpins present-day science and technology is no more than 300 or 400 years old, in many cases less than a century old. Yet the typical high school curriculum covers mathematics that is at least 500 years old — in fact, much of what is taught is over 2,000 years old. It's as if our literature courses gave students Homer and Chaucer but never mentioned Shakespeare, Dickens, or Proust.

1.1 It's Not Just Numbers

Most people think that mathematics is the study of numbers. That description of mathematics ceased to be accurate about 2,500 years ago! Anyone who has that view of mathematics is unlikely to appreciate that research in mathematics is a thriving, worldwide activity, or to accept a suggestion that mathematics permeates, often to a considerable extent, most walks of present-day life and society. Nor are they likely to know which organization in the U.S. employs the greatest number of Ph.D.s in mathematics. (The answer is the National Security Agency. Exactly what the 30 or so new mathematics Ph.D.s who are hired there each year — the exact number is an official secret — do to earn their paychecks is never made public, but it is generally assumed that the majority of them work on code breaking, to enable the agency to read encrypted messages that are intercepted by monitoring systems.)

In fact, the answer to the question "What is mathematics?" has changed several times during the course of history.

Up to 500 B.C. or thereabouts, mathematics was indeed the study of number. This was the period of Egyptian and Babylonian mathematics.[1] In those civilizations, mathematics consisted almost solely of arithmetic. It was largely utilitarian, and very much of a "cookbook" variety. ("Do such and such to a number and you will get the answer.")

The period from around 500 B.C. to 300 A.D. was the era of Greek mathematics. The mathematicians of ancient Greece were primarily concerned with geometry. Indeed, they regarded numbers in a geometric fashion, as measurements of length, and when they discovered that there were lengths to which their numbers did not correspond (the discovery of irrational lengths), their study of number largely came to a halt.[2] For the Greeks, with their emphasis on geometry, mathematics was the study of number and shape.

In fact, it was only with the Greeks that mathematics came into being as an area of study, and ceased being merely a collection of techniques for measuring, counting, and accounting. Greek interest in mathematics was not just utilitarian; they regarded mathematics as an intellectual pursuit having both aesthetic and religious elements. Around 500 B.C., Thales of Miletus (now part of Turkey) introduced the idea that the precisely stated assertions of mathematics could be logically proved by a formal argument. This innovation marked the birth of the theorem, now the bedrock of mathematics. For the Greeks, this approach culminated in the publication of Euclid's *Elements*, reputedly the most widely circulated book of all time after the Bible.

There was no major change in the overall nature of mathematics, and hardly any significant advances within the subject, until the middle of the 17th century, when Isaac Newton (in England) and Gottfried Leibniz (in Germany) independently invented calculus. In essence, calculus is the study of continuous motion and change. Previous mathematics had been largely restricted to the static issues of counting, measuring, and describing shape. With the introduction of techniques to handle motion and change, mathematicians were able to study the motion of the planets and of falling bodies on earth, the workings of machinery, the flow of liquids, the expansion of gases, physical forces such as magnetism and electricity, flight, the growth of plants and animals, the spread of epidemics, the fluctuation of profits, and so on.

[1] Other civilizations also developed mathematics, for example, the Chinese and the Japanese. But the mathematics of those cultures did not appear to have a direct influence on the development of modern Western mathematics, so in this book I will ignore them.

[2] There is an oft repeated story that the young Greek mathematician who made this discovery was taken out to sea and drowned, lest the awful news of what he had stumbled upon should leak out. As far as I know, there is no evidence whatsoever to support this fanciful tale. Pity. It's a great story.

After Newton and Leibniz, mathematics became the study of number, shape, motion, change, and space.

Most of the initial work involving calculus was directed toward the study of physics; indeed, many of the great mathematicians of the period are also regarded as physicists. But from about the middle of the 18th century there was an increasing interest in mathematics itself, not just its applications, as mathematicians sought to understand what lay behind the enormous power that calculus gave to humankind. Here the old Greek tradition of formal proof came back into ascendancy, as a large part of present-day pure mathematics was developed. By the end of the 19th century, mathematics had become the study of number, shape, motion, change, and space, and of the mathematical tools that are used in this study.

