The first few terms of the series can now be written in the form

$$y(x) = c_0 + c_1x - 2c_0x^2 - c_1x^3 + c_0x^4 + \tfrac{1}{5}c_1x^5 + 0x^6 + \cdots$$
$$= c_0(1 - 2x^2 + x^4 + \cdots) + c_1(x - x^3 + \tfrac{1}{5}x^5 + \cdots).$$

The above procedure will also determine the form of the general term. Let us consider the series expansion about the point $x_0 = 0$ and use the summation notation. The expressions are given by

$$(2)\qquad \begin{cases} y(x) = \sum_{i=0}^{\infty} c_i x^i, \\ y'(x) = \sum_{i=0}^{\infty} i c_i x^{i-1}, \\ y''(x) = \sum_{i=0}^{\infty} i(i-1) c_i x^{i-2}, \end{cases}$$

etc.

For convenience of notation, let us limit our discussion to a second order differential equation.

$$(3)\qquad a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

A substitution of equations (2) into equation (3) yields

$$(4)\qquad a_0(x)\sum i(i-1)c_ix^{i-2} + a_1(x)\sum ic_ix^{i-1} + a_2(x)\sum c_ix^i = 0$$

with all summations going from $i = 0$ to $i = \infty$.*

The next step in the procedure is to multiply out the terms of equation (4) and collect like terms of x. This, of course, depends upon the form of $a_i(x)$, $(i = 0, 1, 2)$. In order to be more specific let us, therefore, assume some form of these functions,

$$(5)\qquad a_0(x) = 2x^2 + 1,\ a_1(x) = 3x,\ a_2(x) = -3.$$

Equation (4) would then take the form

$$(6)\qquad (2x^2 + 1)\sum i(i-1)c_ix^{i-2} + 3x\sum ic_ix^{i-1} - 3\sum c_ix^i = 0.$$

After performing the indicated multiplication we obtain

$$(7)\qquad \sum 2i(i-1)c_ix^i + \sum i(i-1)c_ix^{i-2} + \sum 3ic_ix^i - \sum 3c_ix^i = 0.$$

It is desired to find the coefficient of x^k by taking particular

* The only summations for i that we shall consider in this chapter will be from 0 to ∞, and thus we shall drop these values from our notation when i is the **index of summation**.

values of i. Thus let $i = k$ in the first, third, and fourth term and let $i = k + 2$ in the second term; then the coefficient of x^k becomes

$$2k(k-1)c_k + (k+2)(k+1)c_{k+2} + 3kc_k - 3c_k$$
$$= (k+2)(k+1)c_{k+2} + (2k^2 + k - 3)c_k.$$

We set this coefficient equal to zero and solve for c_{k+2} to obtain

$$(8)\qquad c_{k+2} = -\frac{(2k+3)(k-1)}{(k+2)(k+1)}c_k.$$

We have thus solved for c_j in terms of a c with a lower subscript; such a formula is called a **recursion formula.** By choosing values of $k = 0, 1, 2, \cdots$ we can find all c_j in terms of c_0 and c_1. For our particular choice of $a_i(x)$ we have

$$k = 0:\ c_2 = -\frac{(-3)}{2}c_0 = \frac{3}{2}c_0,$$

$$k = 1:\ c_3 = -\frac{(5)(0)}{(3)(2)}c_1 = 0,$$

$$k = 2:\ c_4 = -\frac{(7)(1)}{(4)(3)}c_2 = -\frac{7}{12}\left(\frac{3}{2}\right)c_0 = -\frac{7}{8}c_0.$$

Since the subscripts in the recursion formula differ by 2, we have quantities with even subscripts in terms of lower even subscripts and similarly for the odd subscripts. Now since $c_3 = 0$ all terms with the larger odd subscripts will also be zero. In this example we need only consider the remaining even subscripts.

$$k = 4:\qquad c_6 = -\frac{(11)(3)}{(6)(5)}c_4 = -\frac{11}{10}\left(-\frac{7}{8}\right)c_0 = \frac{77}{80}c_0,$$

$$k = 2n - 2:{}^*\ c_{2n} = (-1)^{n+1}\frac{3 \cdot 7 \cdot 11 \cdots (4n-1)}{2^n(2n-1)n!}c_0.$$

The solution to our case with an arbitrary c_0 and c_1 can now be written in the form

$$y(x) = c_1x + c_0\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}3 \cdot 7 \cdot 11 \cdots (4n-1)}{2^n(2n-1)n!}x^{2n}\right].$$

* The development of the case $k = 2n - 2$ may require some algebraic gymnastics, and if the student did not obtain sufficient practice along these lines during his study of series in the calculus he should write the first half-dozen terms, the $k = 2n - 4$, and the $k = 2n - 2$ terms, and then obtain the product.

