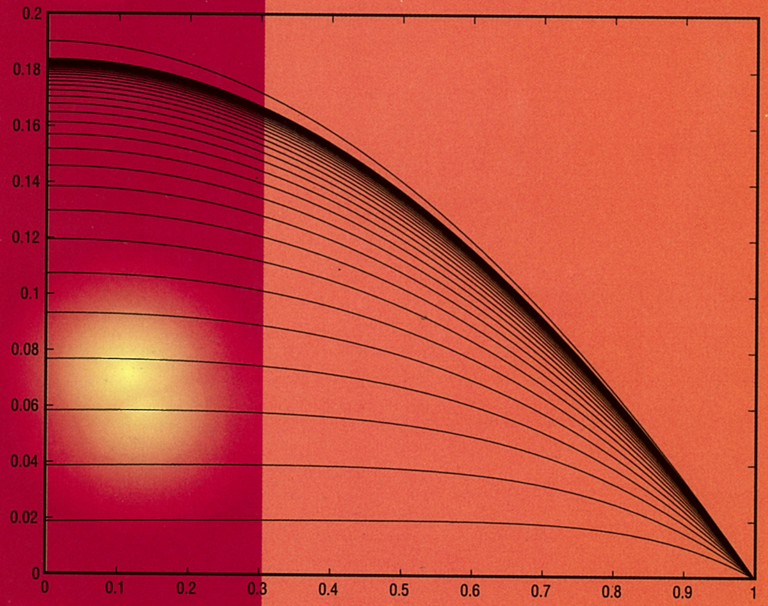
Miguel Antonio Jiménez Pozo
Jorge Bustamante González

TÓPICOS DE TEORÍA DE LA APROXIMACIÓN II



textos Científicos

Benemérita Universidad Autónoma de Puebla

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II

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Miguel Antonio Jiménez Pozo
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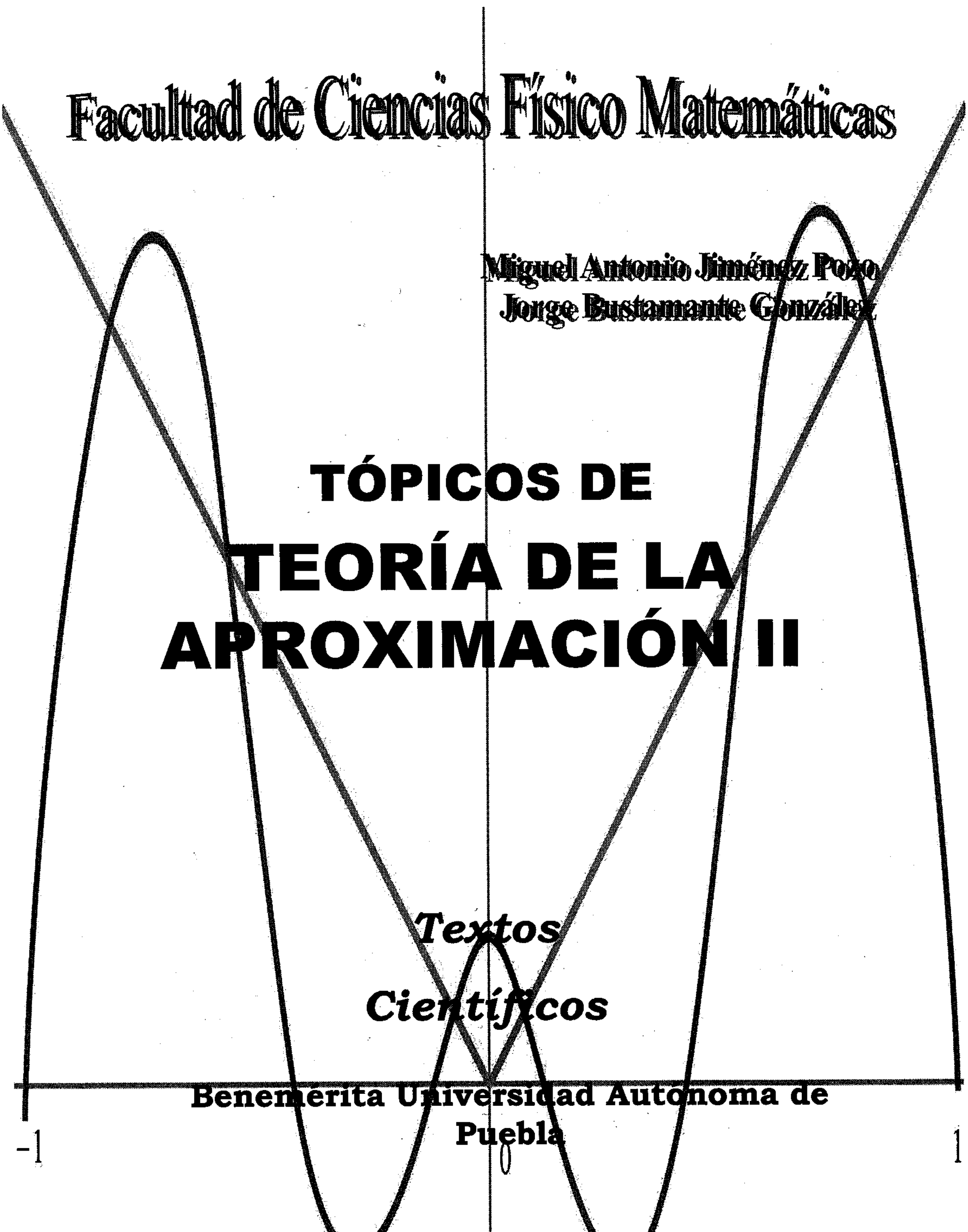

Tópicos de Teoría de la Aproximación II

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Capítulo 1

Preface

The **2nd International Colloquium on Approximation Theory and Related Topics of the University of Puebla** was held in February 2006. Though not excluded a priori, it was not the objective of this Colloquium to request new research papers but original works or surveys that present the state of the art in particular subjects in which the authors are active. This kinds of monographs are useful for our graduate students as well as for researchers in other areas. With this aim in hands we collect here several talks of the Colloquium and we emphasize that all papers in this volume, the ones of the editors included, have been submitted to an international referee procedure.

I would like to thank the University of Puebla, the Organizing Committee of the Colloquium, their participants, the anonymous referees and very special to my friend and also editor of this volume Prof. Dr. Jorge Bustamante, for their help in the organization and further preparation of this volume. On behalf of the Approximation people of the University of Puebla, Mexico, I am very grateful for their indispensable support.

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Capítulo 2

Problems of approximation in Hölder norms

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Abstract: A collection of problems related with approximation in Hölder norms is presented.

This paper is result of some lectures that the author read at the Seminar of Approximation Theory at Puebla State University, in Mexico. Since some graduate students were working on problems involving Lipschitz norms, it was convenient to have a list of open problems. Since we think that this compilation may be useful for other specialists (and for new students), we decided to publish it. In fact, in [17] some historical comments and motivation related to problems in Hölder approximation where developed. Here we pay more attention to open problems and the references related with them.

If we try to develop the approximation theory in mixed norms, then it is natural to ask how the usual results of classical approximation looks like in the new context. Many of the problems to be presented below are stated for functions of a real variable, but they can be extended to the multivariate case.

One introductory question may be the following: if some approximations have a good rate of convergence (in some abstract space of functions) it will be good in another space?. An important idea to be considered is that of avoiding unnecessary repetitions of arguments. We believe that classical approximation provides all element to derive results in Hölder norms.

The paper is organized as follows. In the first section we describe general method to present Hölder spaces in different context and lipschitzian type moduli of smoothness. Then we state and discuss problems taking into account the order in which they usually appear in text books. Thus in the second section we consider questions related with density and existence and unicity of the best approximation. Other sections are devoted to problems of characterization of the rate of convergence of the best approximation, approximation by linear operators, simultaneous approximation, and other questions.

2.1 Abstract Lipschitz spaces

The usual way to present Lipschitz spaces is in the context of metric spaces.

Let (X,d) be a nonempty compact metric space and $C(X)$ the Banach space of all real continuous functions on X provided with the sup norm. If $\alpha \in (0,1]$, we denote by $Lip_\alpha(X)$ the family of all functions $f \in C(X)$ such that

$$\theta_\alpha(f) = \sup_{0<d(x,y)} \frac{| f(x) - f(y) |}{d^\alpha(x,y)} < \infty.$$

A new Banach space is obtained by considering in $Lip_\alpha(X)$ the norm

$$\|f\|_\alpha = \|f\|_\infty + \theta_\alpha(f).$$

For $\alpha \in (0,1)$ the small Lipschitz space is the subspace $f \in Lip_\alpha(X)$ given by all functions f such that

$$\theta_\alpha(f,t) = \sup_{0<s\leq t} \quad \sup_{0<d(x,y)\leq s} \frac{| f(x) - f(y) |}{d^\alpha(x,y)} \longrightarrow 0, \qquad t \to 0.$$

There are several ways to extend the above notions to other situations. For instance, in $\mathbb{R}^n$ the Lipschitz classes can be introduced with the help of the Poisson integral or the Weierstrass integral (see [45], [132], [133]) or in $L_p[0,1]$ they can be defined with the help of differences of order r ([21], [64]). Here we present an abstract approach taken from [14]. This allows us to simplify the document. In this section E denotes a Banach space.

Definition 2.1.1 *A modulus of smoothness on E is a function $\omega : E \times [0,+\infty) \to \mathbb{R}^+$ such that:*

a) For each fixed $t \in (0,+\infty)$, the function $\omega(\cdot,t)$ is a seminorm on E and for all $f \in E$, $\omega(f,0) = 0$;

b) For each fixed $f \in E$, the function $\omega(f,\cdot)$ is increasing on $[0,+\infty)$ and continuous at 0;

c) There exists a constant $C > 0$ such that for each $(f,t) \in E \times [0,+\infty)$, one has

$$\omega(f,t) \leq C\,\|f\|\,.$$

Definition 2.1.2 *Given a real $r > 0$, we say that the modulus of smoothness ω is of order r if $N(E,\omega,r) \neq Ker(\omega)$ and $N(E,\omega,s) = Ker(\omega)$ for all $s > r$, where*

$$Ker(\omega) = \left\{ g \in E : \sup_{t\geq 0} \omega(g,t) = 0 \right\}$$

and

$$N(E,\omega,r) = \left\{ f \in E : \quad \sup_{t>0} \frac{\omega(f,t)}{t^r} < \infty \right\}.$$

Definition 2.1.3 *Given a modulus of smoothness ω on E and a real $\alpha > 0$, we define $\theta_{\omega,\alpha}(f,0) = 0$,*

$$\theta_{\omega,\alpha}(f,t) = \sup_{0<s\leq t} \frac{\omega(f,s)}{s^\alpha} \quad \text{and} \quad \|f\|_{\omega,\alpha} = \|f\|_E + \sup_{t>0} \theta_{\omega,\alpha}(f,t). \tag{2.1}$$

The Hölder space $E_{\omega,\alpha}$ is formed by those $f \in E$ such that $\|f\|_{\omega,\alpha} < \infty$ with the norm $\|f\|_{\omega,\alpha}$. Moreover we denote

$$E^0_{\omega,\alpha} = \left\{ f \in E_{\omega,\alpha} : \quad \lim_{t\to 0} \theta_{\omega,\alpha}(f,t) = 0 \ \right\}. \tag{2.2}$$

Let us explain how some known spaces are obtained using the ideas explained above. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous bounded function. For $r \in \mathbb{N}$, the modulus of smoothness cof order r is defined by

$$\omega_r(f,t) = \sup_{|h|\leq t} \sup_{x\in\mathbb{R}} \mid \Delta_h^r f(x) \mid = \sup_{|h|\leq t} \|\Delta_h^r f\|, \quad t > 0 \tag{2.3}$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x+ih). \tag{2.4}$$

By definition $\omega_r(f,0) = 0$.

The same definition is used for the spaces $C[0,1]$ and $C_{2\pi}$ (2π-periodic continuous functions defined on $\mathbb{R}$, with the sup norm). For $C[0,1]$ we assume that $\Delta_h^r f(x) = 0$ when $x \notin [0,1]$ or $x + rh \notin [0,1]$.

For $1 \leq p < \infty$ and the spaces $L_p[0,1]$ and $L_{p,2\pi}$ we consider similar definitions using the norms

$$\|f\|_p = \left(\int_0^1 \mid f(t) \mid^p dt\right)^{1/p}, \quad \left(\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid f(t) \mid^p)dt\right)^{1/p}\right),$$

in place of the sup norm.

In practically all the results presented here, we have some approximations in a classical space and we want to use it as approximations in a Hölder space. Of course one can state the problem of taking the good approximations in a Hölder space and analyze if they are good in another Hölder space (see [35]), but this kind of problem will not be considered here.

2.2 Density, existence and unicity

We remark that $Lip[0,1]$ is isometric to $L_\infty[0,1]$ via differentiation almost everywhere. In almost all the result considered here approximation in Hölder norms is considered in small Lipschitz spaces.

Assume that $\{P_n\}$ is a family of subspaces of $lip_\alpha(X)$ such that $\cup_{n\in\mathbb{N}} P_n$ is dense in $lip_\alpha(X)$ and $P_n \subset P_{n+1}$. Then the best approximation of $f \in lip_\alpha(X)$ by P_n in Hölder norm is defined by

$$E_{n,\alpha}(f) = \inf_{g\in P_n} \|f - g\|_\alpha.$$

Let us denote by E any one of the following spaces $C_{2\pi}$, $C[0,1]$, $L_p[0,2\pi]$ or $L_p[0,1]$, $(1 \leq p < \infty)$. When we work with periodic (non periodic) functions we consider approximation by trigonometric (algebraic) polynomials. We refer to E as one of the classical spaces. For each integer r the moduli of smoothness of order r is defined as in the previous section. The corresponding small Hölder space (see (2.2)) is denoted by $E^0_{r,\alpha}$, $0 < r < \alpha$. It can be proved that, in all classical cases, polynomials are contained in $E^0_{r,\alpha}$. For the best approximation we use the notation $E_{n,r,\alpha}$.

For each $f \in E^0_{r,\alpha}$ and $n \in \mathbb{N}$ there exists at least one element $g \in P_n$ of best approximation. That is $E_{n,r,\alpha}(f) = \|f - g\|_\alpha$. But, in some cases, this element is not unique. Some characterizations of the elements of the best approximation in Banach spaces are given in [115]. Let us recall some facts before presenting the problems.

The space X is said to be strictly convex if all nontrivial convex combinations of two different unit vector have norm less than 1.

If M is a subset of X, denote by $P_M(x) = \{y \in P_n : \|x - y\| = E_M(x)\}$. The set-valued function P_M, which associates to each $x \in X$ the (possibly empty) set of all its best approximations, is called the metric projection operator. When $P_M(x)$ is a singleton, for all $x \in X$, M is called a Chebyshev set. Historical references on Chebyshev sets are given in [73]. An old problem is to study the connections between the geometric properties of the space and the properties of the Chebyshev sets. For instance, is it true that in a Hilbert space all Chebyshev sets are convex (see [65])? A related question is: find a characterization of Banach spaces for which all Chebyshev sets are convex (or connected). Partial answers to this question are known. A Banach space is called smooth if each point on the unit sphere has a unique support hyperplane. It is not known if in smooth Banach spaces all Chebyshev sets with a continuous metric projections are convex. A selection is function $\psi : X \to M$ such that, $\psi(x) \in P_M(x)$ for each x. When a continuous selections exists? Continuous selection may exist in case when $P_M(f)$ is not a singleton.

Local continuity of the best approximation have been investigated by different authors. For instance in [46], [66] and [99] it was proved that the metric projection operator onto a finite-dimensional Chebyshev subspace of $C[0,1]$ or $L_p[0,1]$ $(1 < p < 2)$ is point-wise *Lip*. Notice that this kind of problem is interesting even when the best approximation is unique. A more difficult problem is to obtain results like these when there is not unicity.

Problem 2.2.1 Let E be one of the classic spaces, find a characterization for the elements of the best approximation in the spaces $E^0_{1,\alpha}$, $0 < \alpha < 1$. In particular, how the Kolmogorov criterium looks like?

It is not clear what we mean by a characterization. For the case of continuous functions the Kolmogorov criterium can be manipulated to obtain the Chebyshev theorem. For the L_p space similar conditions are known. These results have been used to solve other problems, as the study of continuous selection or the Polya algorithm. We are looking for a characterization that help us to analyze this kind of problems.

In order to study the last problem some details of Lipschitz spaces as Banach spaces should be considered. In [90] it is given a description of the dual and bidual space of the small Lipschitz space of 1-periodic complex functions on the real line (see also [143]). The idea is to identify $lip_\alpha(X)$ with a subspace of a space of continuous functions vanishing at infinity on a certain locally compact space and then to use the Riesz representation theorem. This idea has also been used in [70].

Problem 2.2.2 If X is one of the classical spaces, does there exist a continuous selection for the best polynomial approximation for in any the Hölder spaces? Is there a Lipschitz selection (a selections which satisfies a Lipschitz type condition)?. Is there a continuous linear selection?

It is pointed out in [85] and [145] that there exist Lipschitz set-valued maps without a Lipschitz selection.

Problem 2.2.3 Polya's algorithm. Fix $f \in lip_\alpha[0,1] \setminus \Pi_n$ and let $f_{n,p}$ be the element of best approximation for f in $Lip_{p,\alpha}[0,1]$ $(1 < p < \infty)$, by polynomials of degree not greater than n. Does the sequence $\{f_{n,p}\}$ converges in $lip_\alpha[0,1]$ as $p \to \infty$? If the limit exists, say g, is g an element of the best approximation for f in lip_α?

It is probable that, for the last problem, a more convenient norm should be considered in $Lip_{p,\alpha}$, says

$$\|f\|_{p,\alpha} = \left(\|f\|_p^p + (\theta_{p,\alpha}(f))^p\right)^{1/p}.$$

2.3 Rate of convergence of best approximation

Problems related with constructive characteristic of functions were introduced by Bernstein. Let us show an example due to de la Vallée Poussin [86] for $C_{2\pi}$. If $E_n(f) = \mathcal{O}(n^{-m-\alpha})$, $\alpha \in (0,1)$ and $m \in \mathbb{N}$, then

$$f \in W^m H^\alpha = \{g : g \in C_{2\pi}^m, \quad \omega(g^{(m)}, t) = \mathcal{O}(t^\alpha)\}.$$

It is usual that characterizations of functional classes are obtained with the help of the so called direct and inverse theorems. For the case of Hölder approximation this kind of results where given in [15] and [14]. It can be stated as follows. Let r be a positive integer, $\alpha \in (0,r)$ and $X_{r,\alpha}$ the small Lipschitz space corresponding to modulus of smoothness of order r. There exists a constant C_r such that, for each $f \in X_{r,\alpha}$ and every $n \in \mathbb{N}$,

$$E_n(f)_{X_{r,\alpha}} \leq C_r \theta_{r,\alpha}\left(f, \frac{1}{n}\right)$$

and

$$\theta_{r,\alpha}\left(f, \frac{1}{n}\right) \leq C_2 \frac{1}{n^{r-\alpha}} \sum_{k=1}^{n} k^{r-\alpha-1} E_k(f)_{X_{r,\alpha}}. \tag{2.5}$$

In the case of approximation by algebraic polynomials on the interval $[0,1]$, the Lipschitz spaces are defined in terms of the Ditzian-Totik moduli of smoothness with weight $\varphi(x) = \sqrt{x(1-x)}$ [37].

Expressions like the one give in (2.5) are called Stechkin type inequalities. For the case of approximation on classical spaces these inequalities can not be improved. Let us see some details for periodical functions.

Let $\Lambda = \{\lambda_n\}$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \lambda_n = 0$ and denote

$$C(\Lambda) = \{f \in C_{2\pi} : E_n(f) \leq \lambda_n, \quad n \in \mathbb{N} \cup \{0\}\}.$$

It is not difficult to prove that $C(\Lambda)$ is not empty. Moreover, there exists f such that $E_n(f) = \lambda_n$, for all n. This last assertion holds for any Banach space where there is a family of linear subspaces A_n such that, $dim(A_n) = n$, $A_n \subset A_{n+1}$ in which $\cup A_n$ is dense in E (see [139]).

It is known that (see [52], Th. 1) that there exists a positive constant C_1 and, for each $n \in \mathbb{N}$, a constant C_2 such that

$$C_1 \sup_{f\in C(\Lambda)} \omega_r\left(f, \frac{1}{n}\right) \leq \frac{1}{n^r}\sum_{k=1}^{n} k^{r-1}\lambda_{k-1} \leq C_2 \sup_{f\in C(\Lambda)} \omega_r\left(f, \frac{1}{n}\right). \tag{2.6}$$

Problem 2.3.1 Prove the analogous of (2.6) for the case of trigonometric approximation in Hölder norms.

Given a function $\omega : [0,\pi] \to \mathbb{R}$, $\omega(t) > 0$, for $t > 0$, we define

$$H_r(\omega) = \{f \in C_{2\pi}, \omega_r(f,t) \leq \omega(t), t \in (0,\pi]\}.$$

Moreover, if we denote by ω_r^* the corrected majorant of order r of ω (see [119]), then for $n \in \mathbb{N}$ (see [52]),

$$\sup_{f\in H_r(\omega)} E_n(f) \sim \omega_r^*(n^{-1}). \tag{2.7}$$

Problem 2.3.2 Obtain the analogous of (2.7) for the case of trigonometric approximation in Hölder norms.

The results presented in [14] and [15] for L_p spaces cover the case $1 \leq p \leq \infty$. For the classical results concerning best approximation in L_p, $0 < p < 1$, see [67] and [128].

Problem 2.3.3 Obtain the direct and inverse theorems for the best approximation in Hölder norms for L_p spaces, with $0 < p < 1$.

Even in the case of best approximation of trigonometric functions, for $1 < p < \infty$ the estimations given in [15] can be improved. In general, when we use Minkowski's inequality to obtain estimates, we will not obtain good constants. It can be explained because we are dealing with strictly convex spaces. But other approach are available. One of then is a variation of the Riesz-Thorin theorem about interpolation of operators. In particular the ideas given in [69] may be useful to improve the estimation given in [15]. To look for the exact constant (or near exact constants) is necessary for the point of view of applications. For instance, they are very useful in order to design numerical algorithms. For the particular case of L_2 spaces one can use the arguments given in [134], [146] and [147].

We present an example taken from [69]. Let

$$K(t) = \frac{1}{2} + \sum_{k=0}^{\infty} \lambda_k \cos(kt), \qquad \lambda_n \geq 0.$$

be a positive kernel and consider the convolution operator

$$L(f,t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)K(t)dt$$

acting in L_p, $1 < p < \infty$. Fix q such that $1/p + 1/q = 1$. Then, for $1 < p \leq 2$

$$\|f - L(f)\|_p \leq \frac{1}{2^{1/q}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\Delta_t f\|_p^p K(t)dt \right)^{1/p}.$$

Notice that the use of generalized Minkowski inequality only provides the estimate

$$\|f - L(f)\| = \left\| \frac{1}{\pi} \int_{-\pi}^{\pi} \|\Delta_t f(x) K(t) dt \right\|_p \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \|\Delta_t f\|_p^p K(t)dt.$$

Other example is taken from [134]. If $f^{(r-1)}$ is absolutely continuous, then (we consider approximation in $L_{2,2\pi}$)

$$E_n(f) \leq \frac{1}{4n^{r-1}} \int_0^{\pi/n} \omega(f^{(r)}, t)dt.$$

In last years several papers have appeared concerning with weighted approximation. We do not want to present here all details. Assume that w

is a non negative function (a weight) on the interval I (finite or not) and consider the space $L_p(I, w)$ with the norm

$$\|f\|_{p,w} = \left(\int_I \mid f(t) \mid^p w(t)dt\right)^{1/p}.$$

The usual changes are taken in passing to continuous functions. If the weight w is chosen in such a way that polynomials are in $L_p(I, w)$ and they are dense, then a theory of approximation by polynomials can be developed in the new setting. Different ideas from the theory of orthogonal polynomials have been used and, in looking for direct and inverse results, the so called Ditzian-Totik modulus are relevant and Lipschitz spaces appear.

Problem 2.3.4 Obtain direct and inverse results for polynomial approximation in spaces $lip_\alpha(I, w)$ corresponding to a Ditzian-Totik modulus.

Here we do not want to present a long list of references. A good lecture for introducing in this topic is [93].

2.4 Approximation by linear operators

Let us recall that, for a function $f \in L_1[0, 2\pi]$, the Fourier coefficients are defined by

$$a_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\cos(nt)dt \quad \text{and} \quad b_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\sin(nt)dt,$$

$n \in \mathbb{Z}_+$ and (partial) Fourier sums are

$$S_n(f,t) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k(f)cos(kt) + b_k(f)sin(kt)) = \frac{1}{2\pi}\int_0^{2\pi} f(t)D_n(t)dt,$$

where

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} cos(kt) = \frac{\sin((n+1/2)t)}{\sin(t/2)} \tag{2.8}$$

is the so called Dirichlet kernel.

It is known that the Fourier sums of a functions converges to the function (in norm) in the spaces L_p, for $1 < p < \infty$. But this is not the case for continuous functions. Even more this kind of sums does not provide always the best rate of convergence. This explains why different summations methods have been introduced to analyze Fourier series. One of the most popular methods was considered by Fejér in 1904 [42]. The idea was to replace the Fourier sums by the arithmetic means of this sums. Féjer defined

$$\sigma_n(f,x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f,x).$$

The Fejér means of a continuous function f converges uniformly to f.

More general sums were considered by de la Vallée-Poussin. These new sums are defined by

$$V_{n,m}(f,x) = \frac{1}{m+1}\sum_{k=n-m}^{n} S_k(f,x), \quad 0 \le m \le n.$$

The notion of best approximation is very useful in analyzing the rate of convergence of other approximation processes. For instance, in the case of continuous functions one has

$$\mid f(x) - S_n(f,x) \mid \le (4+\log(n+1))E_n(f), \quad \text{(Lebesgue, [87])},$$

$$\mid f(x) - \sigma_n(f,x) \mid \le \frac{C}{n+1}\sum_{k=0}^{n} E_k(f), \quad \text{(Stechkin, [120])},$$

$$\mid f(x) - V_{n,m}(f,x) \mid \le \left(C + \frac{4}{\pi^2}\log\frac{n}{m+1}\right) E_{n-m}(f) \quad \text{(Nikolskii, [100])}, \tag{2.9}$$

and (Zakharov, [148])

$$\mid f(x) - V_{n,m}(f,x) \mid \le \frac{C}{m+1}\sum_{k=n-m}^{n} E_k(f)\left(1+\log\left(1+\frac{n-m}{k-n+m+1}\right)\right).$$

The last estimation also holds for L_p $(1 \le p < \infty)$ spaces. But for $1 < p < \infty$ a sharper inequality is known. That is

$$\|f - V_{n,m}(f)\|_p \le \frac{C_p}{m+1}\sum_{k=n-m}^{n} E_k(f)_p.$$

After the Fejér works different summation methods have been constructed. There are the so called Cèsaro means and general processes of the form

$$M_n(f,\lambda,t) = \frac{a_0(f)\lambda_{n,0}}{2} + \sum_{k=1}^{m(n)} \lambda_{n,k}[a_k(f)cos(kt) + b_k(f)sin(kt)] \tag{2.10}$$

where $\lambda = \{\lambda_{n,k}\}$ is a matrix and some restrictions are needed in order to obtain convergence. We write

$$M_n(f,x,\Lambda) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)M_n(t,\Lambda)dt,$$

where

$$M_n(t,\Lambda) = \frac{\lambda_0^n}{2} + \sum_{k=1}^{n} \lambda_k^n \cos(kt)$$

is the the so called kernel of the operator.

Let us see some examples.

1) When $\lambda_k^n = 1$ $(k = 0, \cdots, n)$ the method agrees with the partial sums of the Fourier series $S_n(f)$ and the kernel is the Dirichlet one (2.8).

2) If

$$\lambda_k^n = 1 - \frac{k}{n}, \qquad k = 0, \cdots, n-1,$$

we obtain the Fejér sums $F_{n,p}$.

3) If p is a non negative integer and

$$\lambda_k^n = \begin{cases} 1 & si \quad k = 0, 1, \cdots, n-p \\ 1 - \frac{k-n+p}{p+1} & si \quad k = n-p+1, \cdots, n, \end{cases}$$

we obtain the de la Vallé-Poussin operator. It can be written in the form

$$V_n(f,x,\Lambda)\frac{1}{p+1}\sum_{k=n-p}^{n} S_k(f,x).$$

4) If $\lambda_k^n = \cos(k\pi/(2n))$, $k = 0, \cdots, n-1$, then we obtain the Rogosinski operators,

$$R_n(f,x,\Lambda) = \frac{1}{2}\left[S_n\left(f, x + \frac{\pi}{2n}\right) + S_n\left(f, x - \frac{\pi}{2n}\right)\right].$$

5) If $s > 0$ the Zygmund operatorsZ_n^s are defined by considering $\lambda_k^n = 1 - (k/n)^s$, $k = 0, \cdots, n-1$.

6) For $\alpha > 0$ the Cèsaro means σ^α are defined by taking

$$\lambda_{n,k} = \frac{A_{n-k}^\alpha}{A_n^\alpha}, \quad \text{where} \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

7) For $\delta > 0$ and $r \in \mathbb{N}$ Riesz means $R_n^{\lambda,\delta}$ are defined by considering

$$\lambda_{n,k} = \varphi\left(\frac{k}{n+1}\right), \quad \varphi(t) = (1-t^r)^\delta.$$

It is known that (Timan, [140])

$$\|f - Z_n^r(f)\|_\infty \le \frac{C_r}{(n+1)^r}\sum_{k=1}^{n}(k+1)^{r-1}E_k(f)_\infty. \tag{2.11}$$

A similar relation holds for the Riesz means.

In [22] sufficient conditions are given in order that a sequence of operators $\{A_n\}$, $A_n : C_{2\pi} \to C_{2\pi}$ satisfies a relation of the type given in (2.11). In some cases the conditions are discovered by studying the saturation properties of the sequence $\{A_n\}$. In [20] necessary and sufficient conditions are given in order to an inequality of the form

$$\|f - A_n(f)\|_\infty \le C_r\omega_r(f, 1/n) \tag{2.12}$$

holds. The following problem is taken form [22].

Problem 2.4.1 Find sufficient and necessary conditions on a sequence $A_n : C_{2\pi} \to C_{2\pi}$ such that the Timan type inequality (2.11) holds.

We do not know wheter there are similar results for operators in L_p spaces.

Problem 2.4.2 How are the results for comparison of method (as the ones presented above) in the case of Hölder approximation?

Trigub proved that there exist positive constants C_1 and C_2 such that

$$C_1\|f - \sigma_n(f)\| \le \|f - \sigma_n^\alpha(f)\| \le C_2\|f - \sigma_n(f)\|.$$

When $\lambda_{n,0} = 1$ (we take $\lambda_{n,n+1} = 0$), it is proved that [148], for $f \in C_{2\pi}$,

$$| f(x) - M_n(f,\lambda,x) | \le C \mid \Delta\lambda_0 \mid \sum_{k=0}^{n} E_k(f)$$

$$+C\sum_{k=0}^{n} \mid \Delta^2\lambda_{k-1} \mid \sum_{i=k}^{n} E_i(f)\left(1 + \log\left(1 + \frac{k}{i-k+1}\right)\right).$$

Related results are given in [49] and [82]. For the Euler summation method see [77].

Some of the summation methods presented above have been analyzed in the context of Hölder spaces: [23] (Fejér operators), [106] (de la Valleé Poussin operators), [31], [59], [95], [107] (Euler, Borel and Taylor means), [29], [32] (Hardy-Littlewood series), [28] ((e, c)-type means) and other methods in [26], [60], [71], [94], [96], [24] and [25].

In [16] it was noticed that, in many cases, convergence in Hölder norms follows from convergence in uniform norm under additional assumptions. A similar assertion holds for L_p spaces. In particular, in [16] quantitative Korovkin type theorems were presented for convolution type operators. For quantitative results in Hölder metrics it is more convenient to use lipschitzian type moduli of smoothness as they were introduced in [15], [13] and [14].