The explosion of mathematical activity that has taken place over the past 100 years or so has been dramatic. The growth has not just been a further development of previous mathematics; many quite new branches have sprung up. At the start of the 20th century, mathematics could reasonably be regarded as consisting of about 12 distinct subjects: arithmetic, geometry, calculus, and so on. Today, between 60 and 70 distinct categories would be a reasonable figure. Some subjects, such as algebra or topology, have split into various subfields; others, such as complexity theory or dynamical systems theory, are completely new areas of study.

Given this tremendous growth in mathematical activity, for a while it seemed as though the only simple answer to the question "What is mathematics?" was to say, somewhat fatuously, "It is what mathematicians do for a living." A particular study was classified as mathematics not so much because of what was studied but because of how it was studied — that is, the methodology used. It was only in the 1980s that a definition of mathematics emerged on which most mathematicians now agree: *mathematics is the science of patterns*. What the mathematician does is examine abstract patterns — numerical patterns, patterns of shape, patterns of motion, patterns of behavior, voting patterns in a population, patterns of repeating chance events, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind. Different kinds of patterns give rise to different branches of mathematics. For example:

- Arithmetic and number theory study the patterns of numbers and counting.
- Geometry studies the patterns of shape.
- Calculus allows us to handle patterns of motion.
- Logic studies patterns of reasoning.
- Probability theory deals with patterns of chance.
- Topology studies patterns of closeness and position.

and so forth.

1.2 Mathematical Notation

One aspect of modern mathematics that is obvious to even the casual observer is the use of abstract notations: algebraic expressions, complicated-looking formulas, and geometric diagrams. The mathematician's reliance on abstract notation is a reflection of the abstract nature of the patterns she studies.

Different aspects of reality require different forms of description. For example, the most appropriate way to study the lay of the land or to describe to someone how to find their way around a strange town is to draw a map. Text is far less appropriate. Analogously, line drawings in the form of blueprints are the appropriate way to specify the construction of a building. And musical notation is the most appropriate medium to convey music, apart from, perhaps, actually playing the piece.

In the case of various kinds of abstract, formal patterns and abstract structures, the most appropriate means of description and analysis is mathematics, using mathematical notations, concepts, and procedures. For instance, the symbolic notation of algebra is the most appropriate means of describing and analyzing general behavioral properties of addition and multiplication.

For example, the commutative law for addition could be written in English as:

When two numbers are added, their order is not important.

However, it is usually written in the symbolic form

$$m + n = n + m$$

Such is the complexity and the degree of abstraction of the majority of mathematical patterns, that to use anything other than symbolic notation would be prohibitively cumbersome. And so the development of mathematics has involved a steady increase in the use of abstract notations.

The first systematic use of a recognizably algebraic notation in mathematics seems to have been made by Diophantus, who lived in Alexandria some time around 250 A.D. His treatise *Arithmetica*, of which only six of the original thirteen volumes have been preserved, is generally regarded as the first algebra textbook. In particular, Diophantus used special symbols to denote the unknown in an equation and to denote powers of the unknown, and he employed symbols for subtraction and for equality.

These days, mathematics books tend to be awash with symbols, but mathematical notation no more is mathematics than musical notation is music. A page of sheet music *represents* a piece of music; the music itself is what you get when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive and becomes part of our experience; the music exists not on the printed page but in our minds. The same is true for mathematics; the symbols on a page are just a *representation* of the mathematics. When read by a competent performer (in this case, someone trained in mathematics), the symbols on the printed page come alive — the mathematics lives and breathes in the mind of the reader like some abstract symphony.

Given the strong similarity between mathematics and music, both of which have their own highly abstract notations and are governed by their own structural rules, it is hardly surprising that many (perhaps most) mathematicians also have some musical talent. Although some commentators make more of this connection in terms of mental ability than I believe is warranted, it is true that for most of the 2,500 years of Western civilization, starting with the ancient Greeks, mathematics and music were regarded as two sides of the same coin. It was only with the rise of the scientific method in the 17th century that the two started to go their separate ways.

For all the historical connections, however, there was, until recently, one very obvious difference between mathematics and music. Although only someone well trained in music can read a musical score and hear the music in her head, if that same piece of music is performed by a competent musician, anyone with a sense of hearing can appreciate the result. It requires no musical training to experience and enjoy music when it is performed.