The solution of a differential equation about a point x_0 other than $x_0 = 0$ can be obtained by a translation of the origin to that point and then finding the power series about the new origin. Thus actually we need to consider only expansions about the point $x_0 = 0$. The translation is given by $x - x_0 = v$ and $D_x^i y = D_v^i y$, $(i = 0, 1, \cdots)$ since $D_x v = 1$.

Illustration. Find a series solution of

$$(x^2 - 2x + 2)y'' + 2(x - 1)y' = 0$$

about the point $x = 1$.

Solution. Translate the origin to $x = 1$ by

$$v = x - 1,\ y'' = D_v^2 y, \text{ and } y' = D_v y.$$

The differential equation becomes

$$(v^2 + 1)D_v^2 y + 2vD_v y = 0.$$

The power series solution is

$$y(v) = \sum c_i v^i$$

so that

$$(v^2 + 1)\sum i(i-1)c_i v^{i-2} + 2v\sum ic_i v^{i-1} = 0$$

or

$$\sum i(i-1)c_i v^i + \sum i(i-1)c_i v^{i-2} + \sum 2ic_i v^i = 0.$$

The coefficient of v^k is

$$k(k-1)c_k + (k+2)(k+1)c_{k+2} + 2kc_k.$$

Set this expression equal to zero and solve for c_{k+2} to obtain

$$c_{k+2} = -\frac{k(k+1)}{(k+2)(k+1)}c_k = -\frac{k}{k+2}c_k.$$

In terms of c_0 and c_1 we have

$k = 0$: $c_2 = 0c_0 = 0$; $\therefore c_{2n} = 0$ for $n > 0$.

$k = 1$: $c_3 = -\frac{1}{3}c_1$.

$k = 3$: $c_5 = -\frac{3}{5}c_3 = -\frac{3}{5}(-\frac{1}{3})c_0 = \frac{1}{5}c_0$.

$k = 2n - 1$: $c_{2n+1} = \pm\dfrac{(2n-1)(2n-3)\cdots 1}{(2n+1)(2n-1)\cdots 1}c_0 = \pm\dfrac{1}{2n+1}c_0$.

Thus

$$y(v) = c_0 + c_1\left[v - \frac{v^3}{3} + \frac{v^5}{5} - \frac{v^7}{7} + \cdots + (-1)^n\frac{v^{2n+1}}{2n+1} + \cdots\right]$$

$$= c_0 + c_1 \text{ arc tan } v.$$

In terms of x we have, $v = x - 1$,

$$y(x) = c_0 + c_1 \text{ arc tan } (x - 1).$$

75. Solutions Near a Regular Singular Point. The discussion of solutions about a regular singular point will be limited to $x_0 = 0$ and for a second order differential equation. The first of these restrictions is not a severe one since we can always translate the origin to the regular singular point x_0. The second restriction is merely an expression of laziness, since higher order equations do not impose theoretical hardships but only involve considerably more labor. Consider the differential equation

(1) $$b_0(x)y'' + b_1(x)y' + b_2(x)y = 0$$

with a regular singular point at $x_0 = 0$. The definition of a regular singular point at $x_0 = 0$ states that equation (1) can be written in the form

(2) $$y'' + p(x)y' + q(x)y = 0$$

in which $p(x)$ can have at the most x to the first power in the denominator and $q(x)$ can have at the most x^2 in the denominator; in other words, we may write

$$p(x) = \frac{P(x)}{x} \quad \text{and} \quad q(x) = \frac{Q(x)}{x^2}$$

where P and Q are rational functions of x which exist at $x = 0$ and thus can be expanded in a power series about $x = 0$. Consequently, we have

(3) $$\begin{cases} p(x) = \dfrac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + \cdots, \\ q(x) = \dfrac{q_0}{x^2} + \dfrac{q_1}{x} + q_2 + q_3 x + \cdots, \end{cases}$$

both valid in the neighborhood of $x = 0$.