A general approach to obtain quantitative results in Hölder norms was given in [72]. The fundamental result can be presented as follows. Let E be a real or complex linear space and

$$\theta : E \times I \to \mathbb{R}_+$$

a family of quasi-seminorms on E, where for each $f \in E$, $\theta(f, \cdot)$ is increasing. The term quasi-seminorm means that there exists a constant C such that, for $f, g \in E$ and $\delta \in I$,

$$\theta(f + g, \delta) \leq C(\theta(f, \delta) + \theta(g, \delta)). \tag{2.13}$$

For $f \in E$ set

$$\theta(f) = \sup\{\theta(f, \delta), \quad \delta \in I\}$$

and denote

$$F = \{f \in E : \theta(f) < \infty, \quad \theta(f, \delta) \to 0, \quad as \quad \delta \to 0\}.$$

In F we consider the norm

$$|||f|||_E = \|f\|_E + \theta(f).$$

Assume that F is a Banach space and there exists a function $\Psi : I \times \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $K > 0$ such that:

(i) For each $\delta \in I$

$$\lim_{t \to 0} \Psi(\delta, t) = \Psi(\delta, 0) = 0.$$

(ii) For each $\delta \in I$, $\Psi(\delta, \cdot)$ is continuous on $\mathbb{R}_+$.

(iii) For each $f \in F$,

$$\theta(f) \leq K\theta(f,\delta) + \Psi(\delta, \|f\|_E). \tag{2.14}$$

If $f_0 \in E$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence in F such that $f_n \to f$ (in E) and

$$\theta(\{f_n\},\delta) = \sup\{\theta(f_n,\delta) : n \in \mathbb{N}\} \to 0, \quad \delta \to 0, \tag{2.15}$$

then

$$\theta(f - f_n) \leq 2CK\theta(\{f_n\},\delta) + \Psi(\delta, \|f - f_n\|_E),$$

where C and K are given by (2.13) and (2.14) respectively.

In order to use this results one needs to construct the function Ψ and to prove inequalities similar to those given in (2.14) and (2.15). For instance, if $E = C[0,1]$ or $E = C(T)$, where $T = \{z \in \mathbb{C} :\mid z \mid= 1\}$, $\alpha \in (0,1)$ and

$$\theta(f,\delta) = \sup_{0<t\leq\delta} \frac{\omega(f,t)}{t^{\alpha}},$$

then $F = lip_{\alpha}$ and we can take

$$\Psi(t,\delta) = \frac{2t}{\delta^{\alpha}}.$$

Then, to obtain convergence in Hölder norm it sufficient to verify condition (2.15). This was accomplished in [72] for convolution type operators.

Let us present another approach taken from [14], where inverse results were also considered. For the reader convenience we only consider the case of approximation by linear operators and assume the notations of the previous section about Hölder abstract spaces.

Teorema 2.4.1 *Let E be a Banach space, W a subspace of E provided with a seminorm $| \circ |_W$, r a positive integer, ω a modulus of smoothness of order r on E, t_0 a positive number and L_t a family of functions $L_t : E \to W$ for which there exists a constant C such that, for all $f \in E$*

$$\|f - L_t f\|_E \leq C\omega(f,t) \quad \text{and} \quad t^r |L_t f|_W \leq C\omega(f,t). \tag{2.16}$$

Assume that, for each n, A_n is a linear subspace of E, $A_n \subset W$.

Let $\{H_n\}$ be a sequence of bounded linear operators, $H_n : E \to$, $H_n(E) \subset A_n$, for which there exists a constant D and a decreasing converging to zero sequence $\{\psi(n))\}$, such that, for each $f \in E$,

$$\|f - H_n f\| \leq D\omega\,(f,\psi(n)) \quad \text{and} \quad \omega(H_n f,t) \leq D\omega(f,t) \qquad (t>0), \tag{2.17}$$

then there exists a constant C_2 such that, for $h \in E^0_{\omega,\alpha}$ one has

$$\|h - F_n h\|_{\omega,\alpha} \leq D_1 \theta_{\omega,\alpha}\,(h,\psi(n))\,. \tag{2.18}$$

This is a very technical result, but it is presented in a such a way that several applications are possible. For classical spaces of continuous functions the modulus ω can be taken as the usual modulus of smoothness of order r and W as the space of all functions g that have continuous derivatives of order r, with the seminorm $\mid g \mid_W = \|g^{(r)}\|_\infty$. In such a case the existence of the functions L_t is known (these functions are used to characterize the modulus in terms of a K-functional). For L_p spaces an analogous remark holds.

For some summation methods (and other approximation processes) one of the conditions in (2.17) (or both) has not been verified. In such a case the problem below is interesting.

Problem 2.4.3 Study the convergence in Hölder norms for approximation processes where one of the conditions given in (2.17) has not been proved.

For instance, in [26], [29], [31] and [141] no one of the conditions were stated. In [56] the first condition was verified, but they consider that the spaces involved are not the same (this last restriction is also assumed in [108]). In [109] a more complicated inequality is used. Similar assertions hold for the results given in [129].

Some properties of Cèsaro can be found in [21]. There are sufficient conditions for the convergence of these means. Let us present one of them.

Fix $\alpha \in [0,1)$ and $\alpha < 1/p < \beta$. If $f \in Lip_{p,\beta}$, then its Fourier series is $(C,-\alpha)$-summable, [62]. Moreover if f is continuous, the result holds for $\beta = 1/p$, [144]. In [61] other sufficient conditions are given for the $(C,-\alpha)$-summability for continuous functions. The results are obtained using the so called maximal modulus of continuity introduced for Sevastyanov in [113]. It is defined as follows.

Let Γ be the family of all segments $\Delta = [a,b] \subset \mathbb{R}$, $\mid \Delta \mid = b-a$. We define

$$\hat{\omega}_1(f,t) = \sup_{\Delta\in\Gamma}\left\{ \sup_{|h|\le t} \frac{1}{\mid \Delta \mid} \int_\Delta \mid f(x+h\mid \Delta \mid) - f(x) \mid dx \right\}.$$

Some properties of this moduli are given in [61]. Here different problems can be sated. We only present two of the possible ones.

Problem 2.4.4 How the maximal modulus of continuity are related with the classical ones?

Problem 2.4.5 When the Cèsaro means are convergent, are they also convergent in the Hölder norms?

There are several papers devoted to describe classes of functions for which the Fourier series (or the Cèsaro sums) converges. It is interesting to know in this classes an strong form of convergence holds. This kind of problem has been considered for the notion of absolute convergence.

Let $\Psi : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function such that $\Psi(0) = 0$. A 2π-periodic function is said to be of first (second) generalized bounded variation if

$$V_\Psi^{(i)}(f) = \sup_a \sup_\Pi \sum_{k=0}^{n-1} \Psi(\mid \Delta^{(i)}(f, x_k, x_{k+1}) \mid) < \infty,$$

where the first supremun is considered over all segments $[a, a + 2\pi]$ and the second one over all partitions $\Pi = \{a = x_0 < x_1 < \cdots < x_n = a + 2\pi\}$. Moreover

$$\Delta^{(1)}(f, x_k, x_{k+1}) = f(x_{k+1}) - f(x_k)$$

and

$$\Delta^{(2)}(f, x_k, x_{k+1}) = \frac{1}{2}\left(f(x_{k+1}) + f(x_k) - 2f\left(\frac{x_{k+1} + x_k}{2}\right)\right).$$

Now we define

$$V_\Psi^{(i)} = \left\{f : V_\Psi^{(i)}(f) < \infty\right\}, \qquad i = 1, 2.$$

A 2π-periodic function F is in class $\Lambda_\Psi^{(i)}$, $i = 1, 2$, if

$$\int_0^{2\pi} \Psi(\mid \Delta_h^{(i)}(f, x) \mid) \leq Ch.$$

If $\Psi(u) = u^p$, then $\Lambda_\Psi^{(1)}$ agrees with the class Lip_1^p.

In [6] some sufficient conditions on Ψ are given in order to the Cèsaro means of any continuous functions $f \in \Lambda_\Psi^2$ converge to f. But the result is not given in a quantitative form.

For $r \in (0, 1)$ set

$$\left(\log \frac{1}{1-r}\right)^k = \sum_{k=0}^{\infty} L_n^{(k)} r^n.$$

The logarithmic means of f are defined by

$$L_r^{(k)}(f, x) = \left(\log \frac{1}{1-r}\right)^{-k} \sum_{n=k}^{\infty} L_n^{(k)} S_n(f, x) r^n.$$

Problems related with logarithmic means have not been studied in details. Mazhar [92] considered this mean in the case $r = 1$. In [114] the asymptotic expansion and saturations classes were analyzed.

Problem 2.4.6 To analyze logarithmic means in the context of Hölder spaces.

In some cases the estimations of convergence of summation methods are given in terms of a remainder and such a remainder is estimated in terms of the modulus of smoothness [104].

In [141] some other summation methods are presented.

2.5 Non periodic functions

The problems to be considered here are related with the previous section. The main results in [14], for Hölder approximation, deal with non periodic functions.

Some linear operators, as the Szász-Mirakyan ones, are not defined for all continuous (or integrable, if we consider the Kantarovich variant) functions. Then, weighted spaces are considered in order to enlarge the domain of the operators. For each case different moduli of smoothness are defined.

Problem 2.5.1 Study the properties of the weighted moduli of smoothness considered in problems with positive linear operators on unbounded intervals. It is not clear what is the best form to introduce these moduli. Obtain results for convergence in Hölder norms for linear positive operators in unbounded intervals.

Let us present some of the weighted moduli that have been used. Let $a > 0$ be a fixed number and set

$$\omega_a(x) = e^{a|x|}, \qquad x \in \mathbb{R}.$$

For a fixed $1 \leq p < \infty$ and $a > 0$ we denote by $L_{p,a}$ the space of all (classes) Lebesgue measurable functions f such that

$$\|f\|_{p,a} = \left(\int_{\mathbb{R}} \mid \omega_a(x) f(x) \mid^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, $L_{\infty,a}$ is the space of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $w_a f$ is uniformly continuous and bounded in $\mathbb{R}$, with the norm

$$\|f\|_{\infty,a} = \sup_{x \in \mathbb{R}} \mid \omega_a(x) f(x) \mid < \infty.$$

If r is a positive integer, Ω^r is the set of all functions $\omega : (0,\infty) \to (0,\infty)$ such that:

(a) ω is increasing and continuous in $(0,\infty)$ and $\omega(0) = 0$;

(b) $\omega(t)t^{-r}$ is increasing for $t \in (0,\infty)$.

For r, $\omega \in \Omega^r$, $1 \leq p \leq \infty$ and $a > 0$ as above, we denote by $H^{\omega,r}_{p,a}$ the family of all $f \in L_{p,a}$ for which

$$\mid f \mid_{p,a,\omega} = \sup_{0<h\leq 1} \frac{\|\Delta^r_h f\|_{p,a}}{\omega(h)} < \infty.$$

In this case the Hölder norm and the space $h^{\omega,r}_{p,a}$ are defined as usual.

Similar space can be constructed for the interval $[0,\infty)$. For $f \in C_a[0,\infty)$ and $n \in \mathbb{N}$, the Szász-Mirakyan operators operators are defined by

$$S_{n,a}(f,x) = \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n+a}\right),$$

where

$$p_k(t) = e^{-t}\frac{t^k}{k!}.$$

In [111] the Szász-Mirakyan operators were considered in Hölder weighted spaces but, in order to present results, the authors considered that $S_{n,a} : h^{\omega,2}_{p,a} \to h^{\mu,2}_{p,b}$, with $\omega, \mu \in \Omega^2$ and other additional restrictions. In particular, the first space can be compactly embedded in the second one.

We think that the results of [111] can be improved by removing condition (2) in the definition of class Ω^r. Moreover, with the ideas of Bustamante and Jiménez theses results can be improved by considering that operators act from one space into itself.

Problem 2.5.2 Verify that the results of [111] can be obtained without assuming condition (2) in the definition of class Ω^2.

It is asserted in [111] that similar results can be obtained for weighted spaces where, in place of the exponential weight $a(t) = e^{-at}$, we use a polynomial weight $b(t) = 1/(1+x^b)$ for some real $b \geq 1$.

Problem 2.5.3 Consider the problem 2.5.2, for polynomial weights.

Problem 2.5.4 Obtain inverse results for Szász-Mirakyan type operators.

2.6 Singular integrals

The Gauss- Weierstrass singular integral

$$W(f,x) = \frac{1}{\sqrt{\pi r}} \int_{-\pi}^{\pi} f(x+t) e^{-t^2/r} dt, \quad r \in (0,1),$$

for the case of periodic functions. The Picard singular integral is defined by

$$P_\xi(f,x) = \frac{1}{2\xi} \int_{-\infty}^{\infty} f(x+t) e^{-|t|/r} dt, \quad \xi > 0, \tag{2.19}$$

and the Poisson-Cauchy singular integral is defined by

$$Q_\xi(f,x) = \frac{\xi}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t)}{t^2+\xi^2} dt, \quad \xi > 0.$$

For $f \in C[0,1]$ or $f \in L_p[0,1]$ $(1 \le p < \infty)$ we also consider the Picard-type singular integral operator (see [48])

$$L_\xi(f,x) = \frac{1}{\xi} \int_{0}^{\infty} f\left(\frac{x}{e^t}\right) e^{-t/\xi} dx, \qquad x \in [0,1].$$

These operators have been studied in [3], [4], [12], [21] [33], [37], [44], [47], [48], [58], [75], [89] and [97].

In [33] and [97] it was proved that there exists a constant C such that

$$\|f - W_\xi(f)\|_\infty \le C \frac{1}{\sqrt{\xi}} \omega_1(f,\xi)_\infty,$$

Moreover, if $0 < \beta < \alpha \le 1$ and $f \in H_\alpha$, then $\|f - W_\xi(f)\|_\beta = O(\xi^{\alpha-\beta-1/2})$. In [44] these results were extended and improved:

$$\|f - W_\xi(f)\|_X \le \left(\frac{5}{4} + \frac{1}{\sqrt{\pi}}\right) \omega_2(f, \sqrt{\xi}) + \frac{1}{2\sqrt{\pi^3}} \xi^{3/2} \|f\|_X.$$

If $f \in H_\omega$, then

$$\|f - W_\xi(f)\|_X \le \left(\frac{5}{4} + \frac{1}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}\right) | \, f \, |_{H_\omega} \xi^{\alpha/2}.$$

The Gauss-Weierstrass singular integral for functions of two variable has been considered in [76].

In order to obtain convergence in the generalized Hölder spaces Firlej and Rempuslka need some additional restrictions. They consider two functions $\omega, \mu \in \Omega^2$ and assume that

$$q(t) = \frac{\omega(t)}{\mu(t)}, \qquad t > 0,$$

is not decreasing. With these conditions on hands they proved that there exists a constant C such that, for $f \in H_\omega$,

$$\|f - W_\xi(f)\|_{H_\mu} \leq C \mid f \mid_{H_\omega} q(\xi^{\alpha/2}).$$

Notice that this last result involved two different Hölder spaces.

In [58] similar results are presented with respect to the Picard singular integral. They proved that

$$\|f - P_\xi(f)\|_X \leq \frac{5}{2}\omega_2(f, \xi)$$

and, if $\omega, \mu \in \Omega^2$ are given as above, then

$$\|f - P_\xi(f)\|_{H_\mu} \leq C \mid f \mid_{H_\omega} q(\xi).$$

In [89] the Picard singular integral was considered in weighted spaces $L_{p,a}$ as they are defined in the previous section. If $\omega \in \Omega^2$, $1 \leq p \leq \infty$ and $a > 0$, we denote by $L_{p,a\omega}$ the family of all $f \in L_{p,a}$ for which

$$\mid f \mid_{p,a,\omega} = \sup_{0<h\leq 1} \frac{\|\Delta_h^2 f\|_{p,a}}{\omega(h)} < \infty.$$

In this space the Hölder norm is defined as usually.

In [4] properties of these operators are proved.

Denote

$$E(x) = \frac{2}{\pi} \int_0^x e^{-t^2} dt.$$

In particular in [4] can find the following estimation

$$\omega_r(P_\xi f, t)_p \leq \omega_r(f, t)_p, \qquad p = 1, \infty$$

and

$$\omega_r(P_\xi f, t)_p \leq \frac{2}{p^{1/p} q^{1/q}} \omega_r(f, t)_p, \qquad 1 < p < \infty,$$

where $(1/p)+(1/q)=1$. For the case of the Poisson-Cauchy one has

$$\omega_r(Q_\xi f,t)_\infty \le \left(\frac{2}{\pi}\tan^{-1}\frac{\pi}{\zeta}\right)\omega_r(f,t)_\infty,$$

and

$$\omega_r(Q_\xi f,t)_p \le \left(\frac{2}{\pi}\tan^{-1}\frac{\pi}{\zeta}\right)\omega_r(f,t)_p, \qquad 1\le p<\infty.$$

Finally, for the Gauss-Weierstrass singular integrals one has

$$\omega_r(W_\xi f,t)_\infty \le E\left(\frac{\pi}{\sqrt{\xi}}\right)\omega_r(f,t)_\infty,$$

and

$$\omega_r(Q_\xi f,t)_1 \le E\left(\frac{\pi}{\sqrt{\xi}}\right)\omega_r(f,t)_1.$$

For $1<p<\infty$ the estimation are much more complicated. For the case of continuous functions the inequalities are sharp.

For the Picard-type singular integral operator it can be proved that, $f\in C[0,1]$

$$\omega_r(L_\xi f,t)_\infty \le \omega_r(f,t)_\infty.$$

For $p=1$, $f\in L_1[0,1]$ and $\xi(0,1)$,

$$\omega_r(L_\xi f,t)_1 \le \frac{1}{1-\xi}\omega_r(f,t)_1.$$

For $1<p<\infty$, $f\in L_p[0,1]$ and $0<\xi<p/2$,

$$\omega_r(L_\xi f,t)_1 \le \left(\frac{2}{q^{1/q}(p-2\xi)^{1/p}}\right)\omega_r(f,t)_p.$$

In [58] an inverse result is considered for the Picard singular integral with respect to the approximation in the norm of the basic space. It is proved that, if $\omega\in\Omega^2$ and

$$\|f-P_\xi(f)\|_X \le \omega(\xi), \qquad \xi\in I,$$

then

$$\omega_2(f,t)_X \le Ct^2\int_t^1 \frac{\omega(s)}{s^3}ds, \qquad t\in(0,1/2).$$

Problem 2.6.1 It is necessary to analyze whether the Picard singular integral has the shape preserving property in the weighed spaces $L_{p,a,\omega}$.

Problem 2.6.2 The translation seems to be a bounded semigroup of operators on the spaces considered above. Thus the moduli can be characterized in terms of K-functionals as usually. This approach can be used to obtain direct and inverse results.

Problem 2.6.3 What kind of inverse results can be obtained for the case of Gauss-Weierstrass and Poisson-Cauchy singular integrals? For the case of the Picard singular integral an inverse result in weighted space was presented in [89]. To solve this problem it is necessary to obtain new Bernstein type inequalities.

Problem 2.6.4 The results related with the Poisson-Cauchy singular integrals presented in [14] and [89] are restricted to functions in class Ω^2. The problem is to remove the condition "$\omega(t)t^{-2}$ *is increasing*" in the definition of the class.

Problem 2.6.5 The results related with approximation in Hölder norms can be improved using the hölderian moduli of Bustamante and Jiménez.

Problem 2.6.6 For some of the operators the saturation classes have not been studied. The results of [21] should be considered here.

Problem 2.6.7 Obtain the results related with simultaneous approximation for the singular integrals considered above.

Problem 2.6.8 Study the singular integrals acting in other weighted spaces.

In [48] the following generalizations of the Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals were introduce: for $\xi > 0$

$$P_{n,\xi}(f,x) = -\frac{1}{2\xi}\sum_{k=1}^{n}(-1)^k\binom{n+1}{k}\int_{-\infty}^{\infty} f(x+kt)e^{-|t|/\xi}dt, \quad x \in \mathbb{R},$$

$$Q_{n,\xi}(f,x) = -\frac{1}{\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}}\sum_{k=1}^{n}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi} \frac{f(x+kt)}{t^2+\xi^2}dt, \quad x \in \mathbb{R},$$

and

$$W_{n,\xi}(f,x) = -\frac{1}{2C(\xi)}\sum_{k=1}^{n}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi} f(x+kt)e^{-t^2/\xi^2}dt, \quad x \in \mathbb{R},$$

where

$$C(\xi) = \int_0^{\pi} e^{-t^2/\xi^2}\,dt.$$

In [48] some estimation of the convergence in uniform norm of the operators $P_{n,\xi}$ and $W_{n,\xi}$ were give, in terms of the usual moduli of order $n+1$. Uniform convergence for the operator $Q_{n,\xi}$ was proved in [5] (the corresponding analysis for L_p spaces, $1 \leq p < \infty$, was also included, but these works do not include estimation in terms of moduli of smoothness).

Problem 2.6.9 To study the properties generalized Gauss-Weierstrass, Picard and Poisson-Cauchy operators in Hölder spaces.

Problem 2.6.10 In [4] some extensions of the singular integral where consider (no positive operators) and the shape preserving property (with respect to moduli of smoothness of different order) was obtained. The problem stated above should also be considered for these operators (including approximation in generalized Hölder spaces).

Problem 2.6.11 Obtain the rate of convergence of the generalized Abel-Poisson operators studied in [39]. We remark that for these operators the shape preserving property has not been proved.

2.7 Simultaneous approximation

In problems of simultaneous approximation we try to estimate the rate of convergence of some approximation of the derivatives of the functions. These problems are more interesting when the approximation of the derivatives are obtained as derivatives of the approximations of the original functions. These problems have been studied in different contexts ([126], [117]).

Problem 2.7.1 To obtain results related with simultaneous approximation in Hölder spaces (see [18]).

2.8 Extremal operators

The construction of extremal operators goes back to papers of Favard ([40] and [41]), who stated the following problem. Given a bounded set $K \subset C_{2\pi}$, try to find a matrix of coefficients $\{\lambda_{n,k}\}$ in such a way that the numbers

$$\sup_{f\in K} \|f - M_n(f, \lambda_{n,k})\|$$

be as small as possible, where the operator $M(f, \lambda_{n,k}, \cdot)$ is defined by (2.10).

For the case when $K = W^r$, Favard obtained the explicit solution. His results motivate a long list of works devoted to construct optimal methods for approximation using different summation processes.

Another type of extremal problems were stated by Kolmogorov [80]. The task is to compute (at least asymptotically) the number

$$d_n(K, X) = \inf_{U \in L_n} \sup_{f \in K} \inf_{u \in U} \|x - u\|,$$

where L_N is the collection of all n-dimensional subspaces of X and X is any one of the classical spaces. Kolmogorov solved the problem for the case $X = L_2$. There are some papers of Favard on this subject.

Some basic ideas are given in the paper of Kolmogorov [79], who found the asymptotic for the deviation of the Fourier sums in the class W^r of all functions for which $| f^{(r)}(x) | \leq 1$ almost everywhere. V. T. Pinkevich [105] considered the case when r is not an integer. Some important generalizations are due to Nikolskii ([102] and [103]. In particular he obtained the asymptotic for the classes $W^r H^\alpha$.

Extremal problems are no simple to solve. In general, the difficult part is to obtain the inverse results. For important ideas see the papers of Nagy [98] and Nikolskii [101]. Nikolskii showed that some problem of minimizing a functional in a normed space can be translated to a problem in the dual space. If X is a Banach space and $x_1, \cdots, x_n \in X$, then

$$\inf_{\alpha_k} \|x - \sum_{k=1}^{n} \alpha_k x_k\| = \sup\{\Lambda(x) : \Lambda \in B(X_1^*), \Lambda(x_k) = 0, k = 1, \cdots, n\}, \tag{2.20}$$

where X^* y $B(X_1^*) = \{\Lambda \in X^* : \|\Lambda\| \leq 1\}$ is the dual space.

Using this approach Korneichuk [83] found the general solution. In particular he presented an exact expression for the class of periodic functions

$$W^m H^\omega = \{f : f \in C^m, \omega(f^{(m)}, t) \leq \omega(t)\},$$

where $\omega(t)$ is a fixed concave majorant. He also considered Lebesgue spaces. For a review of the progress in studying this kind of problems see [84].

Problem 2.8.1 To analyze extremal problems for the case of approximation in Hölder norms of periodic functions.

The solution of Problem 2.2.1 may be useful here.

The problem can be analyzed in more general situations. For instance, in [121] functions of two variables are considered. For other paper on extremal problems see [11], [10], [123], [124] and [55].

In [112] constants related with de la Vallée Poussin were considered in terms of the moduli of continuity of f and f'. For other properties of these operators see [91] (asymptotic formulae), [19] and [7].

2.9 Spline approximation

There are only few results related with approximation by splines in Hölder norms. In [18] approximation by piece-wise linear functions was considered, but not direct or inverse result was given for the best approximation. Moreover the approximation in L_p was not studied. It is possible to extend the ideas of [68] and [43] to approximation in Hölder norms.

2.10 Saturation problems

The notion of saturation has been presented in different forms. For instance, in [150] we find the following definition.

Definition 2.10.1 *Fix $A \subset \overline{\mathbb{R}}$ and let α_0 be a limit point of the set A such that $\alpha_0 \notin A$. Let X be a non empty set and $P = \{p_\alpha\}_{\alpha\in A}$ a family of functionals form X into $\mathbb{R}_+$. The family P is said to be saturated as $\alpha \to \alpha_0$, if the following conditions hold:*

(1) If $\liminf_{\alpha\to\alpha_0} p_\alpha(x) = 0$ for some $x \in X$, then there exists a neighborhood V_{α_0} of α_0 such that $p_\alpha(x) = 0$, for $\alpha \in V_{\alpha_0} \cap A$;

(2) There exists an element $x_0 \in X$ such that

$$0 < \liminf_{\alpha\to\alpha_0} p_\alpha(x_0) \le \limsup_{\alpha\to\alpha_0} p_\alpha(x_0) < \infty.$$

The set $S(M) = \{x \in X : \limsup_{\alpha\to\alpha_0} p_\alpha(x) \le M\}$ is called the saturation (or Favard) class of the family P with constant M, as $\alpha \to \alpha_0$.

The set $\cup_{M=1}^{\infty} S(M)$ is called the saturation class of P, as $\alpha \to \alpha_0$.

Definition 2.10.2 ([130], [21], [78]). *Let $\Lambda = \{\lambda_{n,k}\}$ be a triangular matrix and $X = C_{2\pi}$ or $X = L_p[0, 2\pi]$, $1 \le p < \infty$. Let $U_n(\lambda)$ be a summation method for a series*

$$S(f,x) = \sum_{k=-\infty}^{\infty} c_k(f)e^{ikx}, \quad c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt,$$

in the form

$$U_n(f,x,\Lambda) = \sum_{k=-1-n}^{n+1} \lambda_{n,|k|} c_k(f) e^{ikx}.$$

Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ *be a decreasing function such that* $\lim_{n\to\infty} \varphi(n) = 0$.

Assume that: (1) *the relation* $\|f - U_n(f,\cdot,\Lambda)\| = o(\varphi(n))$ *implies that* f *is* (*a.e.*) *a constant,* (2) *there exists at least a non constant function* $f \in X$ *such that*

$$\|f - U_n(f,\cdot,\Lambda)\| = O(\varphi(n)), \tag{2.21}$$

then we say that the method $U_n(\Lambda)$ *is saturated in* X. *The set of all functions* g *such that* $\|f - U_n(f,\cdot,\Lambda)\| = O(\varphi(n))$ *is called the e saturation class of the method* $U_n(\Lambda)$ *and the function* φ *is called the order of saturation.*

The saturation class is denoted by (X, Λ)

The saturation properties of a great variety of approximation processes can be found in [21]. In [130], [21] and [78]) some sufficient conditions to localize saturation classes are given.

Let $\{\psi(n)\}$ be a sequence of natural numbers. If $f \in L_1$ and the series

$$\sum_{k=-\infty}^{\infty} \frac{1}{\psi(|\, k\,|)} c_k(f) e^{i(kx+\beta(\pi/2)sign(k)}$$

corresponds to the Fourier series of a function in X, this function is denoted by f_β^ψ and is called the (ψ,β) derivative of f.

let us consider some notations:

$$X_0^\psi L_\infty = \{f : f \in C, f_0^\psi \in L_\infty\}$$

$$S_0^\psi = \left\{ f : f \in L, \frac{1}{\psi(|\, k\,|)} c_k(f) = c_k(dg), g \in V \right\}$$

,

$$L_0^\psi L_p = \{f : f \in L_p, f_0^\psi \in L_p\}.$$

The family of all $f \in C_0^\psi L_\infty$ for which $\|f_0^\psi\| \le 1$ is denoted by $C_{0,\infty}^\psi$.

If for a triangular matrix $\Lambda = \{\lambda_{n,k}\}$ there exists a decreasing a converging to zero function $\varphi(n)$ and positive numbers $\psi(k)$, such that

$$\lim_{n\to\infty} \frac{1-\lambda_{n,k}}{\varphi(n)} = \frac{1}{\psi(k)}, \qquad k \in \mathbb{N}, \tag{2.22}$$

then the for method $U_n(\Lambda)$ corresponding to such a matrix, from the relation

$$\|f - U_n(f,\Lambda)\| = o(\varphi(n)), \tag{2.23}$$

it follows that f is (a.e.) constant.

If (2.21) holds for $f \in$, then $f \in X_0^{\varphi}$. That is

$$F(X, \Lambda) \subset X_0^{\varphi}.$$

Under some additional conditions the above relation may be an equality. A similar result can be found in [50], without the assumption on the norms of operators are uniformly bounded. There is in [51] a survey of some works devoted to saturation problems of sequence of operator given by linear methods, with a special attention when it are defined by a matrix.

Problem 2.10.1 Find the analogous of the above result in Hölder spaces.

We remark that the saturation problem in Hölder spaces cannot be solved directly from the corresponding result in classical spaces. In fact, we cannot know whether a function $f \in X$, for which (2.21) holds is the Hölder class.

Problem 2.10.2 Some of the problems presented above can be study in the more general context of orthogonal systems. There is a long list of references on this subject. Here we only recall one of them [63].

2.11 Properties of functions and approximation

An old interesting problem in approximation theory is to find connections between the rate of convergence of the approximations and some properties of the function to be approximated.

For instance, for the case of continuous periodic functions and trigonometric approximation some connections have been established between the best approximation $E_n(f)$ and the properties of (ψ, β)-derivative f_β^ψ ([118], [81] [125] y [127]).

In this section we include the problem of constructive characteristic [135]. For a non empty set $K \subset L_2$ the Kolmogorov number is defined by

$$d_n(K) = \inf_{L \in L_n} \sup_{f \in K} \inf_{\varphi \in L} \|f - \varphi\|,$$

where L_n is the family of all subspace of dimension no greater than n.

In [9] all functions which can be the first order modulus of continuity of a function in L_2 are described. In [135] an analogous result is given and the Kolmogorov numbers are computed for some classes of functions. For the analysis of Lebesgue numbers in Hölder spaces we do not have good references.

Problem 2.11.1 Let r be a positive integer and $w : [0, 2\pi] \to \mathbb{R}$ a continuous increasing function, $w(0) = 0$. When does there exists $f \in lip_{r,\alpha}[0, 2\pi]$ such that $\theta_{r,\alpha}(f, t) = w(t)$?

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx)]b_k \sin(kx)$$

be the Fourier series of a function $f \in L_1$, ¿how can be estimated $\omega_r(f, t)$ in terms of the coefficients a_k and b_k (upper and lower estimation)?

An inequality of the type

$$| a_k |, | b_k | \leq C_r \omega_r \left(f, \frac{1}{n}\right),$$

was known for Lebesgue for $r = 1$. For $r > 1$ Szász obtained an estimation using arguments very similar to the ones considered by Lebesgue [131]. Improved inequalities are given in [1] and [136].

Other type of results have been given using the best approximation (see [53] y [137]). It is clear that some restrictions are needed in the problem. In general we don't know if the series converges, thus some criteria of convergence may be useful in this setting [2]. It is proved in [38] that if the coefficients a_k satisfy $| a_k | \leq k^{-\alpha}$ and the sequence is convex, then $f \in Lip_{1,\alpha}$. Moreover, if the sequence a_k is monotone, then

$$\omega_1 (f, \delta)_1 \leq C_\alpha \delta^\alpha \log(1/\delta), \quad \delta \in (0, 1/2).$$

Reference [137] contains a survey on this kind of problem. Other ideas can be found in [88] and [138].

2.12 Lipschitz-Nikolskii constants

In some cases the error is studied not with o a single function, but considering a class of functions.