For most of its history, however, the only way to appreciate mathematics was to learn how to "sight-read" the symbols. Although the structures and patterns of mathematics reflect the structure of, and resonate in, the human mind every bit as much as do the structures and patterns of music, human beings have developed no mathematical equivalent to a pair of ears. Mathematics can only be "seen" with the "eyes of the mind". It is as if we had no sense of hearing, so that only someone able to sight-read musical notation would be able to appreciate the patterns and harmonies of music.

In recent years, however, the development of computer and video technologies has to some extent made mathematics accessible to the untrained spectator. In the hands of a skilled user, the computer can be used to "perform" mathematics, and the result can be displayed in a visual form on the screen for all to see. Although only a relatively small part of mathematics lends itself to such visual "performance", it is now possible to convey to the layperson at least something of the beauty and the harmony that the mathematician "sees" and experiences when she does mathematics.

Sometimes, the use of computer graphics can be of significant use to the mathematician as well as providing the layperson with a glimpse of the inner world of mathematics. For instance, the study of complex dynamical systems was begun in the 1920s by the French mathematicians Pierre Fatou and Gaston Julia, but it was not until the late 1970s and early 1980s that the rapidly developing technology of computer graphics enabled Benoit Mandelbrot and other mathematicians to see some of the structures Fatou and Julia had been working with. The strikingly beautiful pictures that emerged from this study have since become something of an art form in their own right.

Without its algebraic symbols, large parts of mathematics simply would not exist. Indeed, the issue is a deep one having to do with human cognitive abilities. The recognition of abstract concepts and the development of an appropriate language are really two sides of the same coin.

The use of a symbol such as a letter, a word, or a picture to denote an abstract entity goes hand in hand with the recognition of that entity as an entity. The use of the numeral "7" to denote the number 7 requires that the number 7 be recognized as an entity; the use of the letter m to denote an arbitrary whole number requires that the concept of a whole number be recognized. Having the symbol makes it possible to think about and manipulate the concept.

This linguistic aspect of mathematics is often overlooked. Indeed, one often hears the complaint that mathematics would be much easier if it weren't for all that abstract notation, which is rather like saying that Shakespeare would be much easier to understand if it were written in simpler language.

Sadly, the level of abstraction in mathematics, and the consequent need for notations that can cope with that abstraction, means that many, perhaps most, parts of mathematics will remain forever hidden from the non-mathematician; and even the more accessible parts may be at best dimly perceived, with much of their inner beauty locked away from view.

Inner beauty? What do I mean by that? In what sense is mathematics beautiful?

Mathematical beauty is hard to convey to an outsider. Apart from computer graphical representations of a few mathematical objects, where there is a visual beauty plain for all to see, mathematical beauty is highly abstract — a beauty of abstract form, structure, and logic.

In his 1940 book *A Mathematician's Apology*, the accomplished English mathematician G. H. Hardy wrote:

> The mathematician's patterns, like the painter's or the poet's, must be beautiful, the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics. ... It may be very hard to define mathematical beauty, but that is just as true of beauty of any kind — we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it.

The beauty to which Hardy was referring is a highly abstract, inner beauty, a beauty of abstract form and logical structure, a beauty that can be observed, and appreciated, only by those sufficiently well trained in the discipline. It is a beauty "cold and austere", according to Bertrand Russell, the famous English mathematician and philosopher, who wrote in his 1918 book *Mysticism and Logic*:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

Mathematics, the science of patterns, is a way of looking at the world, both the physical, biological, and sociological world we inhabit and the inner world of our minds and thoughts. Mathematics' greatest success has undoubtedly been in the physical domain, where the subject is rightly referred to as both the queen and the servant of the (natural) sciences. Yet, as an entirely human creation, the study of mathematics is ultimately a study of humanity itself. For none of the entities that form the substrate of mathematics exists in the physical world; the numbers, the points, the lines and planes, the surfaces, the geometric figures, the functions, and so forth are pure abstractions that exist only in humanity's collective mind. The absolute certainty of a mathematical proof and the indefinitely enduring nature of mathematical truth are reflections of the deep and fundamental status of the mathematician's patterns in both the human mind and the physical world.

In an age when the study of the heavens dominated scientific thought, Galileo said,

> The great book of nature can be read only by those who know the language in which it was written. And this language is mathematics.

Striking a similar note in a much later era, when the study of the inner workings of the atom had occupied the minds of many scientists for a generation, the Cambridge physicist John Polkinhorne wrote, in 1986,

> Mathematics is the abstract key which turns the lock of the physical universe.

