Updates on the uniqueness of the $HS_m^n(X)$ hyperspace

Gerardo Hernández Valdez, David Herrera Carrasco, Fernando Macías Romero

Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla

Abstract

Let $n, m \in \mathbb{N}$ with $n \geq m$, and X be a metric continuum. We consider the hyperspaces $C_n(X)$ (respectively, $F_n(X)$) of all nonempty closed subsets of X with at most n components (respectively, n points). The (n,m)-fold hyperspace suspension on X was defined in 2018 by Anaya, Maya, and Vázquez-Juárez, to be the quotient space $C_n(X)/F_m(X)$, denoted by $HS_m^n(X)$. In this work, we present several recent updates on the uniqueness of this hyperspace for some well-known families of continua.

Introduction

Given a continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\},$$

The hyperspaces $F_n(X)$ and $C_n(X)$ are called the *n-fold symmetric product of X* and the *n-fold hyperspace* of X, respectively. In 2018 Anaya, Maya and Vázquez-Juárez, introduced the *(n,m)−fold hyperspace* X, denoted by $HS_m^n(X)$, we mean the quotient space $C_n(X)/F_m(X)$ obtained from $C_n(X)$ by shrinking $F_m(X)$ to a point with the quotient topology [1]. For a continuum X and $n, m \in \mathbb{N}$ satisfying that $n \geq m$, the symbol $q_X^{(n,m)}$ denotes the natural projection $q_X^{(n,m)} : C_n(X) \to HS_m^n(X)$, and F_X^m denotes the element $q_X^{(n,m)}(F_m(X))$. Notice that $q_X^{(n,m)}|_{C_n(X)-F_m(X)} : C_n(X) - F_m(X) \to HS_m^n(X) - \{F_X^m\}$ is a homeomorphism.

Classes of continua with unique hyperspace $HS_m^n(X)$

The following results summarize what we know about the uniqueness of hyperspaces for meshed continua and finite graphs.

Theorem

Let X be a finite graph, $n, m \in \mathbb{N}$ with $n \geq m$. Then X has unique hyperspace $HS_m^n(X)$

Theorem

Let X be a meshed continuum, $n \in \mathbb{N} - \{1, 2\}$, $m \in \mathbb{N} - \{1\}$ with $n \geq m$. Then X has a unique hyperspace $HS_m^n(X)$

Classes of continua without unique hyperspace $HS_m^n(X)$

Recall that a continuum X does not have unique hyperspace $HS_m^n(X)$ if there exists a continuum Y such that $HS_m^n(X)$ and $HS_m^n(Y)$ are homeomorphic, but X and Y are not.

Theorem

If X is a contractible locally connected continuum without free arcs and $n, m \in \mathbb{N}$ such that $n \geq m$, then X does not have unique hyperspace $HS_m^n(X)$.

To illustrate this Theorem, take D_n, D_m dendrites as constructed in [2, 2] with $n, m \in \mathbb{N} - \{1, 2\}$ and $n \neq m$. Observe that these dendrites are not homeomorphic, however $HS_m^n(D_n)$ and $HS_m^n(D_m)$ are, since they are Hilbert cubes ([3, 7.1.10]).

Theorem

Let X be an almost meshed dendrite and $r, m, n \in \mathbb{N}$ such that $n \geq m$ and $r \geq n$. Suppose there exists a contractible closed subset B of $\mathcal{P}(X)$ and pairwise disjoint nonempty open subsets $U_1, \ldots, U_{r+1}$ of X such that $X - B = \cup_{i=1}^{r+1} U_i$ and for each $i \in \{1, 2, ..., r+1\}$, $B \subset_X (U_i)$. Then X does not have unique hyperspace $HS_m^n(X)$.

An application of this result can be found in next section.

A continuum without unique hyperspace $HS(X)$ but with unique hyperspace $HS_1^2(X)$

Example

Let $m \in \mathbb{N}$ and

$$Z_3 = ([-1,1] \times \{0\}) \cup \left(\bigcup_{m \geq 2} \left\{-\frac{1}{m}\right\} \times \left[0, \frac{1}{m}\right]\right) \cup \left(\bigcup_{m \geq 2} \left\{\frac{1}{m}\right\} \times \left[0, \frac{1}{m}\right]\right)$$

The following conditions are satisfied

- Z_3 is an almost meshed locally connected continuum without unique hyperspace $HS(Z_3)$.
- The continuum Z_3 has unique hyperspace $HS_1^2(Z_3)$.

To prove the first part, notice that $\mathcal{P}(Z_3) = \{(0,0)\}$ and Z_3 is not a meshed continuum. Using last Theorem, it follows that Z_3 does not have unique hyperspace $HS(Z_3)$.

Recall that

$$\mathcal{G}(Z_3) = \{x \in Z_3 : x \text{ has a neighborhood in } Z_3 \text{ which is a finite graph}\},$$

Consider a homeomorphism $\phi : \mathcal{G}(Z_3) \to \mathcal{G}(Y)$.

Let

$$\mathcal{G}_L(Z_3) = ([-1,0) \times \{0\}) \cup \left(\bigcup_{m \geq 2} \left\{-\frac{1}{m}\right\} \times \left(0, \frac{1}{m}\right]\right),$$

$$\mathcal{G}_R(Z_3) = (0,1] \times \{0\}) \cup \left(\bigcup_{m \geq 2} \left\{\frac{1}{m}\right\} \times \left(0, \frac{1}{m}\right]\right)$$

Observe that $\mathcal{G}(Z_3) = \mathcal{G}_L(Z_3) \cup \mathcal{G}_R(Z_3)$. Let $\phi(\mathcal{G}_L(Z_3)) = \mathcal{G}_L(Y)$ and $\phi(\mathcal{G}_R(Z_3)) = \mathcal{G}_R(Y)$. Therefore, $\mathcal{G}(Y) = \mathcal{G}_L(Y) \cup \mathcal{G}_R(Y)$. Let $\theta_L \in_Y (\mathcal{G}_L(Y)) - \mathcal{G}_L(Y)$ and $\theta_R \in_Y (\mathcal{G}_R(Y)) - \mathcal{G}_R(Y)$. Then, $\theta_L = \theta_R$.

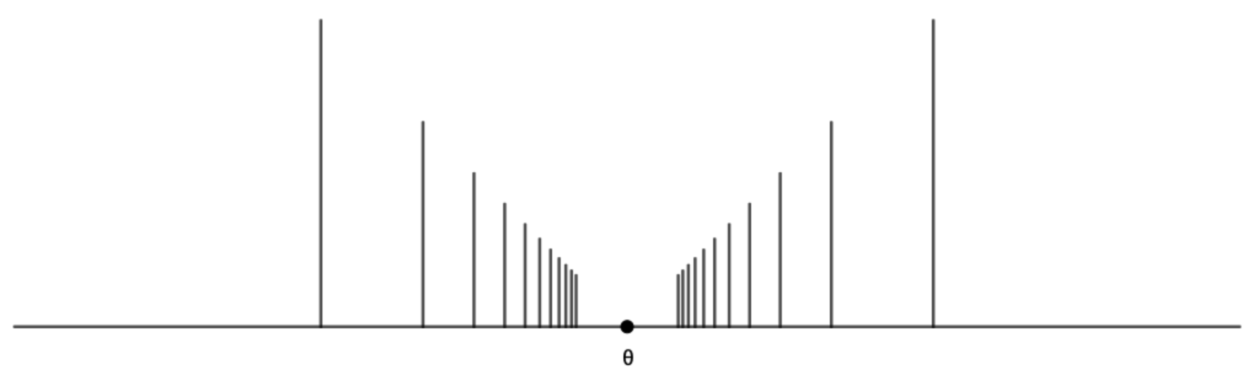

Figure 1: Z_3

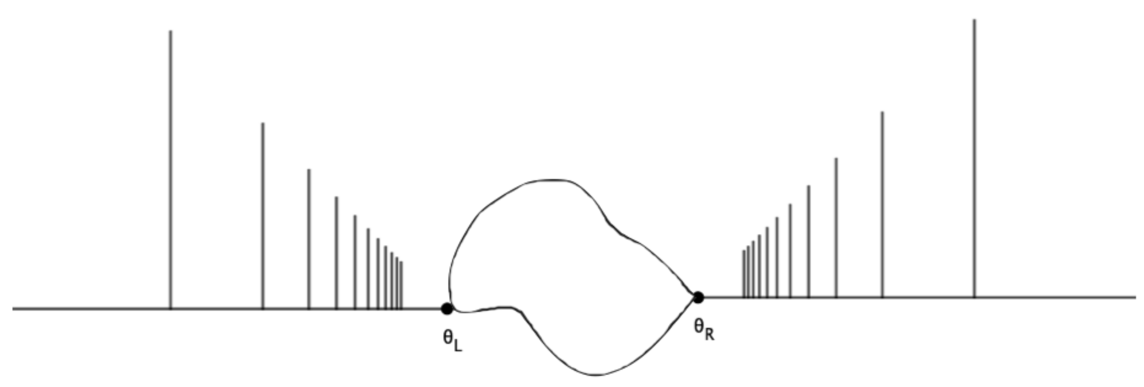

Figure 2: Y

Let $\theta_h \in cl_Y(\mathcal{G}(Y)) - \mathcal{G}(Y)$. We may define a function $\Phi : Z_3 \to Y$ given by

$$\Phi(z) = \begin{cases} \phi(z), & \text{if } z \in \mathcal{G}(Z_3), \\ \theta_h, & \text{if } z = \theta. \end{cases}$$

Thus, Φ is a homeomorphism from Z_3 into Y.

References

[1] F. Vázquez-Juárez J.G. Anaya, D. Maya. The hyperspace $hs_m^n(x)$ for a finite graph x is unique. *Topology Appl.*, 157:428–439, 2018.

[2] W.J. Charatonik J.J. Charatonik. Dendrites. *Aportaciones Mat. Comun.*, 22:227–253, 1998.

[3] S. Macías. On the n−fold hyperspace suspension of continua. *Topology Appl.*, 138:125–138, 2004.

Contact Information

- Email: gerareg09@gmail.com
- Phone: +52 811 666 2888