For instance, let D be a class of functions and A a non empty subset of D. If $\{L_n\}$ is a sequence of operators defined on D and $x \in D$, the error on A at the point x is defined by

$$E(A, L_n, x) = \sup_{f \in A} | f(x) - L_n(f, x) | .$$

If there exists a sequence of positive numbers $\{\psi_n\}$, converging to 0, such that

$$\lim_{n\to\infty} \frac{1}{\psi_n} E(A, L_n, x) = C(A, x),$$

the number $C(A, x)$ is called the Nikolskii constant, corresponding to the order of convergence ψ_n, for the class A relative to the operators L_n. When A is a class of Lipschitz functions, $C(A)$ is called the Lipschitz-Nikolskii constant.

These constants have been studied for many operators (Fejér, Jackson, Abel-Poisson, Bernstein, Szász, Gamma de Müller [110], [36] and [57]).

The Lispchitz-Nikolskii constans have not been studied for Lipschitz norms.

2.13 References

1. S. Aljancić and M. Tomić, *Sur la borne inférieure du module de continuité de la fonction exprimée par les coefficients de Fourier*, Bull. Acad. Serbe Sci. Arts., XL 6, (1967) 39-51.

2. S. Aljancić, *Sur le module de continuité des séries de Fourier particuliéres et sur le module de continuité des series de Fourier transformées par des multiplicateurs de types divers*, Bull. Acad. Serbe Sci. Arts., XL 6, (1967) 13-38.

3. G. A. Anastassiou, *Global smoothness preservation by singular integrals*, Proyecciones Rev. Math. 14, (1995) 83-88.

4. G. A. Anastassiou y S. G. Gal, *General theory of global smoothness preservation by singular integrals, univariate case*, J. Compu. Ana. Appl. 1 (3), (1999) 289-317.

5. G. A. Anastassiou y S. G. Gal, *Convergence of generalized singular integrals to the unit: univariate case*, Math. Ineq. Appl. 3 (4), (2000) 511-518.

6. T. I. Ajobatse, *Function classes and convergence of Fourier series*, Acta Math. Acad. Sci. Hungar. 37 (1-3), (1981) 95-119 (in russian).

7. V. A. Baskakov, *Asymptotic of the approximation of continuous and differentiable functions by the singular integrals of de la Vallée Poussin*, Math. Notes 21, (1977), 433-437.

8. N. I. Belaya, *An algorithm to construct an optimal-accuracy derivative in the class* $C_{2,L,N}$, Izv. Vyssh. Uchebn. Zaved., Mat. 8, (1978) 31-40.

9. O.V. Besov and S. B. Stechkin, *A description of moduli of continuity in* L_2, Tr. Mat. Inst. Akad. Nauk SSSR 134, (1975) 23-25.

10. A.S. Bezlyudnyi, *The approximation of periodic functions of two variables by interpolation trigonometric polynomials*, Dold. Akad. Nauk SSSR 65 (3) (1949).

11. P.T. Bugaets, *An asymptotic estimate the remainder in the approximation of functions of two variables by Fourier sums*, Dokl. Akad. Nauk SSSR 79 (4)(1951).

12. J. Bustamante, *Rate of convergence of singular integrals on Hölder norms*, Math. Nachr. 217, (2000) 5-11.

13. J. Bustamante and C. Castañeda R., *The best approximation in Hölder norm*, Extracta Math. 15 (3), (2000) 563-566.

14. J. Bustamante and C. Castañeda R., *The best approximation in Hölder norm*, J. Approx. Theory 138, (2006) 112-123.

15. J. Bustamante and M. A. Jiménez, *The degree of best approximation in the Lipschitz norm by trigonometric polynomials*, Aportaciones Matemáticas 25, (1999) 23-30.

16. J. Bustamante and M. A. Jiménez, *From Chebyshev to Hölder approximation*, Aportaciones Matemáticas 27, (2000) 23-31.

17. J. Bustamante and M. A. Jiménez, *Trends in Hölder approximation*, Approximation, Optimization and Mathematical Economics, Marc Lassonde Ed., Physica-Verlag, 2001, 81-95.

18. J. Bustamante, D. Mocencahua and Carlos A. López, *Smoothness of the remainder and approximation in Hölder norms*, Aportaciones Matemáticas 27, (2000) 33-45.

19. P. L. Butzer, *On the singular integral of de la Vallée Poussin*, Arch. Math. 7, (1956) 295-309.

20. P. L.Butzer y J. Korevaak, *On Approximation Theory*, Proceedings of the Conference 1963, Birkhauser Verlag, 1964.

21. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation* I, Birkhauser, Basel and Academic Press, New York, 1971.

22. J.-D. Cao, *Stechkin inequalities for summability methods*, Internat. J. Math. Math. Sci., 20, (1997) 93-100

23. P. Chandra, *On the generalized Féjer means in the metric of Hölder spaces*, Math. Nachr. 109, (1982) 39-45.

24. P. Chandra, *Degree of approximation of functions in the Hölder metric*, J. Indian Math. Soc. 53, (1988) 99-114.

25. P. Chandra, *Degree of approximation of functions in the Hölder metric by Borel's means*, J. Math. Anal. and Applic. 149, (1990) 236-248.

26. P. Chandra, *A note on the degree of approximation of continuous functions*, Acta Math. Hung. 62 (1-2), (1993) 2-23.

27. W. Dahmen y E. Görlich, *Best approximation with exponential orders and intermediate spaces*, Math. Z. 148, (1976) 7-21.

28. G. Das, T. Ghosh y B. K. Ray, *Degree of approximation of functions in the Hölder metric by (e, c) means*, Proc. Indian Acad. Sci. (Math. Sci.) 105 (3), (1995) 315-327.

29. G. Das, A. K. Ojha y B. K. Ray, *Degree of approximation of functions associates with Hardy-Littlewood series in Hölder metric by Euler means*, Proc. Indian Acad. Sci. (Math. Sci.), 106 (3), (1996) 227-243.

30. G. Das, T. Ghosh y B. K. Ray, *Degree of approximation of functions by their Fourier series in the generalized Hölder metric*, Proc. Indian Acad. Sci. (Math. Sci.), 106 (2), (1996) 139-153.

31. G. Das, A. K. Ojha y B. K. Ray, *Degree of approximation of functions in the Hölder metric by Borel's means*, Proc. Indian Acad. Sci. (Math. Sci.) 107 (2), (1997) 169-182.

32. G. Das, A. K. Ojha y B. K. Ray, *Degree of approximation of functions associates with Hardy-Littlewood series in the generalized Hölder metric*, Proc. Indian Acad. Sci. (Math. Sci.), 108 (2), (1998) 109-120.

33. E. Deeba, R. N. Mohapatra and R. S. Rodriguez, *On the degree of approximation of some singular integrals*, Rediconti di Mat. 8, (1988) 345-355.

34. S. Delpech, *Modulus of continuity of the Mazur map between unit ball of Orlicz spaces and approximation by Hölder mappings*, Illinois Journal of Mathematics 49(1), (2005), 195-216.

35. Yu. K. Demyanovich, *Estimate of the rate of convergence of best approximation by local functions*, Zapis. Nauch., 1980, 202-215

36. R. DeVore, *Approximation of Continuous Functions by Positive Linear Operators*, Springer Lec. Notes in Math. 293, Spinger-Verlag, 1972.

37. Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, New York, 1987.

38. M. 1. Dyachenko, *On local properties of functions*, Usp. Mat. Nauk, 36 (1), (1981) 205-206.

39. L. P. Falaleev, *Approximation of functions by generalized Abel-Poisson operators*, Sibir, Math. J. 42 (4), (2001) 926-936.

40. J. Favard, *Sur l'approximation des fonctions périodiques par des polynômes trigonométriques*, C. R. Acad. Sci. 203, (1936) 1122-1124.

41. J. Favard, *Sur les meilleurs procèdes d'approximation des certaines classes des fonctions par des polynômes trigonometriques*, Bull. Sci. Math. 2 (60), (1937) 209-224, 243-256.

42. L. Fejér, *Untersuchungen Über Fouriersche Reihen*, Math. Ann. 58, (1904) 501-569 .

43. Z. Finta, *Uniform approximation by means of some piecewise linear functions*, Mathematical Notes Miskolc 3 (2), (2002), 101-112.

44. B. Firlej and L. Rempulska, *Approximation theorems for the Gauss - Weierstrass singular integral*, Rendiconti di Matematica VII (15), Rom (1995) 339-349.

45. T. M. Flett, *Temperatures, Bessel potentials and Lipschitz spaces*, Proc. London Math. Soc. 22, (1971) 385-451.

46. G. Freud, *Eine Ungleichung für Tschebyscheffsche Approximationspolynome*, Acta Sci. Math. Szeged 19, (1958) 162-164.

47. S. G. Gal, *Remark on the degree of approximation of continuous functions by singular integrals*, Math. Nachr. 164, (1993) 197-199.

48. S. G. Gal, *Degree of approximation of continuous functions by some singular integrals*, Rev. Anal. Numér. Théor. Approx. XXVII (2), (1998), 251-261.

49. V. T. Gavrilyuk, *Linear methods of summation of Fourier series and the best approximation*, Ukranian. Matem, J. 15 (4), (1963) 412-418.

50. V. T. Gavrilyuk, *Saturation classes of linear method of summing Fourier series*, Ukranian Math. J., (1988) 486-492.

51. V. T. Gavrilyuk y A. I. Stepantes, *Problem of saturation of linear methods*, Ukranian Math. J., (1988) 255-272.

52. V. E. Geit, *On the exactness of certain inequalities in approximation theory*, Ukranian Math. J., (1971) 768-776.

53. V. E. Geit, *On structural and constructive properties of sine and cosine series with monotone sequence of Fourier coefficients*, Izv. Vyssh. Uchebn. Zaved., Mat. 7, (1969) 39-47.

54. S. K. Girlin, *On optimal accuracy interpolation and minimization of functions of the class* $C_{2,L_1,L_2,\cdots,L_m}$, Izv. Vyssh. Uchebn. Zaved., Mat. 10, (1978) 95-98.

55. M.M. Gorbach, *On the approximation of periodic functions of two variables by Fourier sums*, Dokl. Akad. Nauk UkrRSR, No. 8 (1960).

56. M. Górzeńska and L. Rempulska, *On the limit properties of the Picard singular integral*, Publ. L'Ins. Math. N. Serie 55 (69), (1994), 39-46.

57. E. Görlich y E. L. Stark, *Über beste Konstanten und asymptotische Entwicklungen positiver Faltungsintegrale und deren Zusammmenhang mit dem Saturationsproblem*, J-ber. Deustsch Math.-Verein 72, (1970) 18-61.

58. M. Górzeńska and L. Rempulska, *On the limits properties of the Picard singular integral*, Pub. Inst. Math., N. Serie 55 (69), (1994) 39-46.

59. M. Górzeńska, M. Leśniewicz and L. Rempulska, *On approximation of* 2π*-periodic functions belonging to the Hölder classes*, Fasc. Math. 21, (1990) 87-97.

60. M. Górzeńska, M. Leśniewicz and L. Rempulska, *Approximation theorems for functions of Hölder classes*, Ann. Soc. Math. Pol., ser. I, Commentat. Math. 30 (2), (1991) 301-308.

61. A. I. Grigorev, *Convergence of the Cesaro means of trigonometric Fourier series*, Math. Notes, (1983) 740-747.

62. G. H. Hardy and J. E. Littlewood, *A convergence criterion for Fourier series*, Math. Z. 28, (1928) 612-634.

63. M. Hegedüs y G. Alexits, *Über dire verallgemeinerte de la Valleé - Poussinsche summiertbarkeit allgemeiner orthogonalreihen*, Acta Math. Acad. Sci. Hungar. 20 (3-4), (1969) 281-287.

64. C. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech. 18, (1968) 283-323.

65. J. B. Hiriart-Urruty. *Emsembles de Tchebychev vs emsembles convexes: l'eétat de la situation vu via l'analyse convexe non Lisse*, Ann. Sci. Math. Qu'bec 22 (1), (1998) 47-62.

66. R. Holmes, B. Kripke, *Smoothness of approximation*, Mich. Math. J. 15, (1968) 225-248.

67. V. I. Ivanov, *Direct and converse theorems of approximation theory in the metric of L^p for* $0 < p < 1$, Math. Notes 18 (5), 972-982. FALTA ano

68. V. I. Ivanov, *Approximation in L_p by means of piecewise constant functions*, Math. Notes 44 (1), (1988) 523-532.

69. V. I. Ivanov y S. A. Pichugov, *Approximation of periodic functions in L_p by linear positive methods and multiple modulus of continuity*, Math. Notes. (1987) 925-930.

70. T. M. Jenkins, *Banach spaces of Lipschitz functions on an abstract metric space*, Ph. D. Thesis, Yale Univ., New Have, Conn., 1967.

71. Y. Jiang, *Approximation by a kind of trigonometric polynomials in the Hölder metric*, J. Beijing Norm. Univ. Nat. Sci. 29, (1993), 50-54.

72. M. A. Jiménez, *A tauberian theorem for a class of function spaces*, Monografías de la Academia de Ciencias de Zaragoza 20, (2002) 77-85.

73. M. I. Karlov and I. G. Tsarkov, *Convexity and connectedness of Chebyshev sets and sun*, Fundament. Priklad. Mat. 3 (4), (1997) 967-978 (in russian).

74. P. D.Kathal, A. S. B.Holland y B. N. Sahney, *A class of continuous functions and their degree of approximation*, Acta Math. Acad. Sci. Hungar. 30 (3-4), (1977) 227-231.

75. A. Khan, *On the degree of approximation of K. Picard and E. Poisson - Cauchy singular integrals*, Rendiconti di Math. 2, (1982) 123-128.

76. H. H. Khan and G. Ram, *On the degree of approximation by Gauus Weierstrass integrals*, Internat. J. Math. Math. Sci. Vol. 23 (9), (2000) 645-649.

77. H. H. Khan and G. Ram, *On the degree of approximation*, Facta Univ. (Nis) Ser. Math. Inform. 18, (2003) 47-57.

78. F. I. Kharshiladze, *Saturation classes for certain summation processes*, Dokl. Akad. Nauk SSSR 122(3), 352-355 (1957).

79. A.N. Kolmogorov, *Zur Grussenordnung des Restgliedes Fourierseher Reihen differenzierbarer funktionen*, Ann. Math. 36, (2) (1935).

80. A. N. Kolmogoroff, *Uber die besste Annihrung von Funktionen einer gegebenen Funktionklassen*, Ann. Math. 37, (1936) 107-110.

81. A.A. Konyushkov, *Best approximations by trigonometric polynomials and the Fourier coefficients*, Mat. Sb. 44 (1), (1958) 53-84.

82. R. N. Kovalchuk, *A problem of S. B. Stechkin*, in Questions in Mathematical Physics and the Theory of Functions 1, Kiev (1964), 51-55.

83. N. P. Korneichuk, *Extremal values of functionals and the best approximation on classes of periodic functions*, Izv. Akad. Nauk SSSR, Ser. Mat. 35 (1), (1971) 93-124.

84. N. Korneichuk, *Exact Constants in Approximation Theory*, Cambridge University Press, 1991.

85. K. Przeslawski and D. Yost, *Continuity properties of selectors and Michael's theorem*, Michigan Math. J. 36, (1989) 113-134.

86. J. La Vallé-Poussin, *Ch. Lecons sur l'Approximation des Fonctions d'une Variable Rélle*, Gautier-Villars, Paris (1919).

87. H. Lebesgue, *Sur les intégrales singulières*, Ann. de Toulouse, 1, (1909) 25-117.

88. L. Leindler, *Strong approximation and generalized Lipschitz classes*, in: Functional Analysis and Approximation (Oberwolfach, 1980), Birkhäuser, Basel, (1981) 343-350.

89. A. Leśniewicz, L. Rempulska and J. Wasiak, *Approximation properties of the Picard singular integral in exponential weighted spaces*, Publicacions Matematiques (40), (1996), 233-242.

90. K. de Leeuw, *Banach spaces of Lipschitz functions*, Studia Math. 21, (1961) 55-66.

91. Y. Matsuoka, *Asymptotic formula for Vallée Poussin's singular integrals*, Se. Rep. Kgoshima Univ. 9, (1960) 25-34.

92. M. Mazhar, *Approximation by logarithmic means of a Fourier series*, in Approximation Theory VI, Vol. 2, C. K. Chui, L. L. Schumaker and J. D. Ward (eds.) Academic Press (1989), 421-424.

93. nH. N. Mhaskar, *Introduction to the Theory of Weighted Polynomials Approximation*, World Scientific Pub., 1996.

94. R. N. Mohapatra, *Degree of approximation of Höder continuous functions*, Math. Machr. 140, (1989) 91-96.

95. R. N. Mohapatra and P. Chandra, *Hölder continuous functions and their Euler, Borel and Taylor means*, Math. Chronicle. 11, (1982) 81-96.

96. R. N. Mohapatra and P. Chandra, *Degree of approximation of functions in the Hölder metric*, Acta Math. Hung. 41 (1-2), (1983) 67-76.

97. R. N. Mohapatra and R. S. Rodríguez, *On the rate convergence of singular integrals for Hölder continuous functions*, Math. Nachr. 149, (1990) 117-124.

98. B. Sz. Nagy, *Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen, i. Periodischer Fall*, Ber. Math.-Phys. Acad. d. Wiss., 90, (1938) 103-134.

99. D. J. Newman and H. S. Shapiro, *Some theorems on Ĉtebysev approximation*, Duke Math. J. 30, (1963) 673-681.

100. S. M. Nikolskii, *Some methods of approximation by means of trigonometrical sums*, Izv. Akad. Nauk SSSR, Ser. matem. 4, (1940) 509-520.

101. S. M. Nikolskii, *The approximation of functions by trigonometric polynomials in the mean*, Izv. Akad. Nauk SSSR, Ser. Mat. 10 (3), (1946) 207-256.

102. S.M. Nikolskii, *On some approximation methods by trigonometric sums*, Izv. Akad. Nauk SSSR, Ser. Matem. 4 (6), (1940).

103. S.M. Nikolskii, *The approximation of periodic functions by trigonometric polynomials*, Trudy Matem. Inst. V. A. Steklov, AN SSSR 15 (1945).

104. Yu. L. Nosenko, *Approximation of functions by some means of their Fourier series*, Facta Univ. (Nis) Ser. Math. Inform. 13, (1998) 73-77.

105. V.T. Pinkevich, *On the order of the remainder of the Fourier series of functions which are differentiable in the Weyl sense*, Izv. Akad. Nauk SSSR, Ser. Matem. 4 (3), (1940).

106. J. Prestin, *On the approximation by de la Valleé Poussin sums and interpolatory polynomials in Lipschitz norms*, Analysis Math. 13, (1987) 251-259.

107. J. Prestin, *Trigonometric approximation by Euler means*, Constructive theory of functions, Proc. Int. Conf., Verna-Bulg., (1988) 382-389.

108. P. Pych-Taberska, *Properties of some bivariate approximants*, Collect. Math. 44, (1993) 237-246.

109. P. Pych-Taberska, *On the Durrmeyer-type modification on some discrete approximation operators*, Georgian Math. J. 1 (5), (1994) 523-536.

110. R. K. S. Rathore, *Lipschitz-Nikolskǐ constant for the gamma operator of Müller*, Math. Z. 141, (1975) 193-198

111. L. Rempulska and Z. Walczak, *On some properties of Szasz-Mirakyan operators in Hölder spaces*, Turk. J. Math. 27, (2003) 435-446.

112. F. Schurer and F. W. Steutel, *On the degree of approximation by the operators of de la Valeé Poussin*, Mh. Nath., 87, 1979, 53-64 (archivo: schurer-steul-valle poussin).

113. E. A. Sevastyanov, *On the uniform approximation of functions by Fourier sums*, Mat. Sb., 114, No. 4, 583-610 (1981).

114. X. Shi and Q. Sun, *Approximation of continuous periodic functions by logarithmic means of Fourier series*, Acta Math. Hungar. 63 (2), (1994) 103-112.

115. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, Berlin, 1970.

116. P. M. Soardi, *On nonisotropic Lipschitz spaces*, Lect. Notes Math. 992, Berlin-Heidelberg-New York, Springer, (1983) 115–138.

117. N. N. Sorich, *Joint approximation of periodic functions and their derivatives by the Fourier and Valle Poussin sums in the metric of L*, Ukranian Math. J., (1985) 174-179.

118. S.B. Stechkin, *On the order of best approximations of continuous functions*, Izv. Akad. Nauk SSSR, Ser. Mat. 15 (2), (1951) 219-242.

119. S.B. Stechkin, *On absolute convergence of Fourier series*, (Second report), Izv. Akad. Nauk SSSR, Ser. Matem. 19 (4), (1955) 221-246.

120. S. B. Stechkin, *The approximation of periodic functions by Fejér sums*, Tr. Matem, Inst. AN SSSR 62, (1961) 48-60.

121. A. I. Stepanets, *The approximation of certain classes of differentiable periodic functions of two variable by Fourier sums*, Ukranian Math. J., (1973) 498-506

122. A.I. Stepanets, *Uniform approximations by trigonometric polynomials*, Naukova Dumka, Kiev (1981).

123. A.I. Stepanets, *On a problem of A. N. Kolmogorov in the case of functions of two variables*, Ukr. Matem. Zh. 24 (4) (1972).

124. A.I. Stepanets, *The approximation of functions satisfying the Lipschitz condition by Fourier sums*, Ukr. Matem. Zh. 24 (6) (1972).

125. A.I. Stepanets, *Classification of periodic functions and the rate of convergence of their Fourier series*, Izv. Akad. Nauk SSSR, Ser. Mat. 50 (1), 1986 101-136.

126. A.I. Stepanets and N. N. Sorich, *Joint approximation of periodic functions and their derivatives by Fourier sums*, Mat. Zametki 36 (6), (1984) 873-882.

127. A. I. Stepanets and E. I. Zhukina, *Inverse theorems for the approximation of (ψ, β)-differentiable functions*, Ukranian Math. J.. (1989) 953-959.

128. E. A. Storozhenko, V. G. Krotov and P. Osval'd, *Direct and converse theorems of Jackson's type in spaces L^p, $0 < p < 1$*, Mat. Sb. 98 (3), (1975) 395-415.

129. X-.H. Sun, *Degree of approximation of functions in the generalized Hölder metric*, Indian J. Pure Appl. Math 27 (4), (1996) 407-417.

130. G. Sunouchi, *Characterization of certain classes of functions*, Tôhoku Math. J. 14 (I), (1962) 127-134.

131. O. Szasz, *Fourier series and mean moduli of continuity*, Trans. Amer. Math. Soc. 42, (1937) 366-395.

132. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-spaces I*, J. Math. Mech. 13, (1964) 4074-4080.

133. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-spaces II*, J. Math. Mech. 14, (1965) 821-839.

134. L. V. Taikov, *Best approximation of differentiable functions in the metric of L_2*, Math. Notes, (1977).

135. L. V. Taikov, *Structural and constructive characteristic of functions in L_2*, Math. Notes, (1979) 113-116.

136. S.A. Telyakovskii, *Lower bounds for the integral modulus of continuity of a function in terms of its Fourier coefficients*, Mat. Zametki 52 (5), (1992)107-112.

137. S. A. Telyakovskii, *Estimation of the moduli of continuity of one-variable functions in the metric of L in terms of Fourier coefficients*, Ukrain. Math. J. 46 (5), (1994) 671-678.

138. S. Yu. Tikhonov, *Generalized Lipschitz classes and Fourier coefficients*, Math. Notes 75 (6), (2004) 885-889.

139. A. F. Timan, *Theory of approximation of functions of a real variable*, MacMillan, New York, 1963.

140. M. F. Timan, *Best approximation of functions and linear methods of summability of Fourier series*, Izv. Akad Nauk. SSSR, Ser. Mat. 29, (1965) 587-604.

141. G. Tkebuchava, *Logarithm summability of Fourier series*, Acta Math. Aca. Paedagogicae Nyíregyháziensis 21, (2005) 161-167.

142. R. M. Trigub, *Linear methods of summability and absolute convergence of Fourier series*, Izv. Akad Nauk. SSSR. Ser. Matem. 32, (1968) 24-49.

143. N. Weaver, *Lipschitz algebras*, World Scientific, 1999.

144. K. Yano, *On Hardy and Littlewood's theorem*, Proc. Japan. Acad. 38 (2), (1957) 73-74.

145. D. Yost, *There can be no Lipschitz version of Michael's selection theorem*, In: S. T. L. Choy, J. P. Jesudason, and P. Y. Lee, editors, Analysis Conference, Singapore 1986, North Holland, 1988, 295-299.

146. V. A. Yudin, *Jackson's theorem in L_2*, Math. Notes, (1987) 27-30.

147. A. A. Yudin and V. A. Yudin, *On Jackson's theorem in L_2*, Math. Notes, (1990) 1071-1074.

148. A. A. Zakharov, *Bound of deviation of continuous functions from their de la Valeé-Poussin sums*, Math. Notes, (1968), 45-49.

149. Z. V. Zaritskaya, *Approximaton of functions of two variable by positive linear operators in the L_p metric*, Ukranian Math J., (1973) 298-302.

150. V. V. Zhuk and G. I. Natanson, *Seminorms and continuity modules for functions defined on a segment*, Journal of Mathematical Sciences Vol 118 (1), (2003) 4822-4851.

151. A. Zygmund, *Smooth functions*, Duke Math. J. 12 (1), (1945) 47-76.

152. A. Zygmund, *Trigonometric series*, 1, 2nd ed. Cambridge, University Press, 1959.

Capítulo 3

El algoritmo de Pólya discreto en espacios finitos

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Abstract: We present some results related with the Pòlya algorithm and the Pòlya 1-algoritm respectively. In particular we analyze the order of convergence and the asymptotic development of the approximants.

3.1 Introducción

Dado $(E, \|\cdot\|)$ un espacio normado, K un subconjunto de E (al que llamaremos clase aproximante) y un elemento $h \in E \backslash K$, diremos que $h^* \in K$ es un elemento de mejor $\|\cdot\|$-aproximación (o un mejor $\|\cdot\|$-"aproximante") de h por elementos de K si $\|h^* - h\| \leq \|f - h\|$, para todo $f \in K$.

La definición de mejor $\|\cdot\|$-aproximante depende fuertemente de la norma utilizada en el espacio normado E. Más concretamente, la consideración en E de otras normas, aún en el caso de que se trate de normas equivalentes, supone, por lo general, que sean distintos los mejores $\|\cdot\|$-aproximantes de un mismo elemento h en E. Nosotros utilizaremos como clase aproximante K un subespacio vectorial propio de dimensión finita de E. En este contexto está asegurada la existencia de los mejores $\|\cdot\|$-aproximantes, para cualquier

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h en E. Además, si E es un espacio estrictamente convexo, entonces también está garantizada la unicidad.

Nuestra investigación se desarrolla en los espacios $\ell_p(J)$, $1 \leq p \leq \infty$, donde $J = \{1, 2, ..., n\}$ ó $J = \mathbb{N}$, siendo K un subespacio vectorial propio de dimensión finita de $\ell_1(J)$. En este ambiente, dado $h \in \ell_p(J) \backslash K$, diremos que $h_p \in K$ es un mejor ℓ_p-aproximante de h por elementos de K si $\|h_p - h\|_p \leq \|f - h\|_p$, para todo $f \in K$. En el caso $p = \infty$, los mejores ℓ_∞-aproximantes se llamarán también *mejores aproximantes uniformes.*

En nuestro contexto, la existencia de los mejores ℓ_p-aproximantes está garantizada para $1 \leq p \leq \infty$, habiendo unicidad si $1 < p < \infty$. Además, para $1 \leq p < \infty$, es posible establecer un teorema de Caracterización de los mejores ℓ_p-aproximantes.

Nuestro objetivo central será estudiar el comportamiento de los mejores ℓ_p-aproximantes cuando $p \to \infty$ ó $p \to 1^+$. Este problema es conocido en la literatura como *algoritmo de Pòlya* o 1-*algoritmo de Pòlya*, respectivamente. Debe su nombre a un trabajo de *Pòlya* de 1913 en el que obtiene el mejor aproximante uniforme de una función continua como límite de los mejores L_p−aproximantes, cuando $p \to \infty$.

En 1962, *Cheney y Curtis* plantearon si sería cierta la convergencia del algoritmo cuando K era un subespacio vectorial de $\mathbb{R}^n$. La respuesta afirmativa a esta cuestión la proporcionó *Descloux* en 1963 demostrando que la convergencia tiene lugar hacia un destacado mejor aproximante uniforme denominado *aproximante uniforme estricto*, que notaremos por h^*_∞, y que ya había sido previamente definido por *Rice* en 1962.

El primer resultado sobre el orden de convergencia del algoritmo de Pòlya cuando la clase aproximante es un subespacio vectorial de $\mathbb{R}^n$ se debe a *Egger* y *Huotari*, quienes en 1990 probaron que el orden mínimo de esta convergencia es $1/p$. Posteriormente, *Navas, Marano* y *Quesada*, hacen un estudio completo de la convergencia probando que ésta es de orden a^p/p, para cierta constante a entre 0 y 1. Asimismo, *Navas, Martínez-Moreno* y *Quesada* consiguen un desarrollo asintótico de los h_p, válido para el caso de orden de convergencia $1/p$ y trivial para los demás casos. Estos y otros casos han sido posteriormente completados por los autores. En el presente trabajo damos una visión panorámica del estado actual del problema.

3.2 El algoritmo de Pòlya discreto sobre subespacios

Para $1 \leq p < \infty$, consideraremos el espacio de Banach $\ell_p(n) = (\mathbb{R}^n, \|\cdot\|_p)$ dotado con la p-norma

$$\begin{aligned} \|x\|_p &:= \Big(\sum_{j=1}^{n} |x(j)|^p\Big)^{1/p}, \quad 1 \leq p < \infty, \\ \|x\|_\infty &=: \|x\| = \max_{1\leq j\leq n} |x(j)|. \end{aligned} \tag{3.1}$$

Definition 3.2.1 *Sea $\emptyset \neq K \subset \mathbb{R}^n$. Dado $h \in \mathbb{R}^n \setminus K$, diremos que el elemento $h_p \in K$, $1 \leq p \leq \infty$, es una mejor ℓ_p-aproximación de h por elementos de K o que h_p es un mejor ℓ_p-aproximante de h por elementos de K si verifica que*

$$\|h - h_p\|_p \leq \|f - h\|_p \qquad \forall f \in K.$$

En lo sucesivo nos referiremos al subconjunto K como clase aproximante. A los mejores ℓ_∞-aproximantes de h por elementos de K los llamaremos mejores aproximantes uniformes de h por elementos de K.

Si K es un subconjunto cerrado y no vacío de $\mathbb{R}^n$, la existencia de h_p está garantizada para $1 \leq p \leq \infty$. Si, además, K es un subconjunto convexo y $1 < p < \infty$, entonces existe un único mejor ℓ_p-aproximante.

Por lo general, encontrar un elemento de mejor ℓ_p-aproximación para $p = 1$ ó $p = \infty$ es un proceso complicado. Esto se debe, por una parte, al hecho de que pueden existir más de un mejor aproximante y, por otra, a que las normas $\|\cdot\|_1$ y $\|\cdot\|_\infty$ no son diferenciables. Sin embargo, cuando $1 < p < \infty$ y K es un subespacio vectorial de $\mathbb{R}^n$, podemos hacer uso de técnicas numéricas (método de Newton-Raphson de resolución de sistemas de ecuaciones) para determinar, al menos de forma aproximada, los mejores ℓ_p-aproximantes. Esto justifica el problema de estudiar la convergencia del conjunto de los mejores ℓ_p-aproximantes cuando $p \to \infty$ y cuando $p \to 1$, como un posible método para obtener mejores aproximantes uniformes o ℓ_1-aproximantes, en cada caso.