In today's age, dominated by information, communication, and computation, mathematics is finding new locks to turn. As the science of abstract patterns, there is scarcely any aspect of our lives that is not affected, to a greater or lesser extent, by mathematics; for abstract patterns are the very essence of thought, of communication, of computation, of society, and of life itself.

1.3 Making the Invisible Visible

We have answered the question "What is mathematics?" with the catch phrase "Mathematics is the science of patterns." The second fundamental question about mathematics that we posed at the start of the chapter — "What does it do for us?" — can also be answered with a catchy phrase: *Mathematics makes the invisible visible*.

Let me give you some examples of what I mean by this answer.

Without mathematics, there is no way you can understand what keeps a jumbo jet in the air. As we all know, large metal objects don't stay above the ground without something to support them. But when you look at a jet aircraft flying overhead, you can't see anything holding it up. It takes mathematics to "see" what keeps an airplane aloft. In this case, one way to "see" the invisible is an equation discovered by the mathematician Daniel Bernoulli early in the 18th century.

While I'm on the subject of flying, what is it that causes objects other than aircraft to fall to the ground when we release them? "Gravity," you answer. But that's just giving it a name. It doesn't help us to understand it. It's still invisible. We might as well call it magic. To understand it, you have to "see" it. That's exactly what Newton did with his equations of motion and mechanics in the 17th century. Newton's mathematics enabled us to "see" the invisible forces that keep the earth rotating around the sun and cause an apple to fall from the tree onto the ground.

Both Bernoulli's equation and Newton's equations use calculus. Calculus works by making visible the infinitesimally small. That's another example of making the invisible visible.

Here's another: Two thousand years before we could send spacecraft into outer space to provide us with pictures of our planet, the Greek mathematician Eratosthenes used mathematics to show that the Earth was round. Indeed, he calculated its diameter, and hence its curvature, with 99% accuracy.

Using mathematics, we have been able to look backward and see into the distant past, making visible the otherwise invisible moments when the universe was first created in what we call the Big Bang.

For a more commonplace example, how do you "see" what makes pictures and sound of a football game miraculously appear on a television screen on the other side of town? One answer is that the pictures and sound are transmitted by radio waves — a special case of what we call electromagnetic radiation. But as with gravity, that just gives the phenomenon a name, it doesn't help us to "see" it. In order to "see" radio waves, you have to use mathematics. Maxwell's equations, discovered in the 19th century, make visible to us the otherwise invisible radio waves.

Here are some human patterns:

- Aristotle used mathematics to try to "see" the invisible patterns of sound that we recognize as music.
- He also used mathematics to try to describe the invisible structure of a dramatic performance.
- In the 1950s, the linguist Noam Chomsky used mathematics to "see" and describe the invisible, abstract patterns of words that we recognize as a grammatical sentence. He thereby turned linguistics from a fairly obscure branch of anthropology into a thriving mathematical science.

Finally, using mathematics, we are able to look into the future:

- Probability theory and mathematical statistics let us predict the outcomes of elections, often with remarkable accuracy.
- We use calculus to predict tomorrow's weather.
- Market analysts use various mathematical theories to try to predict the future behavior of the stock market.
- Insurance companies uses statistics and probability theory to predict the likelihood of an accident during the coming year, and set their premiums accordingly.

When it comes to looking into the future, mathematics allows us to make visible another invisible — the invisible of the not yet happened. In that case, our mathematical vision is not perfect. Our predictions are sometimes wrong. But without mathematics, we cannot even see poorly.

Today we live in a technological society, built on mathematics. There are increasingly fewer places on the face of the earth where, when we look around us toward the horizon, we do not see products of our technology — and hence of our mathematics: tall buildings, bridges, power lines, telephone cables, cars on roads, aircraft in the sky, and so forth. Where communicating with one another once required physical proximity, today much of our communication is mediated by mathematics, transmitted in digitized form along wires or optical fibers, or through the ether. Computers — machines that perform mathematics — are not only on our desktops, they are in everything from cell phones to microwave ovens, from interactive games to automobiles, and from children's toys to heart pacemakers. Mathematics — in the form of statistics — is used to decide what food we will eat, what products we will buy, what television programs we will be able to see, and which politicians we will be able to vote for. Just as society burned fossil fuels to drive the engines of the industrial age, in today's information age, the principal fuel we burn is mathematics.