Let us assume that the solution of equation (1) can be written in the form

(4) $$y(x) = \sum_{i=0}^{\infty} c_i x^{i+r} = c_0 x^r + c_1 x^{r+1} + \cdots.$$

Our problem then reduces to finding values for c_i and r. If we substitute the values from equations (3) and (4) into equation (2), we obtain

$$r(r-1)c_0x^{r-2} + (r+1)rc_1x^{r-1} + (r+2)(r+1)c_2x^r + \cdots$$
$$+\left[\frac{p_0}{x} + p_1 + \cdots\right][rc_0x^{r-1} + (r+1)c_1x^r + \cdots]$$
$$+\left[\frac{q_0}{x^2} + \frac{q_1}{x} + \cdots\right][c_0x^r + c_1x^{r+1} + c_2x^{r+2} + \cdots] = 0.$$

After performing the indicated multiplication and collecting the like terms in x, we obtain

$$[r(r-1)c_0 + p_0rc_0 + q_0c_0]x^{r-2} + [(r+1)rc_1 + p_0(r+1)c_1 + p_1rc_0 + q_0c_1 + q_1c_0]x^{r-1} + \cdots = 0.$$

Since this is an identity in x, we have

$$[r(r-1) + p_0r + q_0]c_0 = 0,$$
$$[(r+1)r + p_0(r+1) + q_0]c_1 + [p_1r + q_1]c_0 = 0,$$
etc.

The first of these equations involves only c_0 of the c_i and we shall always take $c_0 \neq 0$; in other words, let c_0 be the coefficient of the first term which exists in the power series. Consequently, we must have

(5) $$r(r-1) + p_0r + q_0 = 0,$$

and we shall call this equation the **indicial equation.** It will be used to determine the values of r, and for a second order it will always be a quadratic in r with two solutions, r_1 and r_2. This fact creates trouble and we are forced to consider individual cases.

Case I. **Difference between r_1 and r_2 nonintegral.** If the difference between r_1 and r_2 is not an integer, we can develop the two series of the form of equation (4) in a straightforward manner.

Illustration. Find a solution of

$$3x^2y'' + x(x-5)y' + 5y = 0$$

about the point $x = 0$.

Solution. Assume the power series

$$y(x) = \sum c_ix^{r+i}$$

and substitute into the differential equation to obtain

$$\sum 3(i+r)(i+r-1)c_ix^{r+i} + \sum (i+r)c_ix^{r+i+1} - \sum 5(i+r)c_ix^{r+i} + \sum 5c_ix^{r+i} = 0.$$

Let $i = 0$ and consider the term which has x to the smallest exponent, in this case x^r,

$$[3r(r-1) - 5r + 5]c_0.$$

The indicial equation is

$$3r^2 - 8r + 5 = (3r-5)(r-1) = 0.$$

Thus $r_1 = 1$ and $r_2 = \frac{5}{3}$.
With $r_1 = 1$, the series is

$$\sum 3(i+1)ic_ix^{i+1} + \sum (i+1)c_ix^{i+2} - 5\sum (i+1)c_ix^{i+1} + 5\sum c_ix^{i+1} = 0.$$

The coefficient of x^k is

$$3k(k-1)c_{k-1} + (k-1)c_{k-2} - 5kc_{k-1} + 5c_{k-1} = 0$$

or

$$(3k-5)(k-1)c_{k-1} = -(k-1)c_{k-2}$$

so that

$$c_{k-1} = -\frac{1}{3k-5}c_{k-2}.$$

The first few terms are

$$k = 2:\ c_1 = -\frac{1}{6-5}c_0 = -c_0,$$
$$k = 3:\ c_2 = -\frac{1}{9-5}c_1 = \frac{1}{4}c_0,$$
$$k = 4:\ c_3 = -\frac{1}{12-5}c_2 = -\frac{1}{7}\left(\frac{1}{4}\right)c_0,$$

and

$$y_1(x) = c_0x[1 - x + \tfrac{1}{4}x^2 - \tfrac{1}{28}x^3 + \cdots]$$
$$= c_0x + c_0\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{1\cdot 4\cdot 7\cdot 10\cdots(3k-5)}x^k.$$