3.2.1 Antecedentes históricos

El primer intento de buscar un mejor aproximante uniforme como límite de los mejores ℓ_p-aproximantes cuando $p \to \infty$ lo realizó *Pòlya* en 1913, [14], en el contexto de encontrar el mejor aproximante uniforme para una función continua, cuando la clase aproximante K está constituida por polinomios de grado fijo. Más concretamente, para $n \in \mathbb{N} \cup \{0\}$, sea Π_n el espacio vectorial de los polinomios de grado a lo sumo n y sea $C[a,b]$ el espacio vectorial de las funciones continuas en el intervalo $[a,b]$, dotado con las normas

$$\|f\|_{L_p[a,b]} := \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{C[a,b]} := \max_{t\in[a,b]} |f(t)|.$$

Dada una función $f \in C[a,b]$, sea $\pi_{n,p}(f)$ la mejor L_p-aproximación de f en $[a,b]$ por elementos de Π_n, es decir,

$$\|f - \pi_{n,p}(f)\|_{L_p[a,b]} \leq \|f - \pi\|_{L_p[a,b]}, \quad \forall \pi \in \Pi_n,$$

y sea $\pi_{n,\infty}(f)$ el mejor aproximante uniforme de f en $[a,b]$ por elementos de Π_n, es decir,

$$\|f - \pi_{n,\infty}(f)\|_{C[a,b]} \leq \|f - \pi\|_{C[a,b]}, \quad \forall \pi \in \Pi_n.$$

Polya probó en 1913 que

$$\forall f \in C[a,b], \qquad \lim_{p\to\infty} \pi_{n,p}(f) = \pi_{n,\infty}(f). \tag{3.2}$$

Desde entonces, a la convergencia anterior se la conoce con el nombre de *algoritmo de Pòlya.*

3.2.2 El algoritmo de Pòlya discreto

En lo que sigue nos centraremos en el estudio del algoritmo de Pòlya para el caso discreto, es decir, cuando la clase aproximante es un subconjunto K de $\mathbb{R}^n$ y las normas ℓ_p vienen dadas por (3.1).

En 1962, *Cheney* y *Curtis*, [1], plantearon si era cierta la convergencia dada en (3.2) cuando la clase aproximante K era un subespacio afín de $\mathbb{R}^n$. En 1963, *Descloux*, [2], probó este resultado, es decir, $\lim_{p\to\infty} h_p = h_\infty^*$, donde h_∞^* representa a un determinado aproximante uniforme de h por elementos de K, definido en 1962 por *Rice*, [23], y conocido con el nombre de *aproximante uniforme estricto.*

El aproximante estricto verifica la siguiente propiedad. Sea H el conjunto de los mejores aproximantes uniformes de h por elementos de K. Dado $g \in H$ consideramos el vector $\tau(g)$ cuyas coordenadas vienen dadas por $|g(j) - h(j)|$, $j = 1, 2, \cdots, n$, ordenadas en orden decreciente. Entonces h_∞^* es el único elemento de H tal que $\tau(h_\infty^*)$ minimiza el orden lexicográfico.

Teorema 3.2.1 (Descloux) *Sea K un subespacio afín de $\mathbb{R}^n$, h un elemento fijo de $\mathbb{R}^n \setminus K$, h_p los mejores ℓ_p-aproximantes de h por elementos de K y h^*_∞ el aproximante uniforme estricto. Bajo estas condiciones, se verifica* $\lim_{p\to\infty} h_p = h^*_\infty$.

3.2.3 Orden de convergencia del algoritmo de Pòlya

El siguiente problema es el de analizar el orden de esta convergencia, es decir, estudiar la rapidez con la que $\|h_p - h^*_\infty\| \to 0$ cuando $p \to \infty$. *Egger* y *Huotari*, [4], probaron que cuando K es un subespacio afín de $\mathbb{R}^n$, el orden de esta convergencia es como mínimo $1/p$.

Teorema 3.2.2 (Egger y Huotari) *Sea K un subespacio afín propio de $\mathbb{R}^n$. Sean h_p, $1 < p < \infty$, los mejores ℓ_p-aproximantes del elemento $h \in \mathbb{R}^n$ por elementos de K y h^*_∞ el mejor aproximante uniforme estricto. Entonces existe $M > 0$, tal que*

$$\|h_p - h^*_\infty\| \leq \frac{M}{p} \qquad \forall p \geq 1. \tag{3.3}$$

Navas en su tesis doctoral, [12], hace un estudio completo de la convergencia del algoritmo de Pòlya cuando la clase aproximante es un subespacio afín de $\mathbb{R}^n$, cuyos resultados aparecen publicados en [11,16,17]. En [11], los autores dan condiciones necesarias y suficientes sobre K para que $p\,\|h_p - h^*_\infty\| \to 0$ as $p \to \infty$ y en [17] se prueba el siguiente resultado que caracteriza el orden de convergencia del algoritmo de Pòlya cuando la clase aproximante es un subespacio afín de $\mathbb{R}^n$.

Teorema 3.2.3 (Navas y Quesada) *Sea K un subespacio afín propio de $\mathbb{R}^n$. Entonces existen números reales a, L_1 y L_2, con $0 \leq a \leq 1$ y $L_1, L_2 > 0$, tales que*

$$L_1\, a^p \leq p\,\|h_p - h^*_\infty\| \leq L_2\, a^p. \tag{3.4}$$

El número a que figura en (3.4) se obtiene en función de las coordenadas de h^*_∞ y de los vectores de una determinada base que los autores construyen en el teorema. De (3.4) se deduce inmediatamente que: a) si $a = 0$ hay coincidencia entre h_p y h^*_∞, b) si $a = 1$ la convergencia es de orden $1/p$ y si $0 < a < 1$ la convergencia es de orden a^p/p.

3.2.4 Desarrollo asintótico de los mejores ℓ_p-aproximantes

En [18,19] los autores prueban que para cualquier $r \in \mathbb{N}$, existen vectores $\alpha_l \in \mathbb{R}^n$, $1 \leq l \leq r$, tales que

$$h_p = h^*_\infty + \frac{\alpha_1}{(p-1)} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)}, \tag{3.5}$$

donde $\gamma_p^{(r)} \in \mathbb{R}^n$ y $\|\gamma_p^{(r)}\| = \mathcal{O}\big(1/(p-1)^{r+1}\big)$. Por otra parte, teniendo en cuenta que para $p > 1$,

$$\frac{1}{p-1} = \frac{1}{p(1-1/p)} = \frac{1}{p}\sum_{\nu\geq 0}\frac{1}{p^\nu} = \sum_{\nu\geq 1}\frac{1}{p^\nu},$$

reordenando y agrupando términos, el desarrollo (3.5) puede escribirse en forma estándar como

$$h_p = h^*_\infty + \sum_{\nu=1}^{r}\frac{\beta_\nu}{p^\nu} + \widehat{\gamma}_p^{(r)},$$

donde $\|\widehat{\gamma}_p^{(r)}\| = \mathcal{O}(p^{-r-1})$.

En [8], los autores sugieren el desarrollo asintótico

$$h_p = h^*_\infty + \sum_{i=1}^{\infty}\frac{B_i}{p_i},$$

y aplican la serie anterior para obtener buenas estimaciones de h^*_∞ por medio de la aplicación de técnicas de extrapolación de Richardson. Sin embargo, hasta nuestro conocimiento, es en [19] donde por primera vez se ofrece una demostración de que tal desarrollo es correcto.

Por otra parte, de acuerdo con el Teorema 3.2.3, existen números reales a, L_1 y L_2, con $0 \leq a \leq 1$ y $L_1, L_2 > 0$, tales que

$$L_1\, a^p \leq p\,\|h_p - h^*_\infty\| \leq L_2\, a^p. \tag{3.6}$$

Observemos en primer lugar que si $0 \leq a < 1$, entonces (3.6) nos asegura que (3.5) se cumple trivialmente para todo $r \in \mathbb{N}$ tomando $\alpha_l = 0 \in \mathbb{R}^n$ para $1 \leq l \leq r$. Además, en tal caso (3.5) no aporta ninguna información acerca de los mejores ℓ_p-aproximantes.

En un primer intento por encontrar un desarrollo asintótico de los mejores ℓ_p-aproximantes que contemple el caso general, hemos probado en [7] el siguiente resultado (como viene siendo habitual supondremos que $h = 0 \in \mathbb{R}^n$ es el elemento que queremos aproximar).

Teorema 3.2.4 [7] *Sea K un subespacio afín propio de $\mathbb{R}^n$, $0 \notin \mathbb{K}$. Para $1 < p < \infty$, denotemos por h_p el mejor ℓ_p-aproximante de 0 por elementos de K y sea h_∞^* el aproximante uniforme estricto. Sea a el número real dado en* (3.6). *Entonces existe $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, tal que*

$$h_p = h_\infty^* + \frac{a^p}{p-1}\alpha + \gamma_p, \tag{3.7}$$

donde $\gamma_p \in \mathbb{R}^n$ y $\|\gamma_p\| = o(a^p/p)$.

Si $a = 0$, entonces (3.7) conduce a que $h_p = h_\infty^*$ para todo $p > 1$. Luego el caso que nos queda por resolver es cuando $0 < a \leq 1$. Observemos también que (3.7) es un caso particular de (3.5) para $a = 1$.

Por otra parte, teniendo en cuenta (3.6) se deduce inmediatamente que $p\|h_p - h_\infty^*\|/a^p$ está acotado cuando $p \to \infty$ y, por tanto, existirán sucesiones parciales $\{p_k\}_{k\in\mathbb{N}}$, con $p_k \to \infty$ cuando $k \to \infty$, tales que la sucesión $p_k(h_{p_k} - h_\infty^*)/a^{p_k}$ es convergente. Sin embargo, no es una cuestión trivial probar que existe el límite de $p(h_p - h_\infty^*)/a^p$ cuando $p \to \infty$. Esto justifica el interés de nuestro resultado.

Puesto que siempre existirá una expresión de la forma (3.7) para algún α y γ_p en $\mathbb{R}^n$, la única parte que requiere demostración es la estimación del error para γ_p.

Finalmente, hacemos notar que la condición $\alpha \neq 0$ en (3.7), pone de manifiesto que no será posible conseguir una convergencia de $h_p \to h_\infty^*$ de orden $1/p^r$, con $r \in \mathbb{N}$ y $r \geq 2$. Esto supone una clara diferencia con el caso $p \to 1^+$ donde es posible que $h_p \to h_1^*$ cuando $p \to 1^+$, con orden $(p-1)^r$, con $r \in \mathbb{N}$ y $r \geq 2$.

Problema abierto 3.2.1 *Determinar el orden exacto del resto γ_p con objeto de obtener el siguiente término del desarrollo asintótico en* (3.7).

3.3 El 1-Algoritmo de Pòlya

A la vista de los resultados obtenidos para $p \to \infty$, parecía natural preguntarse qué pasa cuando $p \to 1^+$. Nuestro interés será ahora tratar de determinar un mejor ℓ_1-aproximante como límite de los mejores ℓ_p-aproximantes cuando $p \to 1$.

En lo sucesivo, K denotará un subespacio afín propio de $\mathbb{R}^n$ y supondremos, sin pérdida de generalidad, que $h = 0$ y $0 \notin K$. Seguiremos designando por h_p, $1 \leq p \leq \infty$, a los mejores ℓ_p−aproximantes de 0 por elementos de K.

En está sección estudiaremos la convergencia de h_p cuando $p \to 1^+$, caracterizaremos el orden de dicha convergencia y trataremos de obtener un desarrollo de h_p en potencias de $(p-1)$ para valores de p próximos a 1.

3.3.1 Convergencia del 1-Algoritmo de Pòlya

En nuestro contexto de aproximación, la convergencia de los mejores ℓ_p−aproximantes cuando $p \to 1^+$ es consecuencia inmediata de un resultado más general de Landers y Rogge, [9] . Para enunciar adecuadamente el citado resultado necesitamos introducir algo de notación. Sea $(\Omega, \mathcal{A}, \mu)$ un espacio de medida finita. Para $1 \le p < \infty$ denotemos por $L_p := L_p(\Omega, \mathcal{A}, \mu)$ el espacio vectorial de las clases μ-equivalentes de funciones $\mathcal{A}$-medibles $f : \Omega \to \mathbb{R}$, tales que

$$\|f\|_p := \left(\int_\Omega |f|^p \, d\mu \right)^{1/p} < \infty.$$

Para $\emptyset \neq C \subset L_p$ y $f \in L_p$, $1 \le p < \infty$, denotaremos por $\mu_p(f|C)$ el conjunto de los mejores $\|\cdot\|_p$-aproximantes de f por elementos de C, esto es, el conjunto de los elementos $g \in C$ tales que

$$\|f - g\|_p = \inf\{\|f - h\|_p : h \in C\}.$$

Si $p > 1$ y $C \subset L_p$ es un conjunto $\|\cdot\|_p$-cerrado y convexo, entonces para cada $f \in L_p$ existe un único mejor $\|\cdot\|_p$-aproximante de f por elementos de C, es decir, el conjunto $\mu_p(f|C)$ contiene un único elemento. Sea $f \in L_1$ y $C \subset L_1$ un conjunto $\|\cdot\|_1$-cerrado y convexo. Un elemento $m_1(f|C) \in \mu_1(f|C)$ es llamado un mejor $\|\cdot\|_1$-aproximante natural de f por elementos de C si para cada $g \in \mu_1(f|C)$, $g \neq m_1(f|C)$, existe $s(g) > 1$ tal que

$$\|f - m_1(f|C)\|_s < \|f - g\|_s, \qquad \text{para todo} \quad 1 < s \le s(g). \tag{3.8}$$

Obviamente, si existe un mejor $\|\cdot\|_1$-aproximante natural de f por elementos de C éste es único. Con objeto de que tenga sentido (3.8) hemos de exigir la condición de que

$$f \in L_{1+} := \bigcup_{s>1} L_s \quad \text{y} \quad \emptyset \neq \mu_1(f|C) \subset L_{1+}.$$

Teorema 3.3.1 (Landers y Rogge) *Sea $(\Omega, \mathcal{A}, \mu)$ un espacio de medida finita y $C \subset L_1$ un conjunto $\|\cdot\|_1$-cerrado y convexo. Entonces para cada $f \in L_{1+}$ con $\emptyset \neq \mu_1(f|C) \subset L_{1+}$, se tiene*

i) *existe un mejor* $\|\cdot\|_1$*-aproximante natural de* f *por elementos de* C, *a saber,* $m_1(f|C)$,

ii) $m_1(f|C)$ *es el único mejor* $\|\cdot\|_1$*-aproximante de* f *por elementos de* C *que minimiza la integral*

$$\int_\Omega |f-g| \ln |f-g| \, d\mu,$$

entre todos los mejores $\|\cdot\|_1$*-aproximantes de* f *por elementos de* C,

iii) $\mu_s(f|C \cap L_s)$ *converge en* L_1 *a* $m_1(f|C)$ *cuando* $p \to 1$.

Corolario 3.3.1 *Sea* K *un subespacio afín de* $\mathbb{R}^n$, $0 \notin K$. *Sean* h_p, $1 < p < \infty$, *los mejores* ℓ_p*-aproximantes de* 0 *por elementos de* K. *Entonces se cumple que*

$$\lim_{p \to 1^+} h_p = h_1^*, \tag{3.9}$$

donde h_1^* *es un mejor* ℓ_1-*aproximante de* 0 *por elementos de* K, *llamado mejor* ℓ_1*-aproximante natural.*

Por analogía con el caso $p \to \infty$, la convergencia dada en (3.9) es conocida como el **1**-*algoritmo de Pòlya*. El mejor ℓ_1-aproximante natural viene caracterizado por la siguiente propiedad. Si L denota el conjunto de los mejores ℓ_1-aproximantes de 0 por elementos de K entonces h_1^* es el único elemento de L que minimiza la expresión

$$\sum_{j=1}^{n} |\, h_1(j)| \ln |\, h_1(j)|,$$

para todo $h_1 \in L$, donde $0 \ln 0 := 0$.

3.3.2 Orden de convergencia del 1-algoritmo de Pòlya

Un primer resultado sobre el orden de convergencia del 1-algoritmo de Pòlya se debe a Egger y Taylor en 1993, [5, Teorema 2].

Teorema 3.3.2 (Egger y Taylor) *Sea* K *un subespacio afín de* $\mathbb{R}^n$, $0 \notin K$. *Sean* h_p, $1 < p < \infty$, *los mejores* ℓ_p*-aproximantes de* 0 *por elementos de* K *y* h_1^* *el mejor* ℓ_1*-aproximante natural de* 0 *por elementos de* K, *entonces existe una constante* $M > 0$ *tal que*

$$\|h_p - h_1^*\| \leq M(p-1)$$

para todo $p > 1$.

Asimismo en [5], los autores prueban que si hay unicidad del mejor ℓ_1-aproximante, entonces la convergencia es de orden exponencial, más concretamente, existe un número real γ, $0 \leq \gamma < 1$, tal que $\|h_p - h_1^*\| = \mathcal{O}\left(\gamma^{1/(p-1)}\right)$. Sin embargo, el siguiente ejemplo muestra que esta condición no es necesaria para garantizar un orden de convergencia exponencial.

Example 3.3.1 Consideremos el espacio afín

$$K = (0,1,1) + \operatorname{span}\{(0,1,-1),(2,1,0)\}.$$

El conjunto de los mejores ℓ_1-aproximantes de 0 por elementos de K viene dado por

$$L = \{(0, 1+\lambda, 1-\lambda) : |\lambda| \leq 1\}.$$

Dado que la función $\Phi(\lambda) = (1+\lambda)\ln(1+\lambda)+(1-\lambda)\ln(1-\lambda)$, con $\lambda \in [-1,1]$, tiene un mínimo en $\lambda = 0$, se obtiene que $h_1^* = (0,1,1)$. Por otro lado, resulta fácil comprobar que para $p > 1$,

$$h_p = \left(\frac{-4}{1+2^{(2p-1)/(p-1)}}, 1 - \frac{1}{1+2^{(2p-1)/(p-1)}}, 1 - \frac{1}{1+2^{(2p-1)/(p-1)}}\right),$$

y se deduce inmediatamente que

$$\lim_{p\to 1^+} 2^{1/(p-1)}\|h_p - h_1^*\| = 1,$$

y, por tanto, $\|h_p - h_1^*\| = \mathcal{O}\left(\left(\frac{1}{2}\right)^{1/(p-1)}\right)$.

Notación y resultados preliminares

Sea $J := \{1,2,...,n\}$. Para $x \in \mathbb{R}^n$, definimos $Z(x) := \{j \in J : x(j) = 0\}$ y $R(x) := J \setminus Z(x)$. Distinguimos a los conjuntos $J_0 = Z(h_1^*)$ y $J_0^c = R(h_1^*)$, en beneficio de la notación. Sea L el conjunto de los mejores ℓ_1-aproximantes de 0 en K y sea h_1^* el mejor ℓ_1-aproximante natural. Podemos escribir $K = h_1^* + \mathcal{V}$, donde $\mathcal{V}$ es un subespacio lineal propio de $\mathbb{R}^n$. El siguiente resultado puede verse en ([13, 24]) y ha sido un instrumento fundamental en todos nuestros cálculos.

Teorema 3.3.3 (Caracterización de los mejores ℓ_p-aproximantes)
h_p, $1 \leq p < \infty$, es un mejor ℓ_p-aproximante de 0 por elementos de K si y sólo si para todo $v \in \mathcal{V}$

$$\sum_{j\in J} v(j)\,|h_p(j)|^{p-1}\operatorname{sgn}(h_p(j)) = 0, \quad si \quad 1 < p < \infty, \tag{3.10}$$

$$\left|\sum_{R(h_1)} v(j)\mathrm{sgn}(h_1(j))\right| \leq \sum_{Z(h_1)} |v(j)|, \quad si \quad p = 1. \tag{3.11}$$

El siguiente lema establece una propiedad muy interesante del mejor ℓ_1-aproximante natural: si una de sus componentes es cero, entonces dicha componente es también cero en todos los mejores ℓ_1-aproximantes.

Lemma 3.3.1 *Si $h_1 \in L$, entonces $Z(h_1^*) \subseteq Z(h_1)$.*

Las propiedades geométricas que siguen expresan en qué direcciones $v \in \mathcal{V}$ de desplazamiento a partir de h_1^*, se extiende el conjunto convexo L de los mejores ℓ_1-aproximantes.

Lemma 3.3.2 *Sea $v \in \mathcal{V}$, $v \neq 0$. Entonces $\widetilde{h} = h_1^* + \lambda v$, con $\lambda > 0$ suficientemente pequeño, es un mejor ℓ_1-aproximante de 0 en K si y sólo si*

$$\sum_{j \in J_0^c} v(j)\, \mathrm{sgn}(h_1^*(j)) + \sum_{j \in J_0} |v(j)| = 0.$$

Corolario 3.3.2 *Sea $v \in \mathcal{V}$, $v \neq 0$. Si $J_0 \subseteq Z(v)$, entonces se cumple que*

a) $\sum_{j \in J_0^c} v(j)\, \mathrm{sgn}(h_1^*(j)) = 0$,

b) $h_1^* + \lambda v \in L$, *para λ suficientemente pequeño,*

c) $\sum_{j \in J_0^c} v(j) \ln |h_1^*(j)|\, \mathrm{sgn}(h_1^*(j)) = 0$,

de otro modo,

$$\left|\sum_{j \in J_0^c} v(j)\, \mathrm{sgn}(h_1^*(j))\right| < \sum_{j \in J_0} |v(j)|. \tag{3.12}$$

Corolario 3.3.3 (Pinkus, [13]) *h_1^* es el único mejor ℓ_1-aproximante de 0 en K si y sólo si (3.12) se cumple para todo $v \in \mathcal{V}$, $v \neq 0$.*

Orden de convergencia

Sea $\mathcal{V}_0$ el subespacio lineal de $\mathcal{V}$ definido por $\mathcal{V}_0 := \{u \in \mathcal{V} : u(j) = 0, \forall j \in J_0\}$. Notemos que si $J_0 = \emptyset$, entonces $\mathcal{V}_0 = \mathcal{V}$. Los dos siguientes lemas técnicos permiten establecer las condiciones para asegurar un orden de convergencia potencial del tipo $(p-1)^r$, para algún $r \in \mathbb{N}$.

Lemma 3.3.3 *Supongamos que existe un $v \in \mathcal{V}_0$ y $r \in \mathbb{N}$ tal que*

$$\sum_{j \in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0, \qquad 0 \leq k \leq r. \tag{3.13}$$

Entonces

$$\lim_{p \to 1^+} \frac{1}{(p-1)^r} \sum_{j \in J_0^c} v(j) \frac{h_p(j) - h_1^*(j)}{|h_1^*(j)|} \tag{3.14}$$

$$= -\frac{1}{(r+1)!} \sum_{j \in J_0^c} v(j) \ln^{r+1} |h_1^*(j)| \operatorname{sgn}(h_1^*(j)).$$

Teniendo en cuenta que por el corolario 3.3.2 la igualdad (3.13) es cierta para todo $v \in \mathcal{V}_0$ y $r = 1$, podemos establecer el siguiente resultado.

Corolario 3.3.4 *Para todo $v \in \mathcal{V}_0$,*

$$\lim_{p \to 1^+} \frac{1}{p-1} \sum_{j \in J_0^c} v(j) \frac{h_p(j) - h_1^*(j)}{|h_1^*(j)|} = \frac{1}{2} \sum_{j \in J_0^c} v(j) \ln^2 |h_1^*(j)| \operatorname{sgn}(h_1^*(j)).$$

Para $J' \subseteq J$, denotamos por $\|\cdot\|_{J'}$ la restricción de $\|\cdot\|$ al conjunto de índices en J'.

Lemma 3.3.4 *Si $\|h_p - h_1^*\|/(p-1)^r$ está acotado para algún $r \in \mathbb{N}$, entonces $\|h_p\|_{J_0}/(p-1)^r \to 0$ cuando $p \to 1^+$. En particular, $\|h_p\|_{J_0}/(p-1) \to 0$ cuando $p \to 1^+$.*

A partir de los lemas anteriores podemos enunciar el siguiente resultado que establece condiciones necesarias y suficientes para obtener un radio de convergencia más rápido que $1/(p-1)$, establecido en su versión más general

Corolario 3.3.5 *Sea $r \in \mathbb{N}$. Entonces $\lim_{p \to 1} \|h_p - h_1^*\|/(p-1)^r \to 0$ si y sólo si*

$$\sum_{j \in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0, \qquad 1 \leq k \leq r+1,$$

para todo $v \in \mathcal{V}_0$.

Exponemos a continuación las condiciones necesarias y suficientes para obtener un radio exponencial de convergencia. Del Corolario 3.3.2 y Corolario 3.3.5 vemos que para que esto ocurra, es necesario que para todo $v \in \mathcal{V}_0$,

$$\sum_{j\in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0, \qquad \text{para todo } k \in \mathbb{N}\cup\{0\}. \tag{3.15}$$

Sean $0 < d_1 < d_2 < \cdots < d_s$ todos los diferentes valores de $|h_1^*(j)|$ en J_0^c. Consideremos la partición $\{J_l\}_{l=1}^s$ de J_0^c dada por $J_l = \{j \in J_0^c : |h_1^*(j)| = d_l\}$. La condición (3.15) puede sustituirse por otra más fácil de verificar.

Lemma 3.3.5 *Sea $v \in \mathcal{V}$. Las siguientes condiciones son equivalentes*

i) $\sum_{j\in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0$, *para todo* $k \in \mathbb{N}\cup\{0\}$.

ii) $\sum_{j\in J_0^c} v(j) \ln^k |h_1^*(j)| \operatorname{sgn}(h_1^*(j)) = 0, \quad 0 \le k \le s-1.$

iii) $\sum_{j\in J_l} v(j) \operatorname{sgn}(h_1^*(j)) = 0$, $1 \le l \le s$.

iv) $\sum_{j\in J_0^c} v(j)|h_1^*(j)|^{p-1} \operatorname{sgn}(h_1^*(j)) = 0$, *para todo* $p > 1$.

v) *Existe* $1 < p_1 < p_2 < \cdots < p_s$ *tal que*

$$\sum_{j\in J_0^c} v(j)|h_1^*(j)|^{p_k-1} \operatorname{sgn}(h_1^*(j)) = 0, \quad 1 \le k \le s. \tag{3.16}$$

Corolario 3.3.6 *$h_p = h_1^*$ para todo $p > 1$ si y sólo si*

$$\sum_{j\in J_l} v(j) \operatorname{sgn}(h_1^*(j)) = 0, \qquad 1 \le l \le s$$

para todo $v \in \mathcal{V}$.

Sea $\mathcal{B}_0$ una base de $\mathcal{V}_0$, donde suponemos que $\mathcal{B}_0 := \emptyset$ si $\mathcal{V}_0 = \{0\}$. Sea $\mathcal{B}_1$ un conjunto (posiblemente vacío) de vectores linealmente independientes en $\mathcal{V}$ tal que $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1$ es una base de $\mathcal{V}$ y sea $\mathcal{V}_1 = \operatorname{span}(\mathcal{B}_1)$. Así, podemos escribir $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$.

Teorema 3.3.4 **([20], [21])** *Supongamos que*

$$\sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j)) = 0, \qquad 1 \le l \le s, \tag{3.17}$$

para todo $v \in \mathcal{B}_0$. Si (3.17) también es cierto para todo $v \in \mathcal{B}_1$, entonces $h_p = h_1^*$ para todo $p > 1$. De otro modo, sea

$$\gamma_0 := \max_{\substack{\|v\|=1\\ v\in\mathcal{V}_1}} \frac{\left|\sum_{j\in J_0^c} v(j)\mathrm{sgn}(h_1^*(j))\right|}{\sum_{j\in J_0}|v(j)|} < 1.$$

Entonces, para todo $\gamma_0 < \gamma < 1$ existe $p_0 = p_0(\gamma) > 1$ y $M > 0$ tal que $\|h_p - h_1^*\| < M\,\gamma^{1/(p-1)}$ para $1 < p < p_0$.

Observemos que si h_1^* es el único mejor ℓ_1-aproximante de 0 en K, entonces el Corolario 3.3.3 implica que $\mathcal{B}_0 = \emptyset$ y (3.17) es cierta obviamente.

Corolario 3.3.7 *Si L es un conjunto unitario, existe un γ, $0 < \gamma < 1$, tal que $\|h_p - h_1^*\| < \gamma^{1/(p-1)}$.*

La siguiente tabla resume los resultados obtenidos y da una descripción completa del radio de convergencia de h_p hacia h_1^* si $p \to 1^+$. Para $v \in \mathcal{V}$ denotamos,

$$\Sigma_{J_0^c}^k(v) = \sum_{j\in J_0^c} v(j)\ln^k|h_1^*(j)|\,\mathrm{sgn}(h_1^*(j)), \quad \Sigma_{J_l}(v) = \sum_{j\in J_l} v(j)\,\mathrm{sgn}(h_1^*(j))$$

y consideramos

$$r_0 = \max\left\{r \in \{1,...,s-1\} : \Sigma_{J_0^c}^k(v) = 0,\ 0 \le k \le r,\ \forall v \in \mathcal{B}_0\right\}.$$

Condición en $\mathcal{V}$	**Orden de convergencia**
$\Sigma_{J_l}(v) = 0,\quad 1 \le l \le s,\quad \forall v \in \mathcal{B}_0 \cup \mathcal{B}_1$	$h_p = h_1^*$, para todo $p > 1$.
$\Sigma_{J_l}(v) = 0,\quad 1 \le l \le s,\quad \forall v \in \mathcal{B}_0$ $\Sigma_{J_l}(v) \neq 0$, para algún $1 \le l \le s$ y algún $v \in \mathcal{B}_1$	$\mathcal{O}(\gamma^{1/(p-1)})$
$r_0 < s-1$	$\mathcal{O}((p-1)^{r_0})$

Notemos que por el Corolario 3.3.2, $r_0 \ge 1$. Por otro lado, por el Lema 3.3.5, $r_0 = s-1$ es equivalente a $\Sigma_{J_l}(v) = 0$, $1 \le l \le s$, para todo $v \in \mathcal{B}_0$.

3.3.3 Desarrollo de Taylor de los mejores ℓ_p-aproximantes

El objetivo de esta sección es probar que, para todo $r \in \mathbb{N}$, los mejores ℓ_p-aproximantes admiten un desarrollo de Taylor de orden r de la forma

$$h_p = h_1^* + \sum_{l=1}^{r} \alpha_l (p-1)^l + \gamma_p^{(r)}, \tag{3.18}$$

con $\alpha_l \in \mathbb{R}^n$, $1 \leq l \leq r$ y $\gamma_p^{(r)} \in \mathbb{R}^n$ tal que $\|\gamma_p^{(r)}\| = \mathcal{O}\left((p-1)^{r+1}\right)$.

Observemos que siempre es posible escribir h_p en la forma (3.18) cualesquiera que sean los α_l, $1 \leq l \leq r$. Por tanto, el interés de (3.18) estará en probar es que es posible elegir los α_l, $1 \leq l \leq r$ de forma que se obtenga la deseada estimación del error para $\gamma_p^{(r)}$.

Teorema 3.3.5 **[22]** *Para todo $r \in \mathbb{N}$ existen vectores $\alpha_l \in \mathbb{R}^n$, $1 \leq l \leq r$, tales que*

$$h_p = h_1^* + \sum_{l=1}^{r} \alpha_l (p-1)^l + \gamma_p^{(r)}, \tag{3.19}$$

donde $\gamma_p^{(r)} \in \mathbb{R}^n$ y $\|\gamma_p^{(r)}\| = \mathcal{O}\left((p-1)^{r+1}\right)$.

Teniendo en cuenta que $\|h_p - h_1^*\|/(p-1)$ está acotado, la demostración del Teorema 3.3.5 se obtiene por inducción mediante la aplicación de los dos siguientes lemas:

Lemma 3.3.6 *Bajo las condiciones del Teorema* 3.3.5*, sea $r \in \mathbb{N}$ y supongamos que existe $\alpha_l \in \mathcal{V}$, $1 \leq l \leq r-1$, tal que*

$$h_p = h_1^* + \sum_{l=1}^{r-1} \alpha_l (p-1)^l + \gamma_p^{(r-1)}.$$

Si existe $\alpha_r := \lim_{p\to 1} \gamma_p^{(r-1)}/(p-1)^r$, entonces $\|\gamma_p^{(r)}\|/(p-1)^{r+1}$ está acotado, donde $\gamma_p^{(r)} := \gamma_p^{(r-1)} - \alpha_r (p-1)^r$.

Lemma 3.3.7 *Bajo las condiciones del Teorema* 3.3.5*, sea $r \in \mathbb{N}$ y supongamos que existe $\alpha_l \in \mathcal{V}$, $1 \leq l \leq r-1$, tal que*

$$h_p = h_1^* + \sum_{l=1}^{r-1} \alpha_l (p-1)^l + \gamma_p^{(r-1)}.$$

Si $\|\gamma_p^{(r-1)}\|/(p-1)^r$ está acotado, existe $\lim_{p\to\infty} \gamma_p^{(r-1)}/(p-1)^r \in \mathcal{V}$.