And yet, as the role of mathematics in our lives has grown more and more significant, it has become more and more hidden from view, forming an invisible universe that supports much of our lives. Just as our every action is governed by the invisible forces of nature (such as gravity), so too we now live in the invisible universe created by mathematics, subject to invisible mathematical laws.

1.4 This Is Where You Come In

That then, is the big picture of mathematics. Hopefully now you have a good overall sense of what the subject is and why it is important, even if you did not previously. But what does it mean to *do* mathematics? What is it that you, the beginning student of post-calculus math, must learn to do? Here too, the answer we give today is not the one that would have been given in the past.

Up to about 150 years ago, although mathematicians had long ago expanded the realm of objects they studied beyond numbers and algebraic symbols for numbers, they still regarded mathematics as primarily about *calculation*. That is, proficiency in mathematics depended above all at being able to carry out calculations or manipulate symbolic expressions to solve problems. By and large, high school mathematics is still very much based on that earlier tradition.

In the middle of the 19th century, however, a revolution took place. One of its epicenters was the small university town of Göttingen in Germany, where the local revolutionary leaders were the mathematicians Lejeune Dirichlet, Richard Dedekind, and Bernhard Riemann. In their new conception of the subject, the primary focus was not performing a calculation or computing an answer, but formulating and understanding abstract concepts and relationships. This was a shift in emphasis from *doing* to *understanding*. Within a generation, this revolution would completely change the way pure mathematicians thought of their subject. Nevertheless, it was an extremely quiet revolution that was recognized as such only when it was all over. It is not even clear that the leaders knew they were spearheading a major change.

For the Göttingen revolutionaries, mathematics was about "thinking in concepts" (*Denken in Begriffen*). Mathematical objects were no longer thought of as given primarily by formulas, but rather as carriers of conceptual properties. Proving was no longer a matter of transforming terms in accordance with rules, but a process of logical deduction from concepts.

This new approach to mathematics now permeates all university mathematics instruction beyond calculus. And by and large, it is the post-Göttingen approach to mathematics that this little book is all about. Among the new concepts that the revolution embraced are many with which today's university mathematics student must come to grips. You will learn about some of them in the pages that follow.

One such post-Göttingen concept is that of a *function*. Prior to the 19th century, mathematicians were used to the fact that a formula such as $y = x^2+3x-5$ specifies a rule that produces a new number (y) from any given number (x). Then along came the Göttingen revolutionary Dirichlet who said, forget the formula and concentrate on what the function *does* in terms of input–output behavior. A *function*, according to Dirichlet, is any rule that produces new numbers from old. The rule does not have to be specified by an algebraic formula. In fact, there's no reason to restrict your attention to numbers. A function can be any rule that takes objects of one kind and produces new objects from them.

Mathematicians began to study the properties of abstract functions, specified not by some formula but by their behavior. For example, does the function have the property that when you present it with different starting values it always produces different answers? (This property is called *injectivity*. We'll meet it again later.)

This approach was particularly fruitful in the development of real analysis, where mathematicians studied the properties of continuity and differentiability of functions as abstract concepts in their own right. In France, Augustin Cauchy developed his famous (some might say infamous) epsilon-delta definitions of continuity and differentiability — the *epsilontics* that to this day cost each new generation of post-calculus mathematics students so much effort to master. Cauchy's contributions, in particular, indicated a new willingness of mathematicians to come to grips with the concept of infinity. Riemann spoke of their having reached "a turning point in the conception of the infinite".

In the 1850s, Riemann defined a complex function *by its property of differentiability*, rather than a formula, which he regarded as secondary. Karl Friedrich Gauss' residue classes (which you are likely to meet in an algebra course) were a forerunner of the approach — now standard — whereby a mathematical structure is defined as a set endowed with certain operations, whose behaviors are specified by axioms. Taking his lead from Gauss, Dedekind examined the new concepts of ring, field, and ideal — each of which was defined as a collection of objects endowed with certain operations. (Again, these are concepts you are likely to encounter soon in your post-calculus mathematics education.)

Like most revolutions, the Göttingen one had its origins long before the main protagonists came on the scene. The Greeks had certainly shown an interest in mathematics as a conceptual endeavor, not just calculation, and in the 17th century, Gottfried Leibniz thought deeply about both approaches. But for the most part, until the Göttingen revolution, mathematics was viewed primarily as a collection of procedures for solving problems. To today's mathematicians, however, brought up entirely with the post-Göttingen conception of mathematics, what in the 19th century was a revolution is simply taken to be what mathematics is. The revolution may have been quiet, and to a large extent forgotten, but it was complete and far reaching. And it sets the scene for this book, the main aim of which is to provide you with the basic tools you will need to enter this new world of modern mathematics.