With $r_2 = \frac{5}{3}$, the series is

$$x^{\frac{5}{3}}[\sum 3(i+\tfrac{5}{3})(i+\tfrac{5}{3}-1)c_ix^i + \sum (i+\tfrac{5}{3})c_ix^{i+1} - 5\sum (i+\tfrac{5}{3})c_ix^i + 5\sum c_ix^i] = 0.$$

Consider the coefficient of x^k in the bracket

$$3(k+\tfrac{5}{3})(k+\tfrac{5}{3}-1)c_k+(k-1+\tfrac{5}{3})c_{k-1}-5(k+\tfrac{5}{3})c_k+5c_k=0.$$

or

$$c_k=-\frac{1}{3k}c_{k-1}.$$

The first few terms are

$$k=1:\ c_1=-\tfrac{1}{3}c_0,$$

$$k=2:\ c_2=-\frac{1}{6}c_1=\frac{1}{6}\cdot\frac{1}{3}c_0=\frac{1}{2}\cdot\frac{1}{3^2}c_0,$$

$$k=3:\ c_3=-\frac{1}{9}c_2=-\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{3^3}c_0=-\frac{1}{3!3^3}c_0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$k=n:\ c_n=\frac{(-1)^n}{3^n n!}c_0.$$

Since this c_0 is not the same as the c_0 for $r_1=1$, we change the notation to b_0 and the second series is

$$y_2(x)=b_0x^{\frac{5}{3}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}x^n\right].$$

The complete solution is given by

$$y(x)=y_1(x)+y_2(x).$$

Case II. $r_1=r_2$. When the indicial equation has equal roots the above procedure would obtain only $y_1(x)$. To obtain $y_2(x)$, however, something new must be added. In the previous technique we were anxious to obtain the values of r from the indicial equation so that we could treat them separately. In the following technique we shall retain r as long as possible.

Consider the differential equation

$$(6)\qquad L(y)=x^2y''+x(x+3)y'+y=0$$

which has a regular singular point at $x=0$. A division by x^2 reveals that

$$p(x)=\frac{x+3}{x}=\frac{3}{x}+1,$$

$$q(x)=\frac{1}{x^2}.$$

Thus $p_0=3$ and $q_0=1$ so that the indicial equation is

$$r(r-1)+3r+1=r^2+2r+1=0$$

with roots $r_1=r_2=-1$. Let the solution be expressed by

$$(7)\qquad y(x)=\sum c_ix^{r+i}$$

and substitute into equation (6) to obtain

$$L(y)=\sum(r+i)(r+i-1)c_ix^{r+i}+\sum(r+i)c_ix^{r+i+1}+\sum 3(r+i)c_ix^{r+i}+\sum c_ix^{r+i}$$
$$=\sum[(r+i)^2+2(r+i)+1]c_ix^{r+i}+\sum(r+i)c_ix^{r+i+1}.$$

The coefficient of the term containing x^{r+k} is

$$[(r+k)^2+2(r+k)+1]c_k+(r+k-1)c_{k-1}$$

and would yield the indicial equation for $k=0$. Let us avoid this value of k and set this coefficient equal to zero for $k\geq 1$; we have

$$(r+k+1)^2c_k=-(r+k-1)c_{k-1}.$$

We can thus obtain the values of c_k in terms of c_0 and r. The individual values are

$$c_1=-\frac{r}{(r+2)^2}c_0,$$

$$c_2=-\frac{r+1}{(r+3)^2}c_1,$$

$$\cdots\cdots$$

$$c_k=(-1)^k\frac{r(r+1)\cdots(r+k-1)}{[(r+2)(r+3)\cdots(r+k+1)]^2}c_0.$$

The function (7) can thus be written as a function of x and r, namely,

$$(8)\qquad y(x,r)=c_0x^r+\sum_{k=1}^{\infty}c_k(r)x^{r+k}$$

with

$$(9)\qquad c_k(r)=(-1)^k\frac{r(r+1)\cdots(r+k-1)}{[(r+2)(r+3)\cdots(r+k+1)]^2}c_0.$$

We have thus obtained a function which when substituted into the differential equation $L(y)=0$ will make all the terms on the