Sean $0 = \rho_0 < \rho_1 < \cdots < \rho_s$, los diferentes valores de $|h_1^*(j)|$, $1 \leq j \leq n$, y sea J_k, $1 \leq k \leq s$, el conjunto de índices $j \in J_0^c$ tal que $|h_1^*(j)| = \rho_k$. En [21] se prueba que si

$$\sum_{j \in J_k} u(j)\,\mathrm{sgn}(h_1^*(j)) = 0, \quad 1 \leq k \leq s, \qquad \forall u \in \mathcal{V}_0, \tag{3.20}$$

entonces existe $M > 0$ y $0 \leq \gamma < 1$ tal que $\|h_p - h_1^*\| \leq M\,\gamma^{1/(p-1)}$. Por tanto, si (3.20) es cierto, tenemos un orden de convergencia exponencial de h_p hacia h_1^* y, por consiguiente, el desarrollo (3.19) se cumple trivialmente con $\alpha_l = 0 \in \mathbb{R}^n$ para todo $1 \leq l \leq r$. Por tanto, para obtener un desarrollo no trivial de h_p en $p = 1$, hemos de asumir que

$$\sum_{j \in J_k} u(j)\,\mathrm{sgn}(h_1^*(j)) \neq 0$$

para algún $u \in \mathcal{V}_0$ y algún $1 \leq k \leq s$.

3.4 El algoritmo de Pòlya en el espacio de sucesiones

Para $1 \leq p < \infty$, consideraremos el espacio de Banach $\ell_p(\mathbb{N})$ de las sucesiones $x = \{x(j)\}_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ tales que

$$\sum_{j \in \mathbb{N}} |x(j)|^p < \infty,$$

dotado con la p-norma

$$\|x\|_p = \left(\sum_{j \in \mathbb{N}} |x(j)|^p \right)^{1/p}.$$

Asimismo, consideraremos el espacio de Banach $\ell_\infty(\mathbb{N})$ de las sucesiones acotadas en $\mathbb{R}^{\mathbb{N}}$, con la norma uniforme $\|x\| = \|x\|_\infty = \sup_{j \in \mathbb{N}}\{|x(j)|\}$.

Sea $K \neq \emptyset$ un subconjunto de $\ell_1(\mathbb{N})$. Dado $h \in \ell_1(\mathbb{N}) \setminus K$ y $1 \leq p \leq \infty$ diremos que $h_p \in K$ es *un mejor ℓ_p-aproximante de h por elementos de K* si

$$\|h_p - h\|_p \leq \|g - h\|_p \qquad \text{para todo } g \in K.$$

Si $p = \infty$ diremos que h_∞ es *un mejor aproximante uniforme de h por elementos de K*.

Si K es un subespacio vectorial de dimensión finita de $\ell_1(\mathbb{N})$, entonces está garantizada la existencia de mejores ℓ_p-aproximantes para cualquier $h \in \ell_1(\mathbb{N}) \setminus K$. Además, tenemos unicidad del mejor ℓ_p-aproximante si $1 < p < \infty$.

En general, la unicidad del mejor aproximante uniforme no está garantizada. Sin embargo, en [10], *Marano* prueba la existencia de *un único mejor aproximante uniforme estricto* cuando K es un subconjunto compacto, convexo y no vacío de c_0 (el espacio de Banach de las sucesiones $x \in \mathbb{R}^{\mathbb{N}}$ tales que $x(j) \to 0$ cuando $j \to \infty$).

Mediante un proceso inductivo, se prueba en [10] la existencia de una sucesión $\{d_n\}_{n\in\mathbb{N}}$ (posiblemente finita) estrictamente decreciente y una familia numerable (posiblemente finita) de conjuntos $\{J_n\}_{n\in\mathbb{N}}$ finitos y disjuntos de forma que:

1) si $d_s = 0$ para algún $s > 1$, entonces $\mathbb{N} = \bigcup_{l=1}^{s} J_l$;

2) en caso contrario, $d_n \to 0$ cuando $n \to \infty$.

El aproximante uniforme estricto de un elemento $h \in \ell_1(\mathbb{N}) \setminus K$, que notaremos por h_∞^*, viene caracterizado por alguna de las dos siguientes propiedades:

- *Si denotamos* $J_0 = \mathbb{N} \setminus \bigcup_{l\in\mathbb{N}} J_l$, *entonces* h_∞^*, *es el único elemento en* H *tal que*

$$|h(j) - h_\infty^*(j)| = d_k, \quad j \in J_k, \quad k = 1, 2, \ldots$$

  y $h_\infty^*(j) = h(j)$ *si* $j \in J_0$.

- Para cada $g \in H$ consideramos la sucesión $\tau(g) \in \ell_1(\mathbb{N})$ cuyas coordenadas están dadas por $|g(j) - h(j)|$ reordenadas en orden decreciente. Entonces h_∞^* es el único elemento en H tal que $\tau(h_\infty^*)$ minimiza el orden lexicográfico.

Observemos que h_∞^* está bien definido porque para cada $\varepsilon > 0$ y cualquier $g \in H$, el conjunto de índices $j \in \mathbb{N}$ tales que $|h_\infty^*(j) - g(j)| \geq \varepsilon$ es finito. La unicidad de h_∞^* es una consecuencia inmediata de la convexidad del conjunto H.

Como ya hemos comentado, cuando K es un subespacio afín de $\mathbb{R}^n$, la convergencia de h_p hacia h_∞^* fue probada por *Descloux*, [2]. El primer resultado sobre el orden de convergencia de $\|h_p - h_\infty^*\|$ cuando $p \to \infty$ fue establecido por *Egger* y *Huotari*, [4], quienes probaron que existen constantes $M > 0$ y $p_0 \geq 1$ tales que

$$p \|h_p - h_\infty^*\| < M, \tag{3.21}$$

para todo $p \geq p_0$. Posteriormente, *Navas* y *Marano*, [11], establecen condiciones necesarias y suficientes sobre K para conseguir que $p \|h_p - h_\infty^*\| \to 0$ cuando $p \to \infty$ y *Navas* y *Quesada*, [17], prueban que existen constantes $L_1, L_2 > 0$ y $0 \leq a \leq 1$, dependiendo de K, tales que

$$L_1 a^p \leq p \|h_p - h_\infty^*\| \leq L_2 a^p, \tag{3.22}$$

para todo p suficientemente grande.

A lo largo de esta sección, K será un subespacio afín de dimensión finita de $\ell_1(\mathbb{N})$. En este contexto es también conocido, [3, 10], que $\lim_{p \to \infty} h_p = h_\infty^*$. Además, en [3] los autores prueban que el resultado de acotación dado en (3.21) sigue siendo cierto.

Nuestro objetivo será dar una completa descripción del orden de convergencia de $\|h_p - h_\infty^*\|$ cuando $p \to \infty$, generalizando el resultado (3.22) a nuestro contexto de aproximación en el espacio $\ell_1(\mathbb{N})$.

3.4.1 Notación y resultados previos.

Para $J \subseteq \mathbb{N}$ denotaremos mediante $J^c = \mathbb{N} \setminus J$. Además, para cada $v \in \ell_1(\mathbb{N})$ definimos $\|v\|_J = \max_{j \in J} |v(j)|$. Observemos que $\|v\|_{\mathbb{N}} = \|v\|$. Sea K un subespacio afín propio de dimensión finita de $\ell_1(\mathbb{N})$, $0 \notin K$. Denotaremos por h_∞^* (respectivamente h_p, $1 < p < \infty$) el aproximante uniforme estricto (respectivamente, el mejor ℓ_p-aproximante) de 0 por elementos de K. Asumiremos, sin pérdida de generalidad, que $\|h_\infty^*\| = 1$. En lo que sigue expresaremos K en la forma $K = h_\infty^* + \mathcal{V}$, donde $\mathcal{V}$ es un subespacio vectorial de dimensión finita de $\ell_1(\mathbb{N})$ con $m := \dim(\mathcal{V}) > 0$. Sea J_0 el conjunto de índices en $\mathbb{N}$ definido por

$$J_0 := \{j \in \mathbb{N} : h_\infty^*(j) = 0\}$$

y consideremos el subespacio vectorial $\mathcal{V}_0$ de $\mathcal{V}$ dado por

$$\mathcal{V}_0 = \{v \in \mathcal{V} : v(j) = 0, \text{ para todo } j \in \mathbb{N} \setminus J_0\}. \tag{3.23}$$

Pongamos $d_0 = 0$. Por medio de un proceso inductivo, para cada $n \in \mathbb{N}$ tal que $\mathbb{N} \neq \cup_{l=0}^{n-1} J_l$, definimos $d_n = \|h_\infty^*\|_{\mathbb{N} \setminus \cup_{l=0}^{n-1} J_l}$ y $J_n = \{j \in \mathbb{N} : |h_\infty^*(j)| = d_n\}$. Observemos que, al ser h_∞^* una sucesión en $\ell_1(\mathbb{N})$, entonces, para cada $n \in \mathbb{N}$, J_n es un conjunto finito de índices.

- Si $\mathbb{N} = \bigcup_{l=0}^{n-1} J_l$ para algún $n \in \mathbb{N}$, entonces obtenemos una sucesión finita estrictamente decreciente $\{d_l\}_{l=1}^{n-1}$ junto con una familia finita de conjuntos finitos y disjuntos $\{J_l\}_{l=1}^{n-1}$. En tal caso, tomaremos $d_n = 0$.

- En caso contrario, obtendremos una sucesión, $\{d_n\}_{n\in\mathbb{N}}$, estrictamente decreciente y una familia numerable de conjuntos finitos y disjuntos $\{J_n\}_{n\in\mathbb{N}}$, de tal forma que

$$d_n \downarrow 0 \text{ cuando } n \to \infty \quad \text{y} \quad \bigcup_{l=0}^{\infty} J_l = \mathbb{N}.$$

Para todo $v \notin \mathcal{V}_0$, definimos

$$r(v) = \min\{n \in \mathbb{N} : v(j) \neq 0, \text{ para algún } j \in J_n\}. \tag{3.24}$$

Lemma 3.4.1 *Sea $\mathcal{V}$ el espacio vectorial y $\{J_n\}_{n\in\mathbb{N}}$ la familia de conjuntos de índices definidos anteriormente. Entonces, existen un entero $r \geq 0$, una sucesión finita estrictamente creciente de números enteros $\{\sigma(k)\}_{k=0}^{r}$, con $\sigma(0) = 0$, y subespacios vectoriales $\mathcal{V}_k$ de $\mathcal{V}$, $0 \leq k \leq r$, tales que $\mathcal{V} = \bigoplus_{k=0}^{r} \mathcal{V}_k$, verificando la siguiente propiedad:*

(p1) *si $v \in \mathcal{V}_k \setminus \{0\}$, entonces $v(j) = 0$ para todo $j \in \bigcup_{l=1}^{\sigma(k)-1} J_l$ y $v(j) \neq 0$ para algún $j \in J_{\sigma(k)}$.*

Lemma 3.4.2 *Si $v \notin \mathcal{V}_0$, entonces existen $j_1, j_2 \in J_{r(v)}$ (véase (3.24)) tales que $v(j_1)h_\infty^*(j_1) > 0$ y $v(j_2)h_\infty^*(j_2) < 0$.*

3.4.2 Orden de convergencia

El teorema de Egger y Huotari

El siguiente Teorema fue inicialmente probado por *Egger y Houtari*, [3]. Sin embargo, en [15] presentamos una demostración alternativa (y en nuestra opinión más sencilla), basada en la fórmula de caracterización y cuyo interés principal radica en que no es necesario usar el hecho de que $h_p \to h_\infty^*$ cuando $p \to \infty$ para concluir que $p\,\|h_p - h_\infty^*\|$ está acotado cuando $p \to \infty$. De hecho, la convergencia de $h_p \to h_\infty^*$ cuando $p \to \infty$ se obtendrá como una consecuencia inmediata de nuestro resultado.

Teorema 3.4.1 [3, 15] *Sea K un subespacio afín finito dimensional de $\ell_1(\mathbb{N})$ tal que $0 \notin K$. Sea h_p, $1 < p < \infty$, el mejor ℓ_p-aproximante de 0 por elementos de K y h_∞^* el aproximante uniforme estricto de 0 por elementos de K. Entonces existen constantes positivas M y C tales que, para todo $p > C$, $p\,\|h_p - h_\infty^*\| \leq M$.*

El siguiente resultado es consecuencia inmediata del Teorema 3.4.1 y es conocido como el *Algoritmo de Pòlya* para sucesiones, que resulta convergente si K es un subespacio afín de dimensión finita de $\ell_1(\mathbb{N})$,

Corolario 3.4.1 *Sea K un subespacio afín finito dimensional de $\ell_1(\mathbb{N})$ tal que $0 \notin K$. Sea h_p, $1 < p < \infty$, el mejor ℓ_p-aproximante de 0 por elementos de K y h_∞^* el aproximante uniforme estricto de 0 por elementos de K. Entonces se cumple que $\lim_{p\to\infty} h_p = h_\infty^*$.*

Orden de convergencia más rápido que $1/p$

En algunos casos podemos precisar aún más el orden de convergencia diciendo que $h_p \to h_\infty^*$ cuando $p \to \infty$ con orden de convergencia exacto $1/p$. El siguiente Teorema establece condiciones necesarias y suficientes en K para asegurar que $p\,\|h_p - h_\infty^*\| \to 0$ cuando $p \to \infty$.

Teorema 3.4.2 [15] *Sea K un subespacio afín finito dimensional de $\ell_1(\mathbb{N})$ tal que $0 \notin K$. Sea h_p, $1 < p < \infty$, el mejor ℓ_p-aproximante de 0 por elementos de K y h_∞^* el aproximante uniforme estricto de 0 por elementos de K. Entonces,*

$$p\,\|h_p - h_\infty^*\| \to 0,$$

cuando $p \to \infty$ si, y sólo si, para todo $1 \leq k \leq r$ y cada $v \in \mathcal{V}_k$,

$$\sum_{j\in J_{\sigma(k)}} v(j)\,\mathrm{sgn}(h_\infty^*(j)) = 0, \tag{3.25}$$

donde los espacios $\mathcal{V}_k$, $1 \leq k \leq r$, y la sucesión $\{\sigma(k)\}_{k=1}^r$ vienen dados en el Lema 3.4.1.

Coincidencia de los mejores ℓ_p-aproximantes

El siguiente teorema establece las condiciones necesarias y suficientes sobre K para garantizar que $h_p = h_\infty^*$ para todo $p > 1$.

Teorema 3.4.3 [15] *Sea K un subespacio afín finito dimensional de $\ell_1(\mathbb{N})$ tal que $0 \notin K$. Sea h_p, $1 < p < \infty$, el mejor ℓ_p-aproximante de 0 por elementos de K y h_∞^* el aproximante uniforme estricto de 0 por elementos de K. Entonces, $h_p = h_\infty^*$ para todo $p > 1$ si, y sólo si, para todo $v \in \mathcal{V} \setminus \mathcal{V}_0$ y todo $n \in \mathbb{N}$, se cumple que*

$$\sum_{j\in J_n} v(j)\,\mathrm{sgn}(h_\infty^*(j)) = 0. \tag{3.26}$$

3.4.3 Orden de convergencia exponencial

El Teorema 3.4.4. nos permitirá caracterizar completamente el orden de convergencia de $h_p \to h_\infty^*$ cuando $p \to \infty$, extendiendo el resultado (3.22) al caso en que K es un subespacio afín de dimensión finita de $\ell_1(\mathbb{N})$.

A continuación introducimos la notación que será utilizada en lo sucesivo. Consideremos el espacio

$$\mathcal{W}_0 = \bigoplus_{k=1}^{r} \mathcal{V}_k,$$

donde $\mathcal{V}_k$, $1 \leq k \leq r$, son los subespacios vectoriales de $\mathcal{V}$ construidos en el Lema 3.4.1 y notemos $m_0 = \dim(\mathcal{W}_0)$. Sea $\mathbf{I} = \{1, \ldots, m_0\}$ y $\mathcal{B} = \{v_1, \ldots, v_{m_0}\}$ una base de $\mathcal{W}_0$ elegida de tal manera que si $i \in \mathbf{I}$, entonces $v_i \in \mathcal{V}_k$ para algún $k \in \{1, \ldots, r\}$. Sea $\{I_k\}_{k=1}^r$ la partición de $\mathbf{I}$ dada por

$$I_k = \{i \in \mathbf{I} : v_i \in \mathcal{V}_k\}$$

y designemos $m_k = \mathrm{card}(I_k)$. Puesto que $\mathcal{V} = \mathcal{W}_0 \oplus \mathcal{V}_0$, dado cualquier vector $v \in \mathcal{V}$ existirán dos vectores, unívocamente determinados, $\Lambda_v = (\lambda_v(i))_{i\in\mathbf{I}} \in \mathbb{R}^{\mathbf{I}}$ y $w_v \in \mathcal{V}_0$, tales que,

$$v = \sum_{i\in\mathbf{I}} \lambda_v(i) v_i + w_v.$$

Teniendo en cuenta que $w_v(j) = 0$ para todo $j \in \hat{J} = \bigcup_{k=1}^r J_{\sigma(k)}$, entonces

$$\|\Lambda_v\| := \max_{i\in\mathbf{I}} |\lambda_v(i)|$$

es una norma sobre $\mathcal{V}|_{\hat{J}}$.

Para $1 \leq k \leq r$ y $l \in \mathbb{N}$, definimos

$$\Sigma(l,k) = \max_{i\in I_k} \Big| \sum_{j\in J_l} v_i(j)\,\mathrm{sgn}(h_\infty^*(j)) \Big|$$

y sea a el número real dado por

$$a = \max_{\substack{l\in\mathbb{N}\\ 1\leq k\leq r}} \left\{ \frac{d_l}{d_{\sigma(k)}} : \Sigma(l,k) \neq 0 \right\}, \tag{3.27}$$

donde asumimos que $a = 0$ si $\Sigma(l,k) = 0$ para todo $l \in \mathbb{N}$ y $1 \leq k \leq r$.

Remark 3.4.1 El número a dado en (3.27) está bien definido dado que el cociente $d_l/d_{\sigma(k)}$ es decreciente con respecto a $l \in \mathbb{N}$ y creciente con respecto a k $(1 \leq k \leq r)$. Observemos también que si $\Sigma(l,k) \neq 0$, entonces $l \geq \sigma(k)$ (recordemos que si $i \in I_k$, entonces $v_i(j) = 0$ para todo $j \in \cup_{l=1}^{\sigma(k)-1} J_l$). Esto implica que $d_l \leq d_{\sigma(k)}$ y, en consecuencia, $0 \leq a \leq 1$.

Teorema 3.4.4 [15]*Sea K un subespacio afín finito dimensional de $\ell_1(\mathbb{N})$ tal que $0 \notin K$. Sea h_p, $1 < p < \infty$, el mejor ℓ_p-aproximante de 0 por elementos de K y h_∞^* el aproximante uniforme estricto de 0 por elementos de K. Entonces existen constantes positivas L_1, L_2 y $p_0 > 1$ tal que, para todo $p > p_0$,*

$$L_1 a^p \leq p \|h_p - h_\infty^*\| \leq L_2 a^p, \tag{3.28}$$

donde a es el número real definido en (3.27).

3.5 Bibliografía

1. Cheney, E. W.; Curtis, P. C., *Research Problem 33: Convex Bodies,* Bull. Amer. Math. Soc., **68**, 305, (1962).

2. Descloux, J., *Approximations in L^p and Chebyshev Approximations,* J. Soc. Indust. Appl. Math., **11**, 4, 1017–1026, (1963).

3. Egger, A.; Huotari, R., *Polya Properties in Sequences Spaces,* Approx. Theory Appl. **7**, 3, 77–85 (1991).

4. Egger, A.; Huotari, R., *Rate of Convergence of the Discrete Polya Algorithm,* J. Approx. Theory, **60**, 1, 24–30, (1990).

5. Egger, A.; Taylor, G. D., *Rate of Convergence of the Discrete Polya-1 Algorithm,* J. Approx. Theory, **75**, 3, 312–324, (1993).

6. Fernández-Ochoa, J, "El algoritmo de Pòlya en subespacios vectoriales de dimensión finita". Tesis Doctoral. Universidad de Jaén, 2005.

7. Fernández-Ochoa, J; Martínez-Moreno, J.; Quesada, J., *Asymptotic Expression of the Best ℓ_p-Approximation from Linear Subspaces,* J. Approx. Theory **140**, 2, 147–153, (2006).

8. Fletcher, R.; Grant, J.; Hebden, M., *Linear minimax approximation as the limit of best L_p-approximation,* SIAM. J. Numer. Anal., **11**, 1, 123–136, (1974).

9. Landers, D.; Rogge, L., *Natural choice of L_1 approximants*, J. Approx. Theory, **33**, 3, 268–280, (1981).

10. Marano, M. , *Strict approximation on closed convex sets*, Approx. Theory and its Appl., **6**, 1, 99-109, (1990).

11. Marano, M.; Navas, J., *The Linear Discrete Polya Algorithm*, Appl. Math. Lett., **8**, 6, 25–28, (1995).

12. Navas, J., "Orden de convergencia del algoritmo de Pòlya sobre subespacios y su extensión a conjuntos convexos". Tesis Doctoral. Universidad de Jaén. Octubre 2001.

13. Pinkus, A., "On L^1-approximation". Cambridge University Press, (1988).

14. Pòlya, G., *Sur un algoritme toujours convergent pour obtenir les polynomes de meilleure approximation de Tchebycheff pour une fonction continue quelconque*, C. R. Acad. Sci. París, **157**, 840–843, (1913).

15. Quesada, J.; Fernández-Ochoa, J.; Martínez-Moreno, J.; Bustamante, J., *The Polya algorithm in sequence spaces*, J. Approx. Theory **135**, 2, 245–257, (2005).

16. Quesada, J.; Navas, J., *On rate of convergence of the linear discrete Polya algorithm*, A Joint Mathematical European-Arabic Conference. Granada, (2000).

17. Quesada, J.; Navas, J., *Rate of Convergence of the Linear Di Algorithm*, J. Approx. Theory, **110**, 1, 109–119, (2001).

18. Quesada, J.; Martínez-Moreno, J.; Navas, J., *On best p-approximation from affine subspaces: Asymptotic expansion,* C. R. Acad. Sci. Paris, Ser. I, **334**, 12, 1077–1082, (2002).

19. Quesada, J.; Martínez-Moreno, J.; Navas, J., *Asymptotic behaviour of best ℓ_p-approximations from affine subspaces*, J. Approx. Theory, **118**, 2, 275–289, (2002).

20. Quesada, J.; Martínez-Moreno, J.; Navas, J.; Fernández-Ochoa, J., *Rate of convergence of best $p-$approximations from affine subspaces as $p \to 1$*, C. R. Acad. Bulg. Sci. **55**, 9, 13–18, (2002).

21. Quesada, J.; Martínez-Moreno; J., Navas, J.; Fernández-Ochoa, J., *Rate of convergence of linear discrete Pòlya 1-algorithm*, J. Approx. Theory, **118**, 2, 316–329, (2002).

22. Quesada, J.; Martínez-Moreno, J.; Navas, J.; Fernández-Ocho J., *Taylor expansion of best ℓ_p-approximations about $p = 1$*, Appl. Math. Lett., **16**, 7, 993-999 (2003).

23. Rice, J. R., *Tchebycheff approximation in a compact metric space*, Bull. Amer. Math. Soc., **68**, 405–410, (1962).

24. Singer, I., Best Approximation in Normed Linear Spaces by Elements of Linear Subspace, Springer - Verlag, Berlin, (1970).

Capítulo 4

Modelos Matemáticos en Electroencefalografía Inversa

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Abstract: We present here some results about the inverse problem of identification of bioelectric sources in the brain and in the cerebral cortex. The start points are measures for the electric potentials generated on the scalp. The obtained results guarantee the existence and unicity of the solution of this problem and allow us to propose stable algorithms for their solution.

We present some examples in the case of a simple geometrical model of the human head.

Resumen: En este trabajo se presentan algunos resultados básicos sobre el problema inverso de identificación de fuentes bioeléctricas, concentradas en el volumen y en la corteza cerebral, a partir de las mediciones de potencial sobre el cuero cabelludo. Estos resultados garantizan la existencia y unicidad de este problema de identificación y permiten proponer algoritmos para su solución. Para el caso de fuentes concentradas en el volumen cerebral, se presenta un algoritmo estable para un caso en que la cabeza se modela por geometrías simples.

Words key: EEG, Problemas Inversos y mal planteados, Regularización.

4.1 Introducción

Actualmente se tiene un gran interés por la investigación sobre los métodos no destructivos para la detección de alguna(s) característica(s) desconocida(s) de un sistema a partir de información parcial sobre la respuesta del sistema a una señal de entrada. Ejemplos de esto son la Tomografía Médica, la Tomografía de Procesos Industriales y la Geofísica Inversa.

En la Tomografía Médica (TM) se trata de identificar una fuente bioeléctrica a partir de alguna manifestación externa producida por esa fuente y que es medida por medio de un dispositivo electrónico. Ejemplos de estos son los métodos de resonancia magnética y los de imagen radio-isotópica. Estos métodos proporcionan imágenes funcionales del flujo de sangre y del metabolismo esenciales para diagnosticar e investigar el cerebro, corazón, hígado, riñón, hueso y otros órganos del cuerpo humano. Estudios sobre el daño cerebral durante el envejecimiento normal y anormal han sido respaldados por los avances en la resonancia magnética (RM), mientras que las técnicas avanzadas de visualización de imágenes han permitido mejorar procedimientos de cuidado en situaciones médicas de gran dificultad tales como cirugía cerebral. En la investigación del cáncer los nuevos métodos han permitido los estudios de la eficacia de la terapia sin la necesidad de la biopsia; la radiografía y los métodos de RM se aplican al sistema esquelético y a la microscopia del tejido.

Dentro de la TM, una técnica que ha surgido recientemente para la reconstrucción de actividad eléctrica cardiaca o cerebral a partir de potenciales eléctricos medidos fuera del cerebro y del corazón es la llamada Visualización de Fuentes Eléctricas (ESI por sus siglas en inglés: Electrical Source Imaging) la cual consiste en "reconstruir" la actividad cerebral o cardiaca a partir de los potenciales medidos fuera del cerebro o del corazón. La electroencefalografía y la electrocardiografía se utilizan para determinar la ubicación de fuentes de actividad bioeléctrica en el cuerpo, respectivamente, a partir de mediciones de voltajes y corrientes superficiales y se usa también en el diagnóstico y guía terapéutica relativa a la epilepsia y anormalidades de la conducción en el corazón ([18], [22]).

En la Tomografía de Procesos industriales se pretende reconstruir o identificar alguna característica de un sistema que puede ser eléctrica, mecánica química, etc.

En particular, la Tomografía de Capacitancia Eléctrica (TCE) y la Tomografía de Resistencia (TR), se enmarcan dentro de la Tomografía de Procesos. Actualmente, no se tiene un algoritmo implementado que permita resolver el problema de la TCE en tiempo real. La solución de éste problema para la TCE permitirá determinar la calidad del petróleo en el momento en que es extraído de los pozos. Un problema importante es el de desarrollar algoritmos e implementar programas de computadora (para esos algoritmos) que permitan visualizar un mapa de permitividades eléctricas de la sección de flujo en un corte transversal de una tubería. Más detalles de este problema así como una amplia bibliografía al respecto se encuentra en [8] y [23].

En este reporte presentamos algunos resultados sobre el problema de la visualización de fuentes bioeléctricas en el cerebro a partir de mediciones de potencial sobre el cuero cabelludo el cual se denomina Electroencefalografía Inversa. Estos resultados se obtienen por separado para el caso de fuentes en el volumen y en la corteza cerebral respectivamente, y garantizan la existencia y unicidad de solución del problema de identificación con lo que se pueden proponer algoritmos de identificación.

4.2 Problemas inversos y mal planteados.

Los problemas de identificación en general pertenecen a un grupo conocido como problemas inversos. Los problemas que se enunciaron en la sección anterior son ejemplos de problemas inversos. A continuación daremos la definición de problema bien planteado y posteriormente su relación con los problemas inversos.

Sean X, Y espacios de Hilbert, $K : X \to Y$ un operador y la ecuación

$$Kx = y \tag{4.1}$$

Se dice que la solución de la ecuación (1) es un problema bien planteado si:

1. Para cada y en Y existe una solución en X.

2. Para cada y en Y la solución es única en X.

3. La solución x del problema (1) depende continuamente de la variación de los datos y.

Cuando no se satisface alguna de las condiciones anteriores se dice que el problema es mal planteado. En términos generales, los problemas de identificación son mal planteados. También muchos problemas inversos que aparecen en las aplicaciones son mal planteados.

4.3 Métodos de regularización

Cuando los problemas son mal planteados, en cuanto a que presentan alguna inestabilidad numérica en su solución, esta puede ser tratada utilizando alguna técnica de regularización, que en lo que sigue explicaremos en que consiste. Consideraremos problemas inversos que están expresados en la forma operacional: $Kx = y$ donde K es un operador lineal compacto entre dos espacios de Hilbert X e Y separables. Puede suponerse que el operador K es uno a uno ya que en caso de no serlo, es posible reemplazar X por el complemento ortogonal del núcleo de K. Encontrar el núcleo de K puede llegar a ser muy complicado.

Supondremos que existe una solución de la ecuación no perturbada $Kx = y$. En la práctica sólo se conoce una aproximación y^δ del lado derecho, donde δ es conocido y se sabe además que la distancia entre ellas es menor que δ, es decir, $\|y - y^\delta\|_Y < \delta$. En este caso se espera encontrar una solución aproximada x^δ a la solución exacta x. Si el problema es sensible al error en las mediciones, es decir, es mal planteado, entonces deben aplicarse métodos de regularización tales como el método de regularización de Tikhonov y el método de regularización de Lavrentiev que son dos de los métodos más comúnmente usados para encontrar soluciones aproximadas de manera estable de la ecuación (4.1) ([2, 12 y 15]) bajo las condiciones mencionadas.

4.3.1 Método de regularización de Tikhonov

Una gran cantidad de problemas inversos pueden formularse como ecuaciones operacionales de la forma:

$$Kx = y$$

donde K es un operador compacto entre los espacios de Hilbert X y Y separables. Por lo dicho arriba supondremos que el operador K es uno a uno. Se supone además que existe una solución exacta $\bar{x} \in X$ de la ecuación con datos exactos $\bar{y}$, es decir, que $\bar{y} \in K(X)$.

En la práctica, en lugar de $\bar{y}$ se tiene una medición aproximada $y^\delta \in Y$, tal que

$$\|\bar{y} - y^\delta\|_Y < \delta. \tag{4.2}$$

A continuación se describe el método de regularización de Tikhonov que permite obtener una solución aproximada a $\bar{x}$ a partir de y^δ.

Consideremos para cada $\alpha > 0$, el llamado funcional de Tikhonov asociado a la ecuación (1):

$$J_\alpha(x) = \|Kx - y\|^2 + \alpha\|x\|^2. \tag{4.3}$$

Es conocido que J_α alcanza el mínimo en un único x^α el cual es la solución de la ecuación normal ([12], [14])

$$\alpha x^\alpha + K^*Kx^\alpha = K^*y$$

Denotaremos por $J_{\alpha,\delta}(x)$ al funcional de Tikhonov correspondiente a y^δ en lugar de y. Entonces, el famoso teorema de Tikhonov dice que si se toma $\alpha(\delta) > 0$ de manera que $\alpha(\delta) \to 0$ y $\frac{\delta^2}{\alpha(\delta)} \to 0$ cuando $\delta \to 0$, entonces se cumple que

$$\|\bar{x} - x^{\alpha(\delta)}\| \to 0 \tag{4.4}$$

Una forma de elegir $\alpha(\delta)$ cuando δ es un valor fijo, es de tal manera que satisfaga la ecuación $\|Kx^{\alpha(\delta)} - y^\delta\| = \delta$. Este método de elegir el parámetro de regularización es llamado el principio de discrepancia de Morozov ([12]) y garantiza que si K tiene rango denso en Y y $\|y - y^\delta\| \leq \delta < \|y^\delta\|$ entonces la solución $x^{\alpha(\delta)}$ cumple (4.4).