Although the Göttingen view of mathematics now dominates the field at the post-calculus university level, it has not had much infuence on high school mathematics — which is why you need this little book to help you make the transition. There was one attempt to introduce the new approach into school classrooms, but it went terribly wrong and soon had to be abandoned. This was the so-called "New Math" movement of the 1960s. What went wrong was that by the time the Göttingen message had made its way from the mathematics departments of the leading universities into the schools, it was badly garbled. To mathematicians before and after 1850, both calculation and understanding had always been important. The 1850 revolution merely shifted the *emphasis* as to which of the two the subject was really about and which was the supporting skill. Unfortunately, the message that reached the nation's school teachers in the 1960s was often, "Forget calculation skill, just concentrate on concepts." This ludicrous and ultimately disastrous strategy led the satirist Tom Lehrer to quip, in his song *New Math*, "It's the method that's important, never mind if you don't get the right answer." (Lehrer, by the way, is a mathematician, so he knew what the initiators of the change had intended.) After a few sorry years, the "New Math" (which was already over 100 years old) was dropped from the syllabus. Such is the nature of educational policy making in free societies that it is unlikely that such a change could ever be made in the foreseeable future, even if it were done properly the second time around. It's also not clear (at least to me) that such a change would be altogether desirable. There are educational arguments (which in the absense of hard evidence either way are hotly debated) that say the human mind has to achieve a certain level of mastery of computation with abstract mathematical entities before it is able to reason about their properties.

1.5 The Stuff of Modern Mathematics

In modern, pure mathematics we are primarily concerned with *statements* about *mathematical objects*.

Mathematical objects are things such as integers, real numbers, sets, functions, etc. Examples of mathematical statements are:

(1) There are infinitely many prime numbers.

(2) For every real number a, the equation $x^2 + a = 0$ has a real root.

(3) $\sqrt{2}$ is irrational.

(4) If $p(n)$ denotes the number of primes less than or equal to the natural number n, then as n becomes very large, $p(n)$ approaches $n/\log_e n$.

Not only are we interested in statements of the above kind, we are, above all, interested in knowing which statements are true and which are false. (The truth or falsity in each case is demonstrated not by measurement or experiment, as in most other sciences, but by a *proof*, of which more in due course). For instance, in the above examples, (1), (3), and (4) are true whereas (2) is false.

The truth of (1) is easily proved. We show that if we list the primes in increasing order as

$$p_1, p_2, p_3, \ldots, p_n, \ldots$$

then the list must continue forever. We all know what the first few members of the sequence are: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11, \ldots$ Consider the list up to stage n:

$$p_1, p_2, p_3, \ldots, p_n$$

Let

$$p = (p_1 . p_2 . p_3 . \ldots . p_n) + 1$$

If p is not a prime, there must be a prime $q < p$ such that q divides p. But none of $p_1, \ldots, p_n$ divides p, for the division of p by any one of these leaves a remainder of 1. So, either p must itself be prime, or else there is a prime $q < p$ that exceeds p_n. Either way we see that there is a prime greater than p_n. Since this argument does not depend in any way upon the size of n, it follows that there are infinitely many primes.

Example (2) can easily be proved to be false. Since the square of no real number is negative, the equation $x^2 + 1 = 0$ does not have a real root.

We give a proof of (3) later. The only known proofs of example (4) are extremely complicated — far too complicated to be included in an introductory text such as this.

Clearly, before we can prove whether certain statements are true or false, we must be able to understand precisely what a statement says. Above all, mathematics is a very *precise* subject, where exactness of expression is required. This already creates a difficulty, for in real life our use of language is rarely precise. Now, systematically to make the entire English language precise (by defining *exactly* what each word is to mean) would be an impossible task. It would also be unnecessary. It turns out that by deciding exactly what we mean by a few simple words, we can obtain enough precision to enable us to make all (or at least most) of the mathematical statements we need. The point is that in mathematics, we do not use all of the English language. Indeed, when we restrict our attention to mathematical *statements* themselves (as opposed to our attempts to explain them), we need only a very small part of the English language.