Counting Colorings with Burnside's Lemma: A Combinatorial Approach to Dihedral Symmetries

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Abstract

Burnside's Lemma offers a powerful technique for counting distinct configurations by averaging the number of fixed points under group actions. This poster explains how to utilize the lemma to count non-equivalent colorings of a square's vertices using three colors, considering symmetries from the dihedral group D_4. Each transformation in D_4—including rotations and reflections—acts on the set of colorings, and we compute the number of different configurations under these actions. After presenting the full calculation for D_4, the method is extended conceptually to general dihedral groups D_n, which represent the symmetries of regular n-gons. This poster exemplifies how algebraic structures can simplify combinatorial counting and highlights the intersection between group theory and visual enumeration problems.

Burnside's Lemma

Counts the number of distinct configurations of a set under the symmetries of a group.

Some useful definitions:

If G is a group acting on a set A, and $a \in A$ and $g \in G$ are some fixed elements.

- **the stabilizer** of a is the set in G defined by

$$G_a = \{h \in G : h \cdot a = a\}.$$

- **the fixed points** of g is the set in A defined by

$$fix(g) = \{b \in A : g \cdot b = b\}.$$

- **the orbit** of a is the set in A defined by

$$Ga = \{h \cdot a : h \in G\} = \{b \in A : b = h \cdot a \text{ for some } h \in G\}.$$

Burnside's Lemma:

$$\text{number of orbits } = \frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{|G|}\sum_{a\in A}|G_a|$$

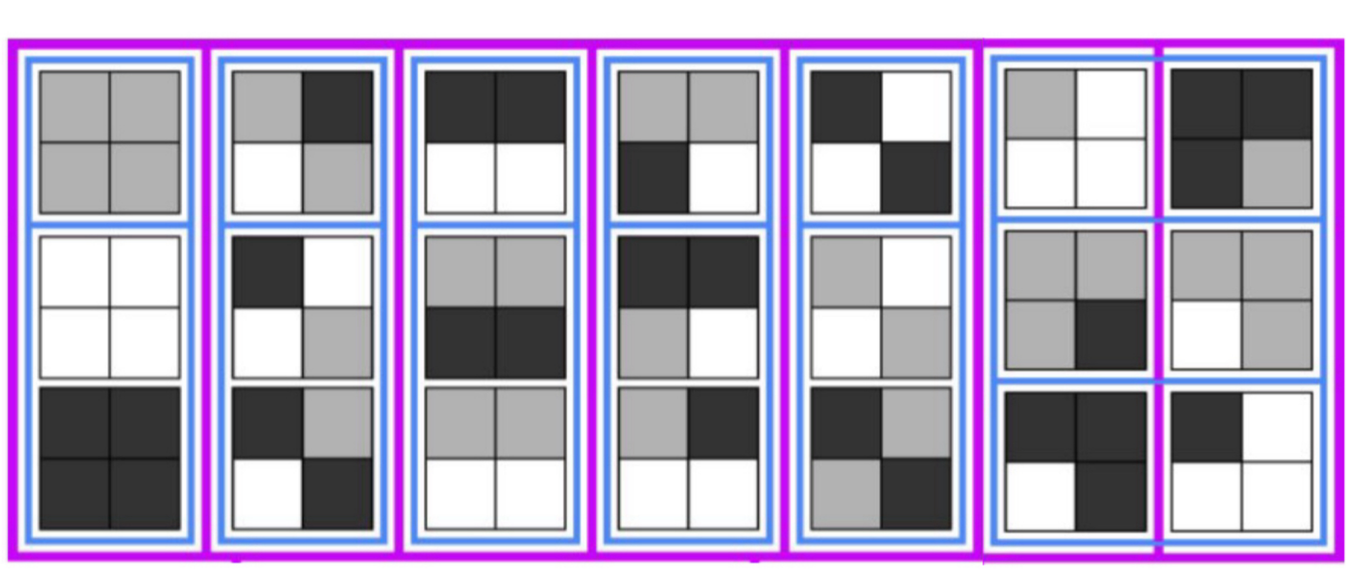
D_4: Vertex colorings of a square with 3 colors

$$D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}, \quad |D_4| = 8$$

$$\mathcal{A} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{\square, \blacksquare, \blacksquare\}, i, j \in \{1,2\} \right\}$$

Transformation	Fix	#
1=rot$_{0°}$	(1) (2) (3) (4)	3^4
r=rot$_{90°}$	(1 2 3 4)	3^1
r$^2 = rot_{180°}$	(1 3) (2 4)	3^2
r$^3 = rot_{270°}$	(1 4 3 2)	3^1
s=vertical reflection	(1 2) (3 4)	3^2
sr= horizontal reflection	(1 4) (2 3)	3^2
sr^2 = diagonal reflection 1	(1) (2 4) (3)	3^3
sr^3 = diagonal reflection 2	(1 3) (2) (4)	3^3

$$\frac{1}{|D_4|}\sum_{g\in D_4}|fix(g)| = \frac{1}{8}(3^4 + 3 + 3^2 + 3 + 3^2 + 3^3 + 3^2 + 3^3) = 21$$



Hence, the number of essentially different colorings is 21.

D_n Vertex colorings of a n-gon with k colors

The method used for D_4 applies to any dihedral group D_n.

$$D_n = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}, \quad |D_n| = 2n.$$

We will use Polya's Enumeration Theorem that complements Burnside's Lemma.

Polya's Enumeration Theorem: Each symmetry can be represented as a permutation of the vertices, which decomposes into cycles. A coloring is fixed under that symmetry if all the vertices in each cycle share the same color. Thus, the number of fixed colorings is $k^{(\text{\# of cycles})}$.

For k colors:

1. Rotations:
   Let $m_{\tilde{r}}$ be the number of vertices shifted by a rotation $\tilde{r}$. Giving

   $k^{\gcd(m_{\tilde{r}}, n)}$, the number of fixed colorings

   for the corresponding rotation.
2. Reflections
   - If n is odd:
     Reflection has 1 fixed vertex (a cycle of length 1) and $(n-1)/2$ mirrored pairs (cycles of length 2). The number of fixed colorings is

     $$k^{(n+1)/2}$$
   - If n is even:
     Reflections through two opposite vertices fix $k^{\frac{n}{2}+1}$ colorings.
     Reflections through two opposite edges fix $k^{n/2}$ colorings.

Number of Distinct Colorings $= \frac{1}{|D_n|}\sum_{g\in D_n}|fix(g)| =$

$$= \frac{1}{2n}\Big(\sum_{\tilde{r}\in\{1,r,,\ldots,r^{n-1}\}} k^{\gcd(m_{\tilde{r}},n)} + \begin{cases} nk^{\frac{n+1}{2}}, & n \text{ odd} \\ \frac{n}{2}k^{\frac{n}{2}+1} + \frac{n}{2}k^{\frac{n}{2}}, & n \text{ even} \end{cases}\Big).$$

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