4.4 Algunos aspectos básicos del EEG.

El método de la electroencefalografía no ha sido tradicionalmente un método de visualización de imagénes, sin embargo es uno de los métodos no destructivos más conocidos en la investigación del cerebro y se basa en el registro de su actividad eléctrica a través de un electroencefalograma (EEG) y ha demostrado ser una herrramienta clínica muy valiosa. Por medio de este es posible detectar tumores cerebrales, condiciones epilépticas, enfermedades infecciosas, retardo mental, daño cerebral, etc. Además los potenciales obtenidos como respuesta a un estímulo (potenciales evocados) se muestran prometedores en el diagnóstico y tratamiento de enfermedades del sistema nervioso central ([18]).

4.4.1 EEG humano. Aspectos generales.

Historia

El cerebro vivo muestra una actividad eléctrica la cual puede ser registrada de la superficie de la cabeza (EEG). Esta actividad fue registrada por primera vez por Caton (1875; 1877) en Liverpool en monos y conejos. El primer registro de esta actividad en un humano fué publicado por Neminsky (1913), el cual no tuvo gran efecto sobre el pensamiento y conceptualización en la investigación cerebral.

El verdadero padre del EEG fue Hans Berger, profesor de Psiquiatría en Jena. Usando varios galvanómetros registró actividad rítmica de la cabeza de varios individuos. En su primer reporte, publicado en 1929, él señala la existencia de ondas alfa. Además, muchos de los fenómenos del EEG fueron descritos por él en algunos de sus reportes y en una monografía, publicados en 1937 y 1938, respectivamente.

Registro del EEG

El EEG se obtiene a partir de electrodos colocados en el cuero cabelludo de forma tal que la impedancia llegue a ser tan baja como sea posible. Después de muchas discusiones entre especialistas, un comité internacional recomienda el llamado sistema 10-20 para la colocación de los electrodos, es decir, entre 10 y 20 electrodos deben ser colocados en posiciones estándar como muestra esquemáticamente la figura 1 ([18], [22]). Actualmente se tienen sistemas de registro con 256 electrodos.

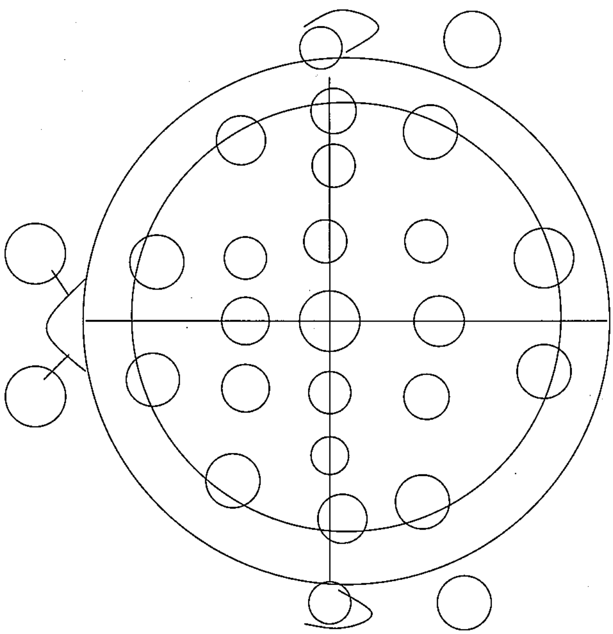

Figure 4.1: El sistema 10-20 para la colocación de los electrodos

Las características de los patrones individuales del EEG de un paciente, se encuentran probando en diferentes sitios, es decir, colocando los electrodos en diferentes puntos sobre el cuero cabelludo. Además el individuo examinado debe ser colocado sobre una silla confortable, relajado y con los ojos cerrados. La luz debe ser tenue y debe estar aislado al ruido. Los disturbios eléctricos que alteran fuertemente el EEG pueden aparecer por diferentes circunstancias, como por ejemplo, los electrodos están débilmente colocados, fallas en la máquina del EEG, el paciente tiene sus ojos y cuerpo en movimiento o por el ruido producido por el registro simultáneo de la actividad eléctrica del corazón.

Por esta razón, un buen registro del EEG depende de la experiencia y habilidad del técnico ([22]).

Evaluación del EEG

Tradicionalmente el EEG ha sido evaluado por inspección visual de las curvas obtenidas en los registros, tomando ciertos criterios tales como las formas de las ondas características de ciertos procesos cerebrales como por ejemplo los derivados de los ataques epilépticos, algunos tumores cerebrales y otros. Como primer paso las curvas se clasifican en 3 categorías: normal, patológico e intermedio. Entonces como segundo paso el observador trata de dar respuesta a muchas cuestiones específicas referentes al paciente bajo observación. Esta clasificación ha tenido gran éxito desde el punto de vista clínico; el especialista puede fiarse de este método debido a su experiencia ([22]).

Algunos científicos esperan que en un futuro no muy lejano, la interpretación del EEG a través de la experiencia del observador pueda ser reemplazada completamente por la evaluación computarizada ([15]).

4.5 Modelo de medio conductor para la actividad electroencefalográfica

En esta sección se presenta un modelo para estudiar un problema de identificación el cual ha sido usado ampliamente ([1, 4, 10, 13, 19, 21]). Se supone en este modelo que la cabeza $(\overline{\Omega})$ está compuesta por dos zonas disjuntas según lo muestra la figura 2, a saber:

1.Ω_1 – Cerebro   2.Ω_2 – Restantes regiones que componen la cabeza

Supondremos que $\Omega = \bigcup_{i=1}^{2} \Omega_i$ y que cada una de las componentes Ω_i tiene una conductividad constante σ_i, $i = 1, 2$ donde $\sigma_1 \neq \sigma_2$.

Figure 4.2: Modelo de la cabeza en capas conductoras

Mediante S_i se denotarán las superficies que componen las fronteras de las regiones Ω_i, $i = 1, 2$:

$$\partial\Omega_1 = S_1, \quad \partial\Omega_2 = S_1 \cup S_2$$

donde S_1 denota a la corteza cerebral y S_2 el cuero cabelludo.

Supondremos que las corrientes que pueden producirse en la región Ω se deben únicamente a la actividad eléctrica del cerebro y pueden ser de dos tipos: óhmicas e impresas. Las primeras se deben al movimiento de cargas iónicas a través del fluido extracelular en el cerebro y las segundas a las corrientes de difusión a través de las membranas neuronales ([18, 20]).

Denotaremos mediante $\vec{J}$ la densidad volumétrica de corrientes impresas en Ω_1 y mediante $\vec{j}$ la densidad superficial de corrientes impresas en S_1 (corteza cerebral). Es conocido que la densidad volumétrica total de corrientes en la región Ω_1 ocupada por el cerebro es ([20], p. 88): $\vec{J}_T^1 = \vec{J} + \sigma_1 \vec{E}_1$ donde $\vec{E}_1$ es el campo eléctrico generado por la actividad bioeléctrica y por la ley de Ohm, el término $\sigma_1 \vec{E}_1$ denota la densidad de corrientes óhmicas. En la región Ω_2 sólo consideraremos estas últimas, es decir, despreciaremos la actividad bioeléctrica.

Se puede demostrar que ([10]) es posible despreciar el término $\dfrac{\partial \rho_i}{\partial t}$ en la ecuación de continuidad: $\nabla \cdot \vec{J}_T^i + \dfrac{\partial \rho_i}{\partial t} = 0$ donde $\vec{J}_T^i$ y ρ_i denotan, respectivamente, la densidad de corriente y carga, en cada región Ω_i, $i = 1, 2$.

De esta forma obtenemos el modelo:

$$\nabla \cdot \left(\vec{J} + \sigma_1 \vec{E}_1 \right) = 0 \qquad \text{en} \quad \Omega_1 \tag{4.5}$$

$$\nabla \cdot (\sigma_2 \vec{E}_2) = 0 \qquad \text{en} \quad \Omega_2 \tag{4.6}$$

Se puede considerar que para el campo magnético $\vec{B}$ generado por la actividad eléctrica del cerebro, se cumple que $\frac{\partial \vec{B}}{\partial t} = 0$ ([20], p. 206). Por lo tanto, existe un potencial electrostático u tal que $\vec{E_i} = \nabla u_i$, $i = 1, 2$. Este potencial u satisface la ecuación:

$$\Delta u_1 = -\frac{1}{\sigma_1} \nabla \cdot \vec{J} \qquad \text{en} \quad \Omega_1 \tag{4.7}$$

$$\Delta u_2 = 0 \qquad \text{en} \quad \Omega_2 \tag{4.8}$$

Si denotamos por S_1 la frontera de Ω_1, S_2 la frontera exterior a Ω_2 y n_i, $i = 1, 2$ el vector normal unitario exterior a S_i, entonces las condiciones de continuidad del potencial eléctrico y de las componentes normales de la corriente en cada superficie $S_j \quad (j = 1, 2)$ nos dan las condiciones de contorno que se escriben en la forma ([21]):

$$\left.\begin{array}{rcll} u_1 & = & u_2 & \text{en} \quad S_1 \\ \sigma_1 \dfrac{\partial u_1}{\partial n_1} & = & \sigma_2 \dfrac{\partial u_2}{\partial n_1} + \vec{j} \cdot n_1 & \text{en} \quad S_1 \\ \dfrac{\partial u_2}{\partial n_2} & = & 0 & \text{en} \quad S_2 \end{array}\right\}$$

respectivamente, donde $\dfrac{\partial u_i}{\partial n_j}$ denota la derivada normal de u_i en S_j con respecto al vector n_j.

En lo que sigue estudiaremos el problema de contorno (4.7)-(4.9), al que llamaremos **Problema de Contorno Electroencefalográfico** (PCE) y utilizaremos la notación $f(x) = -\dfrac{1}{\sigma_1} \nabla \cdot J(x)$ y $j(x) = -(\vec{j} \cdot n)(x)$. En el caso en que $j \equiv 0$, el PCE es llamado Problema de Contorno Electroencefalográfico Volumétrico (PCEV) y en el caso en que $f \equiv 0$, es llamado Problema de Contorno Electroencefalográfico Cortical (PCEC).

Ahora enunciaremos algunas definiciones necesarias para plantear el **Problema Inverso Electroencefalográfico PIE** en términos del PCE.

Definición 4.5.1 *(Problema Directo asociado al PCE) Llamaremos Problema Directo asociado al PCE al problema que consiste en hallar la solución $u(x)$ del PCE cuando están dados $f(x)$ y $j(x)$. En el caso en que $j \equiv 0$, el problema directo asociado al PCE es llamado Problema Directo Electroencefalográfico Volumétrico (PDEV) y en el caso en que $f \equiv 0$, es llamado Problema Directo Electroencefalográfico Cortical (PDEC).*

Definición 4.5.2 *(Problema Inverso asociado al PCE,* **PIE***) Dada una función V definida sobre S_2 encontrar $f(x)$ y $j(x)$ de manera que para la solución $u(x)$ del problema directo correspondiente a $f(x)$ y $j(x)$ se cumpla que: $u_2|_{S_2} = V$. En el caso en que $j \equiv 0$, el problema inverso asociado al PCE es llamado Problema Inverso Electroencefalográfico Volumétrico (PIEV) y en el caso en que $f \equiv 0$, es llamado Problema Inverso Electroencefalográfico Cortical (PIEC).*

Observación: Nótese que en el PIE se trata de obtener $\nabla \cdot \vec{J}$ y $\vec{j} \cdot n_1$ a partir de mediciones del potencial generado sobre el cuero cabelludo. Sin embargo, lo importante es obtener $\vec{J}$ y $\vec{j}$ ya que precisamente el soporte de estas densidades nos permiten identificar las zonas activas tanto en el volumen del cerebro como en la corteza cerebral. Además, el problema de determinar $\vec{J}$ a partir de $\nabla \cdot \vec{J}$ no tiene solución única. Basta notar que si $\nabla \cdot J = 0$ entonces para cualquier función armónica v en Ω_1, la densidad $\vec{J} = \nabla v$ satisface la ecuación $\nabla \cdot \vec{J} = 0$. Luego para la determinación de $\vec{J}$ a partir de $\nabla \cdot \vec{J}$ se deben imponer otras condiciones adicionales.

4.6 Espacios de funciones. Traza de funciones

4.6.1 Espacios de Sobolev

Para cada subconjunto Ω_0 abierto y acotado del espacio euclidiano $\Re^n$, consideremos las siguientes clases de funciones:

$C(\overline{\Omega_0})$: Funciones complejas y continuas definidas sobre $\overline{\Omega_0}$.

$C^m(\overline{\Omega_0})$: Funciones complejas, con derivadas continuas sobre Ω_0 hasta el orden m, todas las cuales pueden ser prolongadas continuamente a $\overline{\Omega_0}$ $(1 \leq m \leq \infty)$.

Recordemos que el soporte de una función u definida sobre Ω_0 y con valores complejos es el complemento en Ω_0 del mayor subconjunto abierto donde u se anula y lo denotaremos por *supp u*.

Con $C_0(\Omega_0)$ denotamos al subconjunto de aquellas funciones en Ω_0 con soporte compacto. Similarmente:

$$C_0{}^m(\Omega_0) = C^m(\Omega_0) \bigcap C_0(\Omega_0), m \geq 1$$

y

$$C_0{}^\infty(\Omega_0) = C^\infty(\Omega_0) \bigcap C_0(\Omega_0).$$

Si $u : A \longrightarrow B$ y $C \subset A$, denotamos con $u|_C$ la *restricción* de u a C.

Mediante $L_p(\Omega_0)$ denotaremos la clase de todas las funciones complejas medibles $u(x)$ definidas sobre Ω_0 y tales que :

$$\int_\Omega |u(x)|^p dx < \infty$$

Es conocido que la aplicación:

$$|u|_p = (\int_{\Omega_0} |u(x)|^p dx)^{1/p}$$

es una norma en $L_p(\Omega_0)$ que provee a este conjunto de funciones de una estructura de espacio de Banach.
Diremos que la frontera $S_0 = \partial\Omega_0$ de la región $\Omega_0 \subseteq \Re^n$ pertenece a la clase $C^k (k \geq 1)$, si para cualquier punto $x_0 \in S_0$ existe una vecindad $U(x_0)$ de x_0 y un homeomorfismo:

$$f : \Re^{n-1} \longrightarrow U(x_0) \cap S_0$$

tal que $f \in C^k(\Re^{n-1})$.

Diremos que Ω_0 es un abierto con frontera regular o suave, si $\partial\Omega_0$ es una variedad orientable de clase C^∞ y dimensión $n-1$.

Para cada $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ de enteros no negativos con $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, denotamos por D^α la derivada parcial:

$$\frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} ... \partial {x_n}^{\alpha_n}}$$

Sea Ω_0 un conjunto abierto en $\Re^n$ y $m \geq 0$ un entero. Se tiene que $C^m(\overline{\Omega_0})$ es el espacio lineal de restricciones a $\overline{\Omega_0}$ de funciones de ${C_0}^m(\Re^n)$. Sobre $C^m(\overline{\Omega_0})$ se define un producto escalar por:

$$\langle u, v \rangle_{H^m(\Omega_0)} = \sum \left\{ \int_{\Omega_0} D^\alpha u.\overline{D^\alpha v} : |\alpha| \leq m \right\}$$

Definición 4.6.1 *Se llama Espacio de Sobolev de orden m y se denota por $H^m(\Omega_0)$ al completamiento del espacio lineal $C^m(\overline{\Omega_0})$ con la norma $\|\cdot\|_{H^m(\Omega_0)}$.*

Notemos que la norma de $H^0(\Omega_0)$ y la norma de $L_2(\Omega_0)$ coinciden sobre $C(\overline{\Omega_0})$.

Definición 4.6.2 *(Derivada en sentido de Sobolev) La función $v(x)$ en Ω_0 se llama derivada de orden α en sentido de Sobolev de la función $u(x)$ en Ω_0 y se denota por $D^\alpha u$, si para cualquier función $\varphi \in {C_0}^\infty(\Omega)$ se satisface la igualdad*

$$\int_\Omega v(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \varphi(x) dx$$

De la definición dada vemos que si $u(x) \in C^k(\overline{\Omega_0})$, entonces $u(x)$ posee en Ω_0 derivadas en sentido de Sobolev hasta el orden k que coinciden con sus derivadas en sentido usual. Sin embargo, hay funciones para las cuales existe la derivada en sentido de Sobolev sin ser derivables en el sentido usual. Notemos también, que a diferencia de la definición clásica de derivada, la derivada en sentido de Sobolev $D^\alpha u$ de orden $|\alpha|$ se define por la igualdad dada, independientemente de las derivadas generalizadas de orden inferior y es además un concepto global y no local como la derivada en sentido usual. Se tiene el siguiente teorema que caracteriza a los elementos de $H^m(\Omega_0)$.

Teorema 4.6.1 *Una función $u \in L_2(\Omega_0)$ pertenece a $H^m(\Omega_0)$ si y solo si, se puede encontrar una sucesión $u_k \in C^k(\overline{\Omega_0})$ tal que $u_k \to u$ en $L_2(\Omega_0)$ y las sucesiones $D^\alpha u_k$ son de Cauchy en $L_2(\Omega_0)$ para todo α con $|\alpha| \leq m$.*

A continuación se enuncian sin demostración los siguientes teoremas relativos a las propiedades de los espacios de Sobolev ([3]).

Teorema 4.6.2 *(Densidad de $C^\infty(\overline{\Omega})$ en $H^m(\Omega)$) Supongamos que la frontera $\partial\Omega$ del dominio Ω pertenece a la clase C^m. El conjunto de funciones de clase $C^\infty(\overline{\Omega})$ y por tanto $C^m(\overline{\Omega})$ es siempre denso en el espacio $H^m(\Omega)$)*

Teorema 4.6.3 *(Compacidad de la inmersión entre espacios de Sobolev) Sea Ω una región acotada de $\Re^n$ con frontera regular y $k_1 > k_2$ ($k_1, k_2 \in N$). Entonces la inmersión $H^{k_1}(\Omega) \longrightarrow H^{k_2}(\Omega)$, es compacta, es decir todo conjunto acotado en $H^{k_1}(\Omega)$ es relativamente compacto en $H^{k_2}(\Omega)$.*

Teorema 4.6.4 *(Regularidad de las funciones de $H^k(\Omega)$)Supongamos que Ω es una región acotada de $\Re^n$ con frontera regular. entonces para $k > n/2$, tiene lugar la inclusión: $H^k(\Omega) \subseteq C^m(\overline{\Omega})$ donde $m = k - \left[\frac{n}{2}\right] - 1$. Además la inmersión $H^k(\Omega) \longrightarrow C^m(\overline{\Omega})$ es compacta si se considera en $C^m(\overline{\Omega})$ la norma $\|.\|_\infty$.*

4.6.2 Restricción y traza de funciones sobre la frontera de una región.

Consideremos una función $u \in C(\overline{\Omega}_0)$ donde $\overline{\Omega}_0$ denota la cerradura de una región Ω_0 de $\Re^n$, $n = 1, 2$. Para la función u tiene sentido hablar de su restricción a la frontera $S_0 = \partial\Omega_0$. Sin embargo, para funciones de $L_2(\Omega_0)$, las cuales son iguales si difieren en un conjunto de medida nula, no es posible hablar, en general, de su restricción a S_0. Para poder dar un sentido a las condiciones de contorno cuando los problemas se plantean en $L_2(\Omega_0)$ se introduce el concepto de traza.

Sea $u \in H^1(\Omega_0)$. Existe una sucesión de funciones $u_k \in C^1(\overline{\Omega_0})$ que converge, en la norma de $H^1(\Omega_0)$ a u. Puede demostrarse ([17], p. 154) que la sucesión de restricciones $u_k|_{S_0}$ es de Cauchy en $L^2(S_0)$ y por lo tanto, converge hacia una función $u_0 \in L_2(\Omega_0)$. Además resulta que la función u_0 no depende de la elección de la sucesión $u_k \in C^1(\overline{\Omega_0})$ que converge a u en $H^1(\Omega_0)$.

Por lo anterior, a la función $u_0 \in L_2(S_0)$ se la llama traza de la función $u \in H^1(\Omega_0)$ y se denota mediante $u_0 := tr(u)$.

Además se cumple el siguiente teorema.

Teorema 4.6.5 *El operador* $tr : H^1(\Omega_0) \to L_2(S_0)$ *es compacto.*

Denotamos por $H_0^m(\Omega_0)$ la cerradura en $H^m(\Omega_0)$ de $C_0^\infty(\Omega_0)$.

4.7 Resultados para fuentes volumétricas

4.7.1 Resultados de existencia y unicidad de la solución débil

Consideremos los siguientes espacios de funciones: $L_2(\widetilde{\Omega})$, $L_2(\Omega_i)$ y $L_2(S_i)$, los espacios de funciones de cuadrado integrable definidas sobre $\widetilde{\Omega}$, Ω_i y S_i, respectivamente, $i = 1, 2$ donde $\widetilde{\Omega} = \overline{\Omega}_1 \cup \Omega_2$; $H^1(\widetilde{\Omega})$ el espacio de Sobolev de funciones de $L_2(\widetilde{\Omega})$ cuya primera derivada generalizada pertenece a $L_2(\widetilde{\Omega})$.

En lo que sigue, para cada función v definida en $\widetilde{\Omega}$, v_i denota la restricción de v a Ω_i, $i = 1, 2$, es decir, $v_i = v|_{\Omega_i}$, $i = 1, 2$.

Definición 4.7.1 *La función* $u \in H^1(\widetilde{\Omega})$ *es solución débil del PCEV si se cumple*

$$\int\limits_{\Omega_1} \sigma_1 f v_1 d\Omega_1 = -\left\{ \int\limits_{\Omega_1} \sigma_1 \nabla u_1 \cdot \nabla v_1 d\Omega_1 + \int\limits_{\Omega_2} \sigma_2 \nabla u_1 \cdot \nabla v_2 d\Omega_2 \right\} \tag{4.9}$$

$\forall\, v \in H^1(\widetilde{\Omega})$.

Teorema 4.7.1 *Para que exista solución débil del PCEV con* $f \in L_2(\Omega_1)$ *es necesario y suficiente que se cumpla que:*

$$\int\limits_{\Omega_1} f d\Omega_1 = 0 \tag{4.10}$$

En este caso existe una única solución débil u que es ortogonal a las constantes en el producto escalar de $L_2(\widetilde{\Omega})$ y para lo cual se tiene el estimado

$$\| u \|_{H^1(\widetilde{\Omega})} \leq C \|f\|_{L_2(\Omega_1)} \tag{4.11}$$

donde C es una constante que no depende de f. Cualquier otra solución débil se diferencia de u es una constante.

Si f satisface la condición (4.10) diremos que $f \in L_2^{\perp}(\Omega_1)$.

4.7.2 Resultados de existencia y unicidad del PIEV

Sea A el operador que a cada $f \in L_2^{\perp}(\Omega_1)$ le asocia la traza a S_2 de la única solución débil $u \in H^1(\widetilde{\Omega})$ del PCEV que es ortogonal a las constantes en el producto escalar de $L_2(\widetilde{\Omega})$. Obsérvese que, de la desigualdad (4.11), se deduce que el operador $T : L_2^{\perp}(\Omega_1) \to H^1(\widetilde{\Omega})$ que asocia a f la solución débil u ortogonal a las constantes, es un operador continuo. A es la composición de T con el operador traza a la frontera exterior S_2 (definido en el teorema 4.6.2). Para cada función $u \in H^1(\widetilde{\Omega})$ existe la traza a S_2 ([17]). Ya que el primer operador es continuo de $L_2^{\perp}(\Omega_1)$ en $H^1(\widetilde{\Omega})$ y el segundo es compacto de $H^1(\widetilde{\Omega})$ en $L_2(S_2)$ se deduce que A es compacto.

Se tienen los resultados siguientes que pueden ser encontrados en [5]:

Teorema 4.7.2 $[N(A)]^{\perp} = H_1(\Omega_1)$ *donde $N(A)$ es el núcleo de A y $H_1(\Omega_1)$ es el subespacio de $L_2(\Omega_1)$ formado por las funciones armónicas que son ortogonales a las constantes.*

En el teorema siguiente, Δ^2 denota el operador biarmónico.

Teorema 4.7.3 *Un subespacio propio de $Im(A)$, que es denso en $L_2(S_2)$, está constituido por las funciones $V \in L_2^{\perp}(S_2)$ para las cuales las trazas a S_1 de la solución y su derivada normal, del problema de Cauchy*

$$\begin{array}{rclcl} \Delta u_2 & = & 0 & en & \Omega_2 \\ u_2 & = & V & en & S_2 \\ \frac{\partial u_2}{\partial n_2} & = & 0 & en & S_2 \end{array} \tag{4.12}$$

son tales que si $u_2|_{S_1} = \varphi$ y $\left.\frac{\partial u_2}{\partial n_1}\right|_{S_1} = \psi$, entonces el problema de contorno

$$\begin{array}{rclcl} \Delta^2 u_1 & = & 0 & en & \Omega_1 \\ u_1 & = & \varphi & en & S_1 \\ \frac{\partial u_1}{\partial n_1} & = & \psi & en & S_1 \end{array} \tag{4.13}$$

tiene solución en $H^1(\Omega_1)$.

Estos resultados permiten hallar una solución del PIEV. Se ha comentado anteriormente que este problema presenta una inestabilidad numérica, la cual, en este caso, se deriva del hecho de que A es compacto.

4.8 Caso de círculos concéntricos

4.8.1 Caso de datos exactos

Teorema 4.8.1 *Supongamos que S_1 y S_2 son círculos concéntricos de radios R_1 y R_2 respectivamente. Entonces la condición necesaria y suficiente para que la fuente f pueda ser identificada en $L_2(\Omega_1)$ de manera única a partir del conocimiento exacto de $u_2|_{S_2} = V$ es que se cumpla lo siguiente:*

1. *f se busca en la clase de las funciones armónicas definidas sobre Ω_1.*

2. *Los coeficientes de Fourier de V satisfacen la condición:*

$$\sum_{k=1}^{\infty} k^4 \left(\frac{R_2}{R_1}\right)^{2k} \{|V_k^1|^2 + |V_k^2|^2\} < +\infty \tag{4.14}$$

Además, si se cumplen estas dos condiciones, entonces f viene dada por la fórmula $\Delta u_1 = f$, donde u_1 es la solución del problema (4.7.3).

Supondremos que el dato sin error $u_2|_{S_2} = V$ satisface la condición (4.14) y denotaremos por V_δ el dato con error de manera que $\|V - V_\delta\|_{L_2(S_2)} \leq \delta$.

En este caso propondremos, la correspondiente solución del problema de identificación en la forma:

$$\begin{aligned} & f_\delta(r,\theta) \\ &= \frac{1}{R_1^2}\sum_{k=1}^{N}\left\{k(k+1)\left(\frac{r}{R_1}\right)^k\left[\left(\frac{\sigma_2}{\sigma_1}-1\right)\left(\frac{R_1}{R_2}\right)^k-\left(\frac{\sigma_2}{\sigma_1}+1\right)\left(\frac{R_2}{R_1}\right)^k\right]\right. \\ &\qquad\qquad \left.\times\left(V_{k,\delta}^1 cosk\theta + V_{k,\delta}^2 senk\theta\right)\right\} \end{aligned} \tag{4.15}$$

donde $V_{k,\delta}^1$ y $V_{k,\delta}^2$ son los coeficientes de Fourier y N será elegido en función de δ de manera conveniente.

Si suponemos la condición adicional:

$$\pi \sum_{k=1}^{\infty} k^5 \left(\frac{R_2}{R_1}\right)^{2k} (|V_k^1|^2 + |V_k^2|^2) \leq E \tag{4.16}$$

se obtiene el estimado:

$$\|f - f_\delta\|^2_{L_2(\Omega_1)} \le 4\left(\frac{\sigma_2}{\sigma_1}+1\right)^2 \left\{N^3\left(\frac{R_2}{R_1}\right)^{2N}\delta^2 + \frac{4E}{N}\right\} \tag{4.17}$$

Definamos $N(\delta)$ como el valor entero más cercano al punto donde alcanza el mínimo, con respecto a N, la función en la parte derecha de (4.17). Dicha función es convexa y alcanza el mínimo en la raíz de la ecuación:

$$3N^2\left(\frac{R_2}{R_1}\right)^{2N}\delta^2 + 2N^3\left(\frac{R_2}{R_1}\right)^{2N}\delta^2 ln\left(\frac{R_2}{R_1}\right) = \frac{4E}{N^2} \tag{4.18}$$

En este caso

$$\|f - f_\delta\|_{L_2(\Omega_1)} \le \frac{\sqrt{32E}\left(\frac{\sigma_2}{\sigma_1}+1\right)\sqrt{5+2ln\frac{R_2}{R_1}}}{\sqrt{ln\frac{4E}{3+2ln\frac{R_2}{R_1}} + ln\frac{1}{\delta^2}}} \tag{4.19}$$

Finalmente se tiene el teorema siguiente:

Teorema 4.8.2 *Supondremos que el dato exacto en el plantemiento del problema de identificación asociado a (4.7)-(4.9), donde el término $\vec{j}\cdot n$ se considera nulo, satisface la condición apriori de suavidad (4.16), y que en el caso de que se tenga el dato con error V_δ, $\|V - V_\delta\|_{L_2(S_2)} \le \delta$, se busca una expresión aproximada f_δ de la fuente f en forma de la serie truncada (4.15).*
Sea $N(\delta)$ el entero más cercano a la única raíz real positiva de la ecuación (4.18).
Entonces la expresión (4.15) con $N = N(\delta)$ es una estrategia regularizante para el cálculo estable de f_δ con respecto a la norma en $L_2(\Omega_1)$ si se satisface la condición (4.16).
Para esta elección de $N(\delta)$ se obtiene el estimado (4.19).

Las pruebas de estos resultados pueden consultarse en [5].

4.9 Resultados para el caso de fuentes concentradas en corteza cerebral.

4.9.1 Un modelo simplificado para el estudio del PCEC

Supongamos ahora que en el PCE tenemos $f \equiv 0$, es decir, no tenemos fuentes volumétricas.

En lo que sigue se dan argumentos que permiten estudiar el problema de identificación de fuentes corticales por medio de un modelo simplificado.Algunas consideraciones anatómicas y fisiológicas justifican el modelo propuesto; una de ellas es que la corteza cerebral tiene circunvoluciones con una altura de aproximadamente 2.5 mm y en las crestas y valles la actividad neuronal representa el 1% de la actividad cortical total. Por otra parte, la actividad es generada por completo por neuronas piramidales que producen un efecto similar a las capas dipolares distribuidas en S_1 y en consecuencia, ellas generan una densidad de corriente primaria $\vec{j}$ que puede ser considerada directamente en la dirección del vector normal n_1 a S_1.

De esta forma resulta que la componente normal de la corriente primaria en la superficie S_1 puede ser despreciada en las crestas y valles de la corteza cerebral. En la parte restante de S_1 la componente normal puede considerarse como la componente tangencial en una superficie sin circunvoluciones.

Ahora, tomando en cuenta los argumentos dados arriba, si continuamos denotando por S_1 a la superficie suavizada (que representará a la corteza cerebral) se tiene que en dicha superficie la componente normal de $\vec{j}$ es nula. Si denotamos por u_i los potenciales en Ω_i, $i = 1, 2$ producidos por $\vec{j}$, se obtiene el modelo simplificado que se introdujo en [6].

$$\Delta u_i = 0, \qquad \Omega_i,\ i = 1, 2 \tag{4.20}$$

$$\left.\begin{array}{rclcc} u_1 & = & u_2 & \text{en} & S_1 \\ \sigma_1 \dfrac{\partial u_1}{\partial n_1} & = & \sigma_2 \dfrac{\partial u_2}{\partial n_1} & \text{en} & S_1 \\ \dfrac{\partial u_2}{\partial n_2} & = & 0 & \text{en} & S_2 \end{array}\right\} \tag{4.21}$$

Al problema (4.20) -(4.21) se le llama Problema de Contorno Electroencefalográfico Simplificado y se denota por PCES.

Por medio de la solución del PCES es posible hallar la componente tangencial de $\vec{j}$, en cada dirección tangencial τ sobre S_1, a través de la expresión siguiente

$$\sigma_1 \frac{\partial u_1}{\partial \tau} - \sigma_2 \frac{\partial u_2}{\partial \tau} = \vec{j} \cdot \tau \tag{4.22}$$

4.9.2 Solución clásica y débil del PCES y su representación como suma de potenciales de superficie

Para el análisis de la solución del PCES se estudiará el siguiente problema de contorno más general:

$$\Delta u_i = f_i, \qquad \Omega_i, \ i = 1, 2 \tag{4.23}$$

con las condiciones de contorno (4.21).

Para ello consideremos los espacios de funciones $L_2(\Omega_i)$ y $L_2(S_i)$, respectivamente, para $i = 1, 2$; $H^1(\widetilde{\Omega})$ el espacio de Sobolev de funciones de $L_2(\widetilde{\Omega})$ cuya primera derivada generalizada pertenece a $L_2(\widetilde{\Omega})$ donde $\widetilde{\Omega} = \overline{\Omega_1} \cup \Omega_2$; $H_0^1(\widetilde{\Omega})$ el subespacio de $H^1(\widetilde{\Omega})$ de funciones cuya traza a S_2 es nula; $H^{\frac{1}{2}}(S_i)$; el espacio de Banach de funciones de $L_2(S_i)$, $i = 1, 2$, las cuales son la traza de funciones de $H^1(\widetilde{\Omega})$ provisto la norma

$$\|\varphi\|_{H^{\frac{1}{2}}(S_i)} = \inf\{\|\phi\|_{H^1(\widetilde{\Omega})} : \phi \in H^1(\widetilde{\Omega}), \phi|_{S_i} = \varphi\} \tag{4.24}$$

Definición 4.9.1 *La función $u \in H^1(\widetilde{\Omega})$ es una solución débil del problema (4.23) con las condiciones de contorno (4.21) si su traza a S_2 coincide con φ y se satisface la siguiente identidad:*

$$\sum_{i=1}^{2} \sigma_i \left[\int_{\Omega_i} f_i v_i dx + \int_{\Omega_i} \nabla u_i \nabla v_i dx \right] = 0 \tag{4.25}$$

para todo $v \in H_0^1(\widetilde{\Omega})$.

Teorema 4.9.1 Si $f_i \in L_2(\Omega_i)$, $i = 1, 2$ y $\varphi \in H^{1/2}(S_2)$, la solución débil u del problema (4.23) con las condiciones de contorno (4.21) existe, es única y satisface la desigualdad

$$\|u\|_{H^1(\widetilde{\Omega})} \leq C \left\{ \sum_{i=1}^{2} \|f_i\|_{L_2(\Omega_i)} + \|\varphi\|_{H^{1/2}(S_2)} \right\} \tag{4.26}$$

donde la constante C no depende de φ ni de f_i, $i = 1, 2$.

Del teorema previo se deduce que el operador T definido por medio de $\varphi \to u|_{S_1}$ donde u es la correspondiente solución débil, (es decir, en el caso cuando $f_i \equiv 0 \quad i = 1, 2$), es un operador continuo de $H^{1/2}(S_2)$ en $H^{1/2}(S_1)$ y compacto de $H^{1/2}(S_2)$ in $L_2(S_1)$. Por otra parte, si la superficie

S_2 pertenece a la clase C^1 y $\varphi \in C^1(S_2)$, de (4.26) se obtiene la siguiente desigualdad (Capítulo IV, §1, sección 8, p. 217 de [17]):

$$\|u\|_{H^1(\Omega_0)} \leq C_2 \|\varphi\|_{C^1(S_2)} \tag{4.27}$$

de donde se deduce que

$$\|u_2|_{S_1}\|_{L_2(S_1)} \leq C_3 \|\varphi\|_{C^1(S_2)} \tag{4.28}$$

y en este caso el operador T de $C^1(S_2)$ in $L_2(S_1)$ es un operador compacto.

Este resultado muestra que el problema de recobrar el potencial $u|_{S_1}$ a partir de las mediciones φ sobre S_2 es un problema bien planteado.

Definición 4.9.2 *La función*

$$u \in C(\overline{\Omega_0}) \cap C^2(\Omega_1) \cap C^2(\Omega_2) \cap C^1(\overline{\Omega_1}) \cap C^1_1(\Omega_2) \tag{4.29}$$

donde $C^1_1(\Omega_2)$ es el subespacio de $C^1(\Omega_2)$ de funciones que pueden prolongarse continuamente a S_1, junto con sus primeras derivadas, es una solución clásica del problema (4.23) con condiciones de frontera (4.21) si satisface esas relaciones en el sentido clásico de diferenciación y la traza.

Expresaremos la solución del PCES a través de potenciales de capa simple y doble ([2] y [14]).

En este caso, cuando en el PCES, las superficies S_i, $i = 1, 2$ pertenecen a la clase C^2 y $\varphi \in C(S_2)$, la solución clásica puede buscarse como suma de potenciales de capa simple y capa doble:

$$u(x) = \int\limits_{S_1} \frac{\rho_1(y)}{|x-y|} ds^1_y + \int\limits_{S_2} \frac{\partial}{\partial n^y_2} \left(\frac{1}{|x-y|} \right) \rho_2(y) ds^2_y \tag{4.30}$$

para $x \in \Omega_0$, con densidades continuas ρ_i en S_i, $i = 1, 2$ y

$$\frac{\partial}{\partial n^y_2} \left(\frac{1}{|x-y|} \right)$$

denota la derivada normal de la función $\frac{1}{|x-y|}$ respecto a y.

La función $u(x)$ dada por (4.30) es armónica en Ω_i, $i = 1, 2$. El potencial $u(x)$ es solución del PCES si u satisface las condiciones de frontera (4.21). Después de sustituir (4.30) en esas condiciones y usando las fórmulas de Sojovstky, se obtiene la siguiente relación operacional matricial para las densidades ρ_i $i = 1, 2$:

$$\left[\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - 2\pi \begin{pmatrix} I_{11} & \\ & I_{22} \end{pmatrix}\right] \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \tag{4.31}$$

donde el operador matricial $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ es un operador compacto en $C(S_1) \times C(S_2)$, I_{ii} denota el operador identidad in $C(S_i)$, $i = 1, 2$ y los operadores integrales:

$A_{ij} : C(S_j) \to C(S_i)$ se definen de la manera siguiente:

$$\begin{aligned}
A_{11}(\rho_1)(x) &= -P.V. \int\limits_{S_1} \tfrac{\partial}{\partial n_1^x} \left(\tfrac{1}{|x-y|}\right) \rho_1(y) ds_y \\
A_{12}(\rho_2)(x) &= -\tfrac{\sigma_2}{\sigma_1} \int\limits_{S_2} \tfrac{\partial}{\partial n_1^x} \tfrac{\partial}{\partial n_2^y} \left(\tfrac{1}{|x-y|}\right) \rho_2(y) ds_y \\
A_{21}(\rho_1)(x) &= \int\limits_{S_1} \tfrac{\rho_1(y)}{|x-y|} ds_y \\
A_{22}(\rho_2)(x) &= P.V. \int\limits_{S_2} \tfrac{\partial}{\partial n_2^y} \left(\tfrac{1}{|x-y|}\right) \rho_2(y) ds_y
\end{aligned} \tag{4.32}$$

donde $P.V.$ denota el valor principal de Cauchy de la integral. Nótese que los operadores A_{ii}, $i = 1, 2$ son singulares.

De esta forma se prueba que el PCES permite recobrar, en forma estable, las densidades de carga en S_1 y las densidades de dipolos en S_2, el mapeo potencial en S_1 y la componente superficial de la corriente en direcciones tangenciales a S_1, a partir de mediciones suavizadas del potencial eléctrico medido en S_2. La unicidad de solución de este problema puede encontrarse en [7] y [9].

Los resultados de esta sección sugieren que, a fin de resolver el PIE usando el PCES en el caso cuando se tienen solamente un número finito de mediciones de φ en S_2, las mediciones mencionadas deben interpolarse en una manera estable, en un espacio de funciones suaves como, por ejemplo, $C^1(S_2)$ antes de iniciar el proceso de identificación de las fuentes.

Una dificultad que se presenta en este problema, adicional a la posibilidad de que el problema sea mal planteado, es que hay diversas opiniones acerca de si el mapeo potencial, la densidad de carga, la densidad de dipolos o la distribución de densidad de corrientes primarias (impresas) generadas por la actividad de células corticales representan alguna característica de la actividad bioeléctrica cortical.

Puede ocurrir que, dependiendo del modelo matemático empleado, este problema de identificación puede o no ser mal planteado, lo cual es mostrado en [9].

Las pruebas de estos resultados se encuentran publicados en [9].

4.10 Conclusiones

En este trabajo se presentan algunos resultados para el problema de identificación de fuentes bioeléctricas concentradas en el volumen y en la corteza cerebral.

En el caso de fuentes volumétricas se presentan condiciones bajo las cuales el problema de identificación tiene solución única, y para el caso particular en el que la geometría corresponde a círculos concéntricos, se presenta un algoritmo que permite recuperar la fuente en forma estable. Además se da una estrategia de regularización cuyo parámetro de regularización es el orden del polinomio en el que se truncan las series con las que se representan las fuentes y los potenciales.

Para el caso de fuentes en corteza, por medio de un modelo simplificado junto con el uso de técnicas de la teoría de potencial, se propone un algoritmo de identificación. Los resultados encontrados sugieren que, antes de iniciar el proceso de identificación, las mediciones deben interpolarse en un espacio de funciones suaves.

4.11 Bibliografía

1. Amir A.,1994. Uniqueness of the generators of brain evoked potential maps. IEEE Transactions on Biomedical Engineering. Vol. 41, pp. 1-11.

2. Colton D., Kress R., 1992. Integral equations methods in scattering theory. Kriuger Publishing Company, Malabar Florida.

3. Fraguela A. 1991. Teoría Espectral de Operadores Diferenciales. Publicado por el CINVESTAV I.P.N.

4. Fraguela A. Morín M., Oliveros J., 1998. Planteamiento del Problema Inverso de Localización de los Parámetros de una Fuente de Corriente Neuronal en Forma de Dipolo. Aportaciones Matemáticas, Serie Comunicaciones 25, pp. 41-55. Sociedad Matemática Mexicana.

5. Fraguela A. Morín M., Oliveros J., 2005. Inverse Electroencephalography for volumetric sources. Sometido en el Journal for Mathematics and Computers Simulation.

6. Fraguela A, Oliveros J Grebennikov A., 2001. Planteamiento Operacional y Análisis del Problema Inverso Electroencefalográfico. Revista Mexicana de Física, Vol. 47, N. 2, pp. 162-174.

7. Fraguela A., Oliveros J., Grebénnikov A., 2001. Uniqueness of Solution of the Inverse Electroencephalographic Problem. Proceedings of the Conference: Numerical Analysis and Its Applications, Second International Conference, NAA 2000, Rousse, Bulgaria, June 2000. Eds. L.G Vulkov, Jerzy Wasniewski, Plamen Yamalov, Lectures Notes in the Computer Science, Vol. 1988, Springer Verlag, Berlin, New York.

8. Fraguela A., Oliveros J., Morín M., Gómez S., Cervantes L. 2005. Un algotitmo no iterativo para la Tomografía de Capacitancia Eléctrica. Revista Mexicana de Física, Vol. 51, N.3, pp. 236-242.

9. Fraguela A,, Oliveros J., Morín M., Cervantes L., 2005. Inverse Electroencephalography for Cortical Sources. Journal of Applied Numerical Mathematics, 55,pp. 191-203.

10. Heller L., 1990. Return Current in Encephalography. Variational Principles. Biophysical Journal, Vol. 57, pp. 601-607.

11. Keller J. B.,1976. Inverse Problems. Am. Math. Mon., 83:107-118.

12. Kirsch A., 1996. An Introduction to the Mathematical Theory of Inverse Problems. Springer Verlag.

13. Koptelov, Yu. M., Zakharov, E. V.,1991. Inverse Problems in Electroencephalography and Their Numerical Solving. Ill-posed Problems in Natural Sciences, pp. 543-552, VSP, Utrecht, 1992. 92C55.

14. Kress R., 1989. Linear Integral Equations. Springer Verlag.

15. Lavrentiev M., Romanov V., 1986. Ill Posed Problems in Mathematical Physics and Analysis. American Mathematical Society, Providence Rhode Island.

16. Malmivuo J., 1999. The Theoretical Limits of EEG Methods are not yet Reached. International Journal of Bioelectromagnetism. N. 1, Vol. 1

17. Mijailov V.P., 1982. Ecuaciones Diferenciales en Derivadas Parciales. Editorial Mir, Moscu.

18. Nunez P. L., 1981. Electric Field of the brain. N. Y., Oxford Univ. Press.

19. Pascual-Marqui R., 1999. Review of Methods for Solving the EEG Inverse Problems. International Journal of Bioelectromagnetism. N. 1, Vol. 1.

20. Plonsey R., Fleming D. G., 1969. Bioelectric Phenomena. N.Y. Mc Graw-Hill.

21. Sarvas J., 1987. Basic Mathematical and Electromagnetic Concepts of the Biomagnetic inverse Problem. Phys. Med. Biol., Vol. 32, No. 1, pp. 11-22.

22. Vogel F., 2000. Genetics and the Electroencephalogram. Springer.

23. Yang. W., 2003. Image Reconstruction Algorithm for electrical capacitance tomography, Meas, Sci Tecnol., Vol. 14, R1-R13.

Capítulo 5

Mejor aproximación polinomial en bandas no uniformes

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Abstract: A new concept related with best Chebyshev polynomial approximation is presented. We present some results concerning existence of the new approximants. A characterization theorem is included.

Resumen: Se introduce un concepto nuevo relativo a la mejor aproximación polinomial de Chebyshev. Se anuncia la existencia, unicidad y caracterización de esta mejor aproximación polinomial.

Sea $[a,b]$ un intervalo real. Consideremos funciones $f, h_1, h_2 \in C[a,b]$, $h_1, h_2 \geq 0$, y el conjunto factible de los vectores de $\mathbb{R}^{n+1}$, cuyas coordenadas son los coeficientes de los polinomios p de grado a lo más n, tales que

$$f - h_1 \leq p \leq f + h_2, \text{ sobre } [a,b].$$

Este tipo de conjuntos conduce a interesantes problemas simultáneos de aproximación y de optimización estudiados por Guerra y Jiménez desde el punto de vista de la programación seminfinita [6], y posteriormente por otros matemáticos, por ejemplo [8], [10] y [9].

Inspirados en esos trabajos de aproximación y optimización y particularmente en una generalización del teorema de alternancia de Chebyshev de Guerra-Jiménez, desarrollamos un concepto de aproximación de Chebyshev en bandas no uniformes que presentamos aquí, en el epígrafe 2. En realidad, existen muchas variaciones y extenciones de este famoso teorema de Chebyshev constituyente de los fundamentos de la Teoría de la Aproximación (vea, en nuestro contexto, el artículo relativamente reciente de Tijomirov [13], y la teoría de Kolmogorov en [5]); pero en cada problema concreto se deben comprobar hipótesis y precisar los resultados.

El teorema de Chebyshev clásico caracteriza los polinomios de la mejor aproximación y determina su unicidad. En el contexto de nuestro estudio, debemos precisar previamente lo que entederemos como mejor aproximación en bandas no uniformes y proceder al análisis de las diferentes preguntas que usualmente se formulan en ese tópico de la aproximación. Este estudio se desarrolla satisfactoriamente para h_1 y h_2 estríctamente positivas. En el epígrafe 3 presentamos una colección de ejemplos que muestran la riqueza -pero a la vez la limitación y dificultades- del caso en el que h_1 y h_2 tengan ceros. Este último caso continúa siendo objeto de estudio por nuestra parte y confiamos en presentarlo en forma más o menos completa en una próxima publicación.

Los resultados de Teoría de la Aproximación que aquí utilizamos son clásicos y pueden encontrarse en la mayoría de los textos sobre el tema (por ejemplo, en los textos clásicos [4], [12], o más reciente [5]).

Debemos señalar que durante el periodo de tiempo en que nos ocupabamos de este estudio, se publicó un artículo en el cual se estudian problemas abstractos de la mejor aproximación [1], que tiene puntos de conexión con nuestros resultados para el caso en que h_1 y h_2 no tienen ceros comunes.

5.1 Mejor aproximación de Chebyshev

Sean (X, d) un espacio métrico y $Y \subset X$, $Y \neq \phi$, dado $x \in X$, consideremos el problema de encontrar un elemento $y^* \in Y$, tal que $d(y^*, x) \leq d(y, x)$, para todo $y \in Y$.

No siempre es posible encontrar tales elementos; pero cuando existen se les denomina como *mejor aproximación* de Y a x. Sin embargo, aún teniendo demostrada la existencia, no siempre se puede garantizar la unicidad de estos elementos. Es fácil demostrar, sin embargo, que si X es un espacio vectorial normado, $x \in X$ y Y un subespacio de X, entonces el conjunto de mejores aproximaciones de x a Y es un conjunto convexo.

Recordemos algunos detalles introductorios para el problema que abordaremos en esta exposición .

Sean $X := C\,[a,b]$ y $Y := P_n(x)$, el conjunto de los polinomios de grado a lo más n y la norma

$$\|f\| = \underset{x\in[a,b]}{máx} |f(x)| .$$

Para demostrar la existencia de la mejor aproximación a una función, por polinomios de grado a lo más n, con $n \in \mathbb{N}$ fijo, utilizamos el teorema general siguiente.

Teorema 5.1.1 *Si X es un espacio vectorial normado y $Y \subset X$ un subespacio de X de dimensión finita, entonces dado $x \in X$ existe $y^* \in Y$ tal que*

$$\forall y \in Y, \ \ \|x - y^*\| \leq \|x - y\| .$$

La demostración del teorema precedente es muy sencilla, pues se reduce al hecho de que una función real continua sobre un conjunto compacto, no vacío, alcanza sus valores extremos. Para este y otros resultados preliminares, consulte [4] o [12], entre otros.

Obviamente, si p_n^* es un polinomio de mejor aproximación de P_n a $f \in C\,[a,b]$ y denotamos

$$E_n = E_n(f;[a,b]) := d(f, P_n),$$

entonces $E_n = d(f, p_n^*)$. Utilizaremos esta notación más adelante.

Definition 5.1.1 *Un conjunto de $k+1$ puntos $x_0, \ldots, x_k \in [a,b]$, tales que $a \leq x_0 < x_1 < \cdots < x_k \leq b$, es llamado conjunto alternante para la función $h \in C\,[a,b]$, si*

$$|h(x_j)| = \|h\| , \qquad j = 0, ..., k$$

y

$$h(x_j) = -h(x_{j+1}), \qquad j = 0, ..., k-1.$$

Uno de los resultados pioneros de la Teoría de la Aproximación es el siguiente:

Teorema 5.1.2 (***de Alternancia de Chebyshev***), [3] *y* [2]. *Dados* $n \in \mathbb{N}$ *y* $f \in C[a,b]$; p_n^* *es un polinomio de mejor aproximación de* P_n *a* f *si y sólo si, existe un conjunto alternante de* $n+2$ *puntos para la función* $f - p_n^*$.

Demostraciones contemporáneas de este teorema pueden encontrarse en diversos textos, aunque para nosotros es más cómodo tenerlo como un caso particular del teorema de alternancia de Guerra-Jímenez [7], que revisaremos más adelante al introducir el epígrafe 2. Gracias al teorema de Chebyshev tenemos que los polinomios de mejor aproximación, de grado a lo más n, de una función continua f, se caracterizan por tomar en $n+2$ puntos consecutivos, de manera alternada, los valores de las funciones $f \pm E_n$. Debido a esta caracterización, se puede demostrar la unicidad de la mejor aproximación de f a P_n. Por abuso de lenguaje, diremos que el polinomio toca arriba y abajo, alternadamente, la banda limitada por las funciones $f \pm E_n$. En general no existe [11] una regla para determinar la distribución de los puntos alternantes.

La caracterización del polinomio de mejor aproximación, puede utilizarse en ocasiones para determinar a simple vista este polinomio. Por ejemplo:

En el intervalo $[-a, a]$, consideremos la función $f(x) = sen(nx)$. Mientras que el número de ceros de f en $[-a, a]$ determine que el número de contactos de los bordes de la banda con el eje sea mayor o igual a $m+2$, el polinomio p_m^* de grado a lo más m que mejor aproxima uniformemente a f en $[-a, a]$, es $p_m^* \equiv 0$.

Figure 5.1: El parámetro
λ hace el efecto de brir o cerrar la banda tanto como se necesite.

A veces, la busqueda de p_n^* no es inmediata, pero aún así el teorema de Chebyshev puede servir de mucha ayuda. Desarrollamos a continuación un ejemplo interesante, a manera de conclusión de esta introducción.

Sean $f(x) = |x|/2$, $[a,b] = [-1,1]$ y $E_2 = d(f, P_2)$, entonces el polinomio de grado a lo más 2, $p \in P_2$, que mejor aproxima a f sobre $[-1,1]$, se caracteriza por alternar los valores máximo y mínimo de la función $p-f$ en 4 puntos consecutivos del intervalo. Calculemos dicho polinomio y el valor de E_2.

Supongamos que $p(x) = ax^2 + bx + c$. Por ser f una función par, debemos tener que $b = 0$. Con igual razonamiento sobre la paridad, el polinomio de mejor aproximación de grado a lo sumo $2n$, también es el de mejor aproximación entre los de grado a lo sumo $2n+1$. De aquí deducimos no sólo que $E_2 = E_3$ sino que hay un número impar de puntos de contacto del polimomio de mejor aproximación con los bordes de la banda. Ahora, dado que la función $p-f$ es par y p debe tocar, alternadamente, a las funciones $f \pm E_2$ en 5 puntos consecutivos al menos, pero siempre impar, entonces un punto de contacto debe ser $x = 0$, y más aún, debe alcanzar este contacto en la banda de arriba, pues no es posible tener

$$f(x) - E_2 \leq p_2(x), \quad \text{con } p_2(0) = -E_2$$

en una vecindad de cero; por lo que

$$p(0) = f(0) + E_2.$$

Esto es $c = E_2$.

Ahora, también por la geometria del problema, $p-f$ alcanza su norma en los puntos -1 y 1, es decir

$$\begin{aligned} E_2 &= (p-f)(-1) \\ &= (p-f)(1). \end{aligned}$$

Por lo tanto $a = 1/2$, es así que el polinomio de mejor aproximación es $p(x) = x^2/2 + E_2$.

Calculemos ahora el valor E_2, para esto, observemos que existen al menos dos puntos t y $-t$, $0 < t < 1$, en los que $p-f$ alcanza el valor $-E_2$, es decir, p toca a la función $f - E_2$ en los puntos $-t$ y t. Como estos puntos son interiores a $[-1,1]$ y $p-f$ es de clase C^1, la derivada debe anularse en estos puntos. De aquí deducimos que $t = \pm 1/2$ y de $(p-f)(1/2) = E_2$, que $E_2 = 1/8$ y el polinomio de mejor aproximación es $p(x) = x^2 + 1/8$.

5.2 Aproximación polinomial para banda no uniforme

En este epígrafe introduciremos el concepto de aproximación polinomial en el caso en el que la banda antes formada por la funciones f más/menos

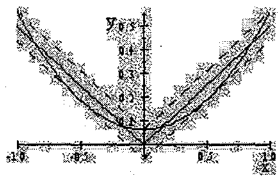

Figure 5.2: Gráfica de las funciones $f, f \pm E_n$ y p^*.

la función constante E_n, se forme al sumar y restar funciones extrictamente positivas.

Hacía mediados de la década de los 90, en [7] y [8], Guerra-Jiménez, presentan resultados que motivan el estudio de la aproximación en bandas no uniformes. Ellos cosideran el conjunto factible constituido por los vectores cuyas coordenadas son los coeficientes de los polinomios reales algebraicos incluidos en la banda delimitada por las funciones $f - h_1$ y $f + h_2$, donde $h_1 > 0$, $h_2 > 0$ y f son continuas, sin que la propia f sea un polinomio de grado a lo sumo n. Demuestran entre otras cosas que dado un número natural n, el conjunto de polinomios de grado a lo más n que se encuentra dentro de esta banda satisface una de las condiciones siguientes:

i) Es vacío.

ii) Es infinito.

iii) Es un monoelemento.

En realidad, esto es una consecuencia inmediata de lo que demuestran realmente: El conjunto de soluciones factibles se reduce a un único elemento si y sólo si los elementos de este conjunto tocan los bordes de la banda arriba y abajo, alternadamente, en $n+2$ puntos del intervalo $[a, b]$. Esto es una extensión del Teorema de Alternancia de Chebyshev. Notemos, sin embargo, que no se menciona aún la idea de mejor aproximación.

Introduzcamos un parámetro $\lambda > 0$, que multiplique a las funciones h_1 y h_2, y que haga el efecto de expander o contraer la banda como se muestra en la figura.

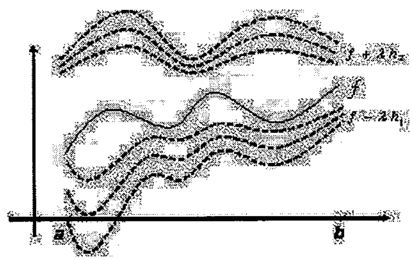

Figure 5.3:

Planteemos ahora el problema de optimizar este parámetro de manera de generalizar la idea de la mejor aproximación uniforme.

Sean f, $h_1, h_2 \in C[a,b]$, tales que $h_1, h_2 > 0$ y $n \in \mathbb{N}$ dado. Para un escalar $\lambda > 0$ definimos el conjunto $M_\lambda = M_\lambda(n, f, h_1, h_2) \subset P_n$ como sigue:

$$M_\lambda := \{p \in P_n : \forall x \in [a,b], (f - \lambda h_1)(x) \leq p(x) \leq (f + \lambda h_2)(x)\}.$$

La idea de introducir el parámetro λ es para poder expandir la banda tanto como sea necesario para permitir la entrada de polinomios y contraerla lo suficiente como para sólo dejar pasar uno, en caso de que esto sea posible. Aquí, al igual que en el caso tradicional, tiene poco interés si f es un polinomio de grado a lo más n.

Observemos que si $\lambda_1 < \lambda_2$, entonces $M_{\lambda_1} \subset M_{\lambda_2}$, es decir $(M_\lambda)_{\lambda > 0}$ es una familia creciente de conjuntos. Por otro lado, para cada $\lambda > 0$ podemos identificar M_λ con un subconjunto de $\mathbb{R}^n$, el cual resulta ser cerrado y acotado, por tanto compacto, con lo que tendriamos que M_λ también lo es.

Afirmamos que $M_\lambda \neq \phi$ para un valor de λ suficientemente grande. En efecto, sea $p \in P_n$. Consideremos la función $p - f$, la cual es continua y por tanto alcanza su valor máximo y mínimo en $[a,b]$; es decir, para todo $x \in [a,b]$,

$$m \leq (p - f)(x) \leq M,$$

en donde

$$m = \underset{x\in[a,b]}{mín}(p - f)(x) \text{ y } M = \underset{x\in[a,b]}{máx}(p - f)(x).$$

Por ser $h_1, h_2 > 0$ y continuas, existe $\lambda^* > 0$, tal que

$$-\lambda^* h_1 < m \quad \text{y} \quad M < \lambda^* h_2;$$

por tanto

$$f - \lambda^* h_1 \leq p \leq f + \lambda^* h_2.$$

Así $M_{\lambda^*} \neq \phi$.

Habiendo definido estos conjuntos y demostrado algunas de sus propiedades, estamos listos para dar la definición de mejor aproximación y polinomio de mejor aproximación.

Definition 5.2.1 *Sean $f, h_1, h_2 \in C[a,b]$, con $h_1, h_2 > 0$ y $n \in \mathbb{N}$ dados. Definimos la mejor aproximación de P_n a f, relativo a h_1 y h_2, como*

$$\lambda_n := inf\{\lambda > 0 : M_\lambda \neq \phi\}.$$

Supongamos ahora que $\lambda' > 0$ es un valor suficientemente grande para el cual $M_{\lambda'} \neq \phi$. Entonces, es de suponer que el valor de λ_n debe estar asociado al conjunto

$$M_{\lambda_n} = \bigcap_{\substack{\lambda < \lambda', \\ M_\lambda \neq \phi}} M_\lambda.$$

Cuando este conjunto sea no vacío, se denominará a sus elementos como polinomios de mejor aproximación a f relativo a h_1 y h_2.

En el caso de la aproximación polinomial uniforme se tiene que, $\lambda_n = E_n(f)$ y que el conjunto M_{λ_n} es no vacío, más aún, consta de un único elemento. La generalización del caso clásico al nuevo concepto introducido, es como sigue:

Teorema 5.2.1 *Sean $f, h_1, h_2 \in C[a,b]$, $h_1, h_2 > 0$ y $n \in \mathbb{N}$. El polinomio de grado a lo más n, p_n^*, es de mejor aproximación a f relativo a h_1 y h_2, si y sólo si, la función $f - p_n^*$ toca alternadamente arriba y abajo en $n+2$ puntos consecutivos de $[a,b]$, a las funciones $-\lambda_n h_1$ y $\lambda_n h_2$.*

Este resultado junto al obtenido por los doctores Guerra y Jiménez, [7], nos permite analizar la unicidad del polimonio de mejor aproximación que formalizaremos en el teorema siguiente.

Teorema 5.2.2 *Sean $f, h_1, h_2 \in C[a,b]$, $h_1, h_2 > 0$ y $n \in \mathbb{N}$. Entonces el polinomio de mejor aproximación de grado a lo más n a f relativo a h_1 y h_2 es único.*

Respecto a la velocidad de convergencia se tiene:

Sean $h_1, h_2 \in C[a,b]$, $h_1, h_2 > 0$, fijas. Dada una función $f \in C[a,b]$, nos interesamos en comparar la sucesión (λ_n^*) de la mejor aproximación relativa a f, h_1, h_2; con la sucesión (E_n) de la mejor aproximación uniforme tradicional.

Definamos

$$M = max \max_{x\in[a,b]} \{h_1(x), h_2(x)\}$$

y

$$m = min \min_{x\in[a,b]} \{h_1(x), h_2(x)\}.$$

Se tiene $0 < m \leq M < \infty$ y el resultado siguiente:

Teorema 5.2.3 *Para toda $f \in C[a,b]$, se tiene*

$$\frac{E_n(f)}{M} \leq \lambda_n^*(f) \leq \frac{E_n(f)}{m}$$

o equivalentemente

$$m\lambda_n^*(f) \leq E_n(f) \leq M\lambda_n^*(f).$$

5.3 Condiciones de interpolación

En este epígrafe introducimos el caso en que h_1 y h_2 pueden tener ceros en el intervalo de estudio. La idea es que al permitirle esta cualidad a dichas funciones y estas tengan un número finito de ceros coincidentes, los polinomios incluidos en la banda tengan que interpolar a la función f que se aproxima, en cada cero común de h_1 y h_2.

Bajo condiciones adicionales, podría intentarse interpolar con una suavidad predeterminada, buscando generalizar la interpolación de Hermite. Por otra parte, también estudiar la aproximación lateral.

A diferencia del caso en el cual h_1 y h_2 son estríctamente positivas, el nuevo problema que enfrentaremos encierra un alto grado de complejidad y dificultad, según podremos observar en los ejemplos que citaremos.

No se trata de presentar en esta ocasión un estudio completo del nuevo caso sobre los problemas de existencia, caracterización, unicidad, etc., como logramos en el epígrafe precedente. Pero los ejemplos anunciados tienen por sí, nos parece, interés propio.

Ocupémonos del planteamiento del problema. Se tienen $h_1, h_2, f \in C[a,b]$, $h_1, h_2 \geq 0$ y $n \in \mathbb{N}$. Nos preguntamos si existen un escalar $\lambda > 0$ y un polinomio $p \in P_n$ tales que

$$f - \lambda h_1 \leq p \leq f + \lambda h_2. \qquad (*)$$

En tales casos pasaríamos a estudiar los otros problemas típicos de la mejor aproximación relativa a este contexto.

Si una de las funciones h_i, $i = 1, 2$, es identicamente nula y la otra no tiene ceros, estaríamos generalizando la aproximación lateral (superior si $h_1 \equiv 0$ e inferior si $h_2 \equiv 0$). Obviamente siempre existe una solución para todo n y $f \in C[a,b]$ dados. No es díficil demostrar que también aquí se tienen resultados de existencia, unicidad y caracterización de la mejor aproximación polinomial.

Más complejo es el caso cuando las funciones h_1 y h_2 tienen ceros comunes.

La interpolación en un conjunto numerable de puntos, tiene interés, por ejemplo, en la teoría de variable compleja. Para el caso de la aproximación real polinomial, no tiene, a nuestro parecer, mayor interés.

En efecto, si las funciones de error h_i, $i = 1, 2$, se anulan simultáneamente en un conjunto infinito de puntos, el problema $(*)$ sólo podría tener solución si la función objeto de estudio, f, coincidiese sobre ese conjunto con un polinomio p, de grado N exactamente. En tal caso, podría existir $\lambda^* > 0$ que fuese solución de $(*)$ con $n \geq N$. Si así fuere, p sería la única solución para todo $\lambda \geq \lambda^*$ y todo $n \geq N$ y no habría solución si $n < N$.

Si el número de ceros comunes de h_1 y de h_2 es finito, puede existir o no, una solución al problema $(*)$.

Ejemplo: Sea $[a,b] = [-1,1]$, supongamos que deseamos encontrar un polinomio p, tal que

$$f - \lambda h_1 \leq p \leq f + \lambda h_2,$$

en donde $f = h_2 = |x|/2$ y $h_1 \equiv 0$.

O sea, tal que

$$\frac{|x|}{2} \leq p \leq (1+\lambda)\frac{|x|}{2}.$$

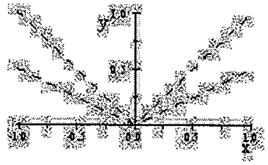

Figure 5.4: La forma de la banda en este problema determina su no solubilidad.

Analíticamente la no solubilidad, se debe a que cualquier polinomio dentro de la banda deberá ser de grado mayor o igual a 2 y tener un cero múltiple en 0, lo cual es incompatible con la desigualdad $|x|/2 \leq p(x)$ en una vecindad del cero.

El ejemplo precedente conduce a pensar, que la existencia de solución del problema está relacionada directamente con la suavidad de las funciones f, h_1 y h_2. Sin embargo, el ejemplo siguente demuestra que con funciones suaves, puede igualmente no haber solución.

Ejemplo: Sean

$$[a.b] = [-1, 1]\,, h_1 = h_2 = x^2 \text{ y}$$

$$f(x) = \begin{cases} |x|^{\frac{3}{2}}\, sen\frac{1}{|x|^{\frac{1}{4}}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

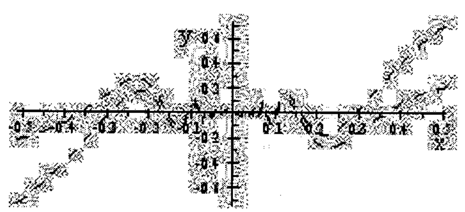

Figure 5.5: Gráfica de los bordes de la banda en una vecindad de cero.

Notemos que $f, h_1, h_2 \in C^1[-1,1]$, que h_1 y h_2 tienen un único cero y sin embargo no existe polinomio alguno que satisfaga

$$f - \lambda h_1 \leq p \leq f + \lambda h_2,$$

independientemente del valor de λ y del grado del polinomio.

En efecto, sean (x_n) y (y_n) definidas por

$$\frac{1}{|x_n|^{\frac{1}{4}}} = \frac{4n+3}{2}\pi \text{ y } \frac{1}{|y_n|^{\frac{1}{4}}} = \frac{4n+1}{2}\pi$$

tanto $x_n \to 0$ como $y_n \to 0$ y las sucesiones están intercaladas de manera que

$$sen\left(\frac{1}{|x_n|^{\frac{1}{4}}}\right) = -1 \text{ y } sen\left(\frac{1}{|y_n|^{\frac{1}{4}}}\right) = 1.$$

Dado $\lambda > 0$ tendríamos que buscar $p \in P_n$, tal que, en particular, se tenga

$$p(x_n) \leq \lambda x_n^2 - x_n^{3/2}$$

y

$$p(y_n) \geq -\lambda y_n^2 + y_n^{3/2}.$$

Pero, para n suficientemente grande se tendrá

$$\lambda x_n^2 - x_n^{3/2} \leq \frac{-x_n^{3/2}}{2}$$

y

$$-\lambda y_n^2 + y_n^{3/2} \geq \frac{y_n^{3/2}}{2}.$$

Es decir, en cualquier vecindad de cero, p tendría infinitos máximos y mínimos y no es constante. Luego p no puede ser un polinomio.

En este caso el comportamiento oscilante de f afectó el problema de existencia.

Podemos tener ejemplos en los cuales, dado n, aún cuando podamos encontrar un polinomio de grado a lo más n y la mejor aproximación, $\lambda > 0$, para los cuales

$$f - \lambda h_1 \leq p \leq f + \lambda h_2,$$

no se tenga unicidad. El ejemplo siguiente lo ilustra.

Ejemplo: Para $[a, b] = [-1, 1]$. Analicemos la desigualdad

$$f - \lambda h_1 \leq p \leq f + \lambda h_2,$$

para $f = |x|/2$, $h_1 = 3|x|/2$ y $h_2 \equiv 0$. Tenemos

$$(1 - 3\lambda)\frac{|x|}{2} \leq p(x) \leq \frac{|x|}{2}.$$

Notemos que si $0 < \lambda < 1/3$, no hay solución para ningún $n \in \mathbb{N}$.

Para valores de $\lambda \geq 1/3$ empezamos a encontrar polimonios dentro de la banda, es decir la mejor aproximación sería $\lambda = \frac{1}{3}$, para todo n. Si $n = 0$ o $n = 1$, el polinomio de mejor aproximación es la constante cero. Si $n > 1$, ya se pierde la unicidad. A manera de ilustración para $n = 2$, $p(x) = x^2/4$ y $p(x) = x^2/2$.

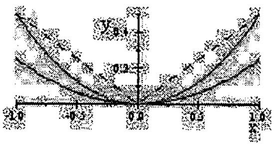

Figure 5.6:

Independientemente de los ejemplos negativos mostrados en este epígrafe, con hipotésis complementarias es posible desarrollar una teoría con resultados similares a los señalados en el epígrafe 2; con la novedad de encontrarnos en un tipo de aproximación en bandas no uniformes, con condiciones de interpolación. Estos resultados, con el tratamiento detallado del epígrafe 2, incluidos en una teoría general, conforman el contenido de un artículo próximo en vías de preparación por estos autores.

5.4 Bibliografía

1. Bustamante J., Moreno S. G., Quesada J. M., *Best approximation and wrappings*, Topology Proc., Vol. 29, 1, 1-22, 2005.

2. Chebyshev P. L., *Théorie des mécanismes connus sous le nom de parallélogrammes*, Mém. Prés. Acad. Imp. Sci. Pétersb. Divers Savants, VII, pp. 539-568, 1854.

3. Chebyshev P. L., *Sur les questions de minima qui se rattachent à la représentation approximative des fontions*, Mém. Prés. Acad. Imp. Sci. Pétersb. Sci. Math. Phys. VII, pp. 199-291, 1859.

4. Cheney, E. W., *Introducction to Approximation Theory*, McGraw-Hill, New York, 1966.

5. DeVore R. A., Lorentz G. G., *Constructive Approximation*, Springer-Verlag, New York, 1993.

6. Guerra F., Jiménez M. A., *A semi-infinite programming approach to a mixed approximation problem*, Parametric Optimization and Related Topics IV, J. Guddat et al, series *Approximation and Optimization*, Peter Lang Verlag, Frankfurt, pp. 135-143, 1997.

7. Guerra F., Jiménez M. A., *On feasibles sets defined through Chebyshev approximation,* Math. Methods Oper. Res., Vol. 47, Issue 2, pp. 255-264, 1998.

8. Hassouni A., Oettli W., *On regularity and optimality in nonlinear semi-infinite programing.* Recent advances, Dordrecht: Kluwer Acad. Publ. Nonconvex Optim. Appl. 57, pp. 59-74, 2001.

9. Jiménez M. A., Juárez E. L, Guerra F., *Transformation of some mixed approximation problems by optimization methods*, Vol 5, 1, pp. 175-190, 2002.

10. Jiménez M. A., Todorov M. I., *Unicity of the solutions of infinite linear inequality systems.* C. R. Acad. Bulgare Sci., Tome 54, Nr. 6, pp. 17-20, 2001.

11. Lorentz G. G., *Distribution of alternation points in uniform polynomial approximation,* Proc. Amer. Math. Soc., Vol. 92, 3, pp. 401-403,1984.

12. Rivlin T. J., *An introduction to the approximation of functions,* Blaisdell Publishing Company, Massachusetts, 1969.

13. Tikhomirov V. M. *Commentary.* J. Approx. Theory, 106, pp. 58-65, 2000.

Capítulo 6

Teoremas directos e inversos en espacios de Lipschitz generalizados

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Abstract: We present direct and inverse theorems in generalized Lipschitz spaces X_p^{α}, $1 \leq p < \infty$, $\alpha = \beta + 1/p$, $0 < \beta < 1$, in terms Lipschitzian moduli of smoothness.

Resumen: Presentamos teoremas directos e inversos en los espacios de Lipschitz generalizados X_p^{α}, $1 \leq p < \infty$, $\alpha = \beta + 1/p$, $0 < \beta < 1$, en términos de módulos de continuidad Lipschitzianos.

6.1 Introducción

Consideremos el grupo multiplicativo $\mathbf{T}$ del círculo unitario en el plano complejo, que podemos identificar homeomórficamente al espacio métrico $X = [0, 2\pi)$, con la distancia definida por

$$d(x, y) = \text{mín }\{|x-y|, \quad 2\pi - |x-y|\}. \tag{6.1}$$

Entonces las funciones reales medibles en X son identificadas con los elementos del espacio lineal de las funciones reales o complejas, 2π-periódicas, medibles sobre $\mathbb{R}$, denotado por $F_{2\pi}$. El espacio de Banach $C(X)$ de funciones continuas reales o complejas sobre X es identificado con el espacio de Banach clásico $C_{2\pi}$. Los espacios de Banach $L^p_{2\pi}$, $1 \leq p < \infty$, son los espacios usuales de funciones 2π periódicas, $p-$integrables con medida de Lebesgue $dx/2\pi$ (por convenio, si $p = \infty$, $L^\alpha_{2\pi} = C_{2\pi}$). También consideramos funciones definidas sobre el grupo producto $X^2 = X \times X$, los correspondientes espacios son denotados por $F_{(2\pi)^2}$, $L^p_{(2\pi)^2}$, etc.

Sea $f \in F_{2\pi}$ y $\alpha > 0$. Para cualquier $\delta > 0$, $1 \leq p \leq \infty$, definimos

$$\Theta^p_\alpha(f, \delta) := \sup \left\{ \frac{||[f]_t - f||_p}{t^\alpha} \quad : \quad 0 < t \leq \delta \right\}, \tag{6.2}$$

donde $[f]_t(x) = f(x+t)$ y $||f||_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}$ es la $p-$norma si $1 \leq p < \infty$ y $f \in L^p_{2\pi}$ o $||f||_p := \sup\{| \ f(x) \ |: \ x \in X\}$ es la norma del supremo si $p = \infty$ y $f \in C_{2\pi}$. Finalmente

$$\Theta^p_\alpha(f) := \sup\{\Theta^p_\alpha(f, \delta) \quad : \quad \delta > 0\}. \tag{6.3}$$

Entonces $f \in L^p_{2\pi}$, $1 \leq p \leq \infty$, satisface una condición de Hölder de orden $\alpha \in (0, 1]$, si $\Theta^p_\alpha(f) < \infty$. Para $\alpha \in (0, 1]$, denotamos la colección de todas estas funciones por Lip^p_α. Es sabido que Lip^p_α es un espacio de Banach con la $p-$norma de Hölder

$$||f||_{p,\alpha} := ||f||_p + \Theta^p_\alpha(f). \tag{6.4}$$

Si $\alpha > 1$ y $\Theta^p_\alpha(f) < \infty$, entonces f es constante. Así que, si $\alpha = r + \beta$, $r \in \mathbb{N}$ y $0 < \beta \leq 1$ se definen los espacios Lip^p_α como las funciones f, para las cuales $f, \ldots, f^{r-1}$ son absolutamente continuas en X y $f^r \in Lip^p_\beta$.

Lip^p_α es normado por

$$||f||_{p,\alpha} := ||f||_p + \Theta^p_\beta(f^{(r)}). \tag{6.5}$$

La fórmula (6.4) esta incluida en (6.5) para $r = 0$.

Otra forma de definir estos espacios es reemplazando la diferencia $[f^{(r)}]_t - f^{(r)}$ por diferencias de orden superior, siguiendo las ideas de Zygmund

[11]. Si α no es entero, ambas vías de definición son equivalentes [5]. Si α es entero, ambos métodos de definición difieren y las diferencias de orden superior son necesarias para obtener teoremas inversos tipo Bernstein.

Algunos problemas requieren de una escala de suavidad muy fina, escala que es proporcionada por los espacios de Besov. Sean $1 \leq p,\ q \leq \infty$, $\alpha > 0$, $r = [\alpha] + 1$, $f \in L^p_{2\pi}$, se define la función

$$\Theta_\alpha(f,\, t)_{p,q} := \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^t \left(\dfrac{\omega_r(f,\, s)_p}{s^\alpha} \right)^q \dfrac{ds}{s} \right)^{1/q} & \text{si } 1 \leq q < \infty \\ \sup\limits_{0<s\leq\pi} \dfrac{\omega_r(f,\, s)_p}{s^\alpha} & \text{si } q = \infty \end{cases} \tag{6.6}$$

donde $\omega_r(f,\, t) = \sup\{|\Delta_r^s f| \ :\ 0 < s \leq t\}$ es el $r-$módulo de suavidad de f y el espacio lineal

$$B^\alpha_{p,q} := \{f \in L^p_{2\pi} \ :\ \Theta_\alpha(f,\, \pi)_{p,q} < \infty\}. \tag{6.7}$$

Ellos son espacios de Banach con la norma

$$||f||_{p,q,\alpha} := ||f||_p + \Theta_\alpha(f,\, \pi)_{p,q,\alpha} \tag{6.8}$$

o alguna otra norma equivalente, por ejemplo, si $p = q < \infty$, con

$$||f||_{p,q,\alpha} = \left(||f||_p^p + \Theta_\alpha(f,\, \pi)^p_{p,p} \right)^{1/q} \tag{6.9}$$

(ver Jiménez–Linares [8]).

Los espacios definidos por (6.7) son llamados espacios de Besov o también espacios de Lipschitz generalizados. Los espacios de Besov estan más vinculados al desarrollo de los espacios de funciones y sus aplicaciones a las Ecuaciones Diferenciales Parciales, Integrales Singulares, Procesamiento de Imágenes, Teoría de Aproximación y otras, que al caso de funciones periódicas. Por tal motivo, aunque no dejaremos de referirnos frecuentemente a la denominación de espacios de Besov, utilizaremos preferentemente la terminología de Butzer–Berens [4], que incluye los espacios de Besov del caso periódico en la categoría de los espacios de Lipschitz generalizados. Para mencionar algunas referencias básicas, acerca de estos espacios, ver Butzer–Berens [4], [10] Triebel, De Vore–Lorentz [5], Bustamante–Jiménez [2] y Anastassious–Gal [1].

Para todo $1 \leq p \leq \infty$, $1 \leq q < \infty$ y $f \in B^{\alpha}_{p,q}$, la función $\Theta_\alpha(f,\cdot)_{p,q}$ es un módulo de continuidad. No lo es, sin embargo, para el caso $q = \infty$.

Como

$$l^{\alpha}_{p,\infty} := \{f \in B^{\alpha}_{p,\infty} : \Theta_\alpha(f,t)_{p,\infty} \longrightarrow 0, \quad t \to 0\}$$

es cerrado en $B^{\alpha}_{p,\infty}$, este es un espacio de Banach por si mismo. En el mismo espíritu que escribimos $L^{\infty}_{2\pi} = C_{2\pi}$ en Teoría de Aproximación, se sobreentenderá que $B^{\alpha}_{p,\infty} := l^{\alpha}_{p,\infty}$. Con estas definiciones, para todo $1 \leq p,\, q \leq \infty$ y $f \in B^{\alpha}_{p,q}$ la función $\Theta_\alpha(f,\cdot)$ es un módulo de continuidad y el operador traslación es continuo en $B^{\alpha}_{p,q}$, los polinomios trigonométricos son densos en cada uno de ellos. Las definiciones hasta aquí desarrolladas son clásicas. Nosotros vamos a modificarlas ligeramente para reintroducir los espacios de Lipschitz desde otro punto de vista que tiene sus ventajas y encontrar teoremas directos e inversos de caracterización, de manera un poco diferente. En efecto, en [3] hemos caracterizado los espacios de Besov de una manera muy general. Lo que aquí presentamos tiene un alcance más restringido; pero en su génesis es algo diferente, las demostraciones también y aparecen algunas ventajas [6]. Para comprender la nueva, observe que la medida utilizada en (6.6) es la medida de Haar en $\mathbb{R}_+$. Pero esto, nos parece, oculta un tanto la realidad, lo que importa es dividir la medida de Lebesgue por el parámetro s para que los espacios de Besov no resulten triviales. Si en (6.6) nos limitamos al caso en el cual $0 < \alpha < 1$ y pasamos dentro de la integral el parámetro s que divide a la nedida de Lebesgue; sería lo mismo que suponer $\alpha = \beta + 1/p$, con $0 < \beta < 1$, $1 \leq p < \infty$, la medida de Lebesgue en lugar de la medida de Haar y con $\omega_r = \omega_1 = \omega$. Según [4] se puede sustituir con equivalencia entre las seminormas resultantes, el módulo de continuidad $\omega(f,s)_p$ por el incremento $\|[f]_s - f\|_p$. Después, continuar con el enfoque desarrollado en [6], según explicaremos a continuación. Para $f \in F_{2\pi}$, $1 \leq p < \infty$ y $\alpha > 0$, se define casi dondequiera una nueva función medible $F_\alpha f$ sobre $X \times X$ mediante

$$F_\alpha(f)(x,y) = \frac{f(y) - f(x)}{d(x,y)^\alpha}, \quad x \neq y. \tag{6.10}$$

Las propiedades de esta transformación pueden consultarse en [6] o en [8]. En esta última referencia, precisamente, desarrollamos el estudio de los espacios X^{α}_p, definidos como sigue: Para $1 \leq p < \infty$, $\alpha = \beta + 1/p$, $0 < \beta < 1$, el espacio X^{α}_p es la clase de funciones $f \in L^p_{2\pi}$, para las cuales $F_\alpha(f) \in L^p_{(2\pi)^2}$

y

$$\Psi_\alpha(f)_p := ||F_\alpha\, f||_{L^p_{(2\pi)^2}} = \left(\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|F_\alpha\, f(x,\, y)|^p\, dx\, dy\right)^{1/p} < \infty. \tag{6.11}$$

Estos espacios son de Banach, homogéneos en el sentido de Katznelson y estrictamente convexos con la norma

$$||f||_{X_p^\alpha} := ||f||_{\alpha,p} := \left(||f||_p^p + \Psi_\alpha(f)_p^p\right)^{1/p}. \tag{6.12}$$

Más aún, es posible verificar que los espacios X_p^α y B_{pp}^α, $\alpha = \beta + 1/p$, $0 < \beta < 1$ coinciden y sus normas son equivalentes. Pero como puede apreciarse en [6] y [8], los espacios X_p^α tienen ciertas ventajas.

6.2 Resultados preliminares

Aunque la vía de obtener teoremas directos mediante la construcción de operadores particulares, no suele ser la mejor en cuanto a constantes multiplicativas de los estimados de aproximación, tiene la ventaja manifiesta de constituir un método constructivo y no existencial. Para el caso de funciones periódicas, el método tradicional consiste en la utilización de operadores de convolución a partir del núcleo de Jackson. En este epígrafe introduciremos un teorema de tipo Tauberiano [7], que utilizaremos para la obtención de teoremas directos en X_p^α.

Supongamos que estamos en presencia de un espacio de Banach $(E,\, ||\cdot||_E)$ y de una familia de seminormas sobre E,

$$\Theta : E \times I \longrightarrow \mathbb{R}_+ \cup \{0\},$$

donde el conjunto de índices I es un intervalo abierto $(a,\, b)$ o semi abierto $(a,\, b]$ y $b = \infty$ es posible. Para cada $f \in E$ fija, la función $\Theta(f,\, \cdot)$ es creciente en I. Luego

$$\forall f \in E, \quad \Theta(f) := \sup\{\Theta(f,\, \delta);\quad \delta \in I\} \tag{6.13}$$

es una seminorma sobre E.

Definimos los subespacios de E

$$\mathrm{F} := \{f \in E\, :\, \Theta(f) < \infty\} \tag{6.14}$$

$$F := \{f \in \mathrm{F}\, :\, \Theta(f,\, \delta) \longrightarrow 0, \quad \delta \to 0\} \tag{6.15}$$

los que quedan seminormados por (6.13) y normados mediante

$$||\cdot||_{\mathrm{F}} := ||\cdot||_E + \Theta(\cdot). \tag{6.16}$$

Queda claro de (6.16) que si una sucesión converge según $||\cdot||_{\mathrm{F}}$; también lo hace en E y con igual límite (el recíproco es falso). Es frecuente la situación en que podemos estudiar la convergencia de una sucesión en dos sentidos diferentes, uno de los cuales es más débil que el otro. Un teorema tauberiano es un resultado del cual se puede derivar la convergencia en el sentido más fuerte, a partir de la convergencia en el sentido más débil y del conocimiento de determinado comportamiento asintótico de los elementos de la sucesión, que se denomina condición tauberiana.

En el caso de la norma (6.16), restringida a F, la condición tauberiana propuesta por Jiménez [7], requiere de las definiciones siguientes.

Definición 1. Un conjunto no vacío $G \subset \mathrm{F}$ es llamado $0-$ **equicontinuo,** si

$$\Theta(G,\, \delta) := \sup\{\Theta(g,\, \delta) \mid g \in G\} \longrightarrow 0, \quad \delta \to 0. \tag{6.17}$$

Definición 2. La familia de seminormas $\Theta(\cdot,\, \delta)$, $\delta \in I$, se dice **admisible,** con respecto a la norma de E, si las condiciones siguientes son satisfechas:

(a) $(F,\, ||\cdot||_{\mathrm{F}})$ es Banach.

(b) Existe $C > 0$ y una función $\mathcal{X} : I \times \mathbb{R}_+ \to \mathbb{R}_+$, tal que, para cada $\delta \in I$ fijo, $\mathcal{X}(\delta,\, \cdot\,)$ es continua en 0 y $\mathcal{X}(\delta,\, 0) = 0$.

(c) $\forall f \in F \quad \forall \delta > 0, \quad \Theta(f) \leq C\,\Theta(f,\, \delta) + \mathcal{X}(\delta,\, ||f||_{\mathrm{F}})$.

Teorema 1. (Tauberiano) *Sean* F y F, *definidas a partir de un espacio de Banach E, por una familia de seminormas admisibles Θ, en el sentido de la Definición 2. Sea $(f_n) \subset F$ una sucesión que converge en E a un elemento f. Si (f_n) es $0-$equicontinua, entonces $f \in F$, (f_n) converge a f en F, y se tiene el estimado de convergencia*

$$\Theta(f_n - f) \leq 2C\,\Theta((f_n),\, \delta) + \mathcal{X}(\delta,\, ||f_n - f||_E). \tag{6.18}$$

Para la demostración ver [7].

6.3 Teorema directo tipo Jackson para X_p^α

Como hemos adelantado, para el caso de $lip(\alpha)_p$, Bustamante–Jiménez [2], utilizaron el módulo de suavidad $\Theta_\alpha(f,\, \delta)_p$ y caracterizaron estas clases de funciones. Es decir, las clases de Hölder definidas tomando supremo en $||\Delta_t f||_p/t^\alpha$. Con ello simplificaron el problema de la caracterización que usualmente considera el módulo de suavidad asociado a la norma y que es más complicado. Para los espacios de Besov de funciones periódicas, tenemos los resultados de Bustamante–Jiménez–Linares [3]. Ahora, estudiaremos el mismo problema cuando definimos los espacios a través de (6.11) y (6.12). Para esto utilizaremos la familia de seminormas

$$\psi_\alpha(f,\,\delta)_p^p := \frac{1}{2\pi^2}\int_0^\delta\int_0^{2\pi}\left|\frac{f(x+t)-f(x)}{t^\alpha}\right|^p dx\,dt, \quad f\in X_p^\alpha, \tag{6.19}$$

que es admisible con la función

$$\mathcal{X}(\delta,\,t) := C\left[t+\frac{t}{\delta^\beta}\right]. \tag{6.20}$$

En [6] y [7], se estudia la relación entre (6.1) y (6.19), en particular se demuestra que

$$\psi_\alpha(f,\,\pi) = \psi_\alpha(f)_p.$$

Lema 1. *Sea $1\le p<\infty$ y $\alpha=\beta+1/p$, con $0<\beta<1$. Sea $f\in X_p^\alpha$. Entonces, para todo $t>0$, se cumple*

$$w(f,\,t)_p \le C_1\,t^\beta\psi_\alpha(f,\,t)_p, \tag{6.21}$$

donde C_1 es una constante que sólo depende de p.

Demostración:

$$\left[\frac{1}{t}w(f,\,t)_p\right]^p := \sup_{0<s\le t}\ \frac{1}{t^p}\ \frac{1}{2\pi}\int_0^{2\pi}|f(x+s)-f(x)|^p\,dx. \tag{6.22}$$

Fijemos s, en el rango $(0,\,t]$ y utilicemos

$$\frac{1}{2\pi}\int_0^{2\pi}|f(x+s)-f(x)|^p dx = \frac{1}{2\pi s}\int_0^s\int_0^{2\pi}|f(x+s+v)-f(x+v)|^p dxdv$$

$$\le \frac{2^p}{2\pi s}\left[\int_0^s\int_0^{2\pi}|f(x+s+v)-f(x+s)|^p dxdv\right.$$

$$+ \left. \int_0^s \int_0^{2\pi} |f(x+s) - f(x+v)|^p \, dx \, dv \right]$$

$$= \frac{2^p}{2\pi s} \int_0^s \int_0^{2\pi} |f(x+v) - f(x)|^p dx dv$$

$$+ \frac{2^p}{2\pi s} \int_0^s \int_0^{2\pi} |f(y) - f(y+v-s)|^p dy dv \tag{6.23}$$

Además

$$\begin{aligned} \int_0^s \int_0^{2\pi} |f(y) - f(y+v-s)|^p dy dv &= \int_{-s}^0 \int_0^{2\pi} |f(y) - f(y+u)|^p dy du \\ &= \int_0^s \int_0^{2\pi} |f(y) - f(y-u)|^p \, dy \, du \\ &= \int_0^s \int_0^{2\pi} |f(x+v) - f(x)|^p \, dx \, dv. \end{aligned}$$

Luego, regresando a (6.23) tendremos

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x+s) - f(x)|^p dx \leq \frac{2^{p+1}}{2\pi s} \int_0^s \int_0^{2\pi} |f(x+v) - f(x)|^p dx dv$$

$$\leq 2^{p+1} s^{p\alpha - 1} \frac{1}{2\pi} \int_0^s \int_0^{2\pi} \left| \frac{f(x+v) - f(x)}{v^\alpha} \right|^p dx dv.$$

Sustituyendo esta última desigualdad en (6.22), obtenemos

$$\left[\frac{1}{t} w(f,\, t)_p \right]^p \leq \sup_{0 < s \leq t} \frac{2^{p+1} \pi s^{p\,\alpha - 1}}{t^p} \left[\frac{1}{2\pi^2} \int_0^s \int_0^{2\pi} \left| \frac{f(x+v) - f(x)}{v^\alpha} \right|^p dx \, dv \right]$$

$$= C t^{p(\alpha - 1) - 1} \psi_\alpha(f,\, t)_p^p.$$

Luego

$$\frac{1}{t} w(f,\, t)_p \leq C \, t^{\beta - 1} \, \psi_\alpha(f,\, t),$$

de donde se obtiene (6.21). $\blacksquare$

Lema 2. *Sea* (K_n) *una sucesión de funciones uniformemente acotadas por una constante* M *en* $L_1(T)$. *Sea* $f \in X_p^\alpha$, $1 \leq p < \infty$, $\alpha = \beta + 1/p$ *con* $0 < \beta < 1$. *Entonces* $(K_n * f)$ *es* $0-$*equicontinua en* X_p^α *y*

$$\psi_\alpha((K_n * f), \delta)_p \leq M\, \psi_\alpha(f, \delta)_p. \tag{6.24}$$

Demostración: En efecto, fijemos $\delta > 0$. Tenemos

$$\begin{aligned}
\psi_\alpha((K_n * f), \delta)_p &= n\mathrm{sup} \left[\frac{1}{\pi} \int_0^\delta ||\Delta_t (K_n * f)||_p^p\, t^{-\alpha p}\, dt \right]^{\frac{1}{p}} \\
&= n\mathrm{sup} \left[\frac{1}{\pi} \int_0^\delta ||K_n * \Delta_t f||_p^p\, t^{-\alpha p}\, dt \right]^{\frac{1}{p}} \\
&\leq n\mathrm{sup} \left[\frac{||K_n||_1^p}{\pi} \int_0^\delta ||\Delta_t f||_p^p\, t^{-\alpha p}\, dt \right]^{\frac{1}{p}} \\
&\leq M\, \psi_\alpha(f, \delta)_p. \quad \blacksquare
\end{aligned}$$

Ahora estamos en condiciones de presentar el teorema de aproximación polinomial trigonométrica en X_p^α. Si $f \in X_p^\alpha$ y $E_n(f)_{\alpha,p} := E_n(F)_{X_p^\alpha}$ denota la mejor aproximación en X_p^α, por polinomios de grado no mayor que n, se tiene:

Teorema 1. *Sean* $1 \leq p < \infty$, $\alpha = \beta + 1/p$, *con* $0 < \beta < 1$ *y* $f \in X_p^\alpha$. *Entonces*

$$E_n(f)_{\alpha,p} = O\left(\psi_\alpha \left(f, \frac{1}{n} \right)_p \right). \tag{6.25}$$

Demostración: Sea (K_n) el núcleo de Jackson. Sabemos que $||K_n||_1 = 1$ y que

$$\forall\, f \in L_p(T), \quad ||K_n * f - f||_p \leq C\, w\left(f, \frac{1}{n} \right)_p,$$

donde C es una constante que no depende de n.

Apliquemos el teorema 1 a $K_n * f$ en X_p^α, con la utilización de (6.19), (6.20) y (6.24), donde $M = 1$. Se obtiene, para todo $\delta > 0$ y todo $n \in \mathbb{N}$, que

$$\begin{aligned}\psi_\alpha(k_n * f - f)_p &\leq 2\psi_\alpha((K_n * f), \delta)_p + \mathcal{X}(\delta, ||K_n * f - f||_p) \\ &\leq 2\psi_\alpha(f, \delta)_p + C_1 \left[||K_n * f - f||_p + \tfrac{||K_n * f - f||_p}{\delta^\beta}\right].\end{aligned}$$

Teniendo presente que para a, b positivos y $1 \leq p < \infty$, se cumple que

$$(a^p + b^p)^{\frac{1}{p}} \leq a + b,$$

tendremos entonces

$$\begin{aligned}||K_n * f - f||_{\alpha,p} &\leq ||K_n * f - f||_p + \psi_\alpha(K_n * f - f)_p \\ &\leq 2\psi_\alpha(f, \delta)_p + (C_1 + 1)||K_n * f - f||_p + C_1 \tfrac{||K_n * f - f||_p}{\delta^\beta}.\end{aligned}$$

Si tomamos $\delta = 1/n$, y sustituimos el estimado de Jackson en la desigualdad anterior, tendremos

$$||K_n * f - f||_{\alpha,p} \leq C_2 \left[\psi_\alpha\left(f, \frac{1}{n}\right)_p + \frac{w\left(f, \frac{1}{n}\right)}{\left(\frac{1}{n}\right)^\beta}\right].$$

Utilizando (6.21) obtenemos

$$||K_n * f - f||_{\alpha,p} \leq C_3\, \psi_\alpha\left(f, \frac{1}{n}\right)_p.$$

Como

$$E_n(f)_{\alpha,p} \leq ||K_n * f - f||_{\alpha,p},$$

tenemos el estimado (6.25). ■

6.4 Teorema inverso de tipo Bernstein para X_p^α

Para obtener este tipo de resultados, utilizaremos la técnica de los K-funcionales. También desarrollamos una desigualdad de tipo Bernstein, relativa a la acotación de derivadas. En este epígrafe hemos contado con sugerencias valiosas del Dr. Jorge Bustamante, que nos complace reconocer.

Si $f \in L_p$ y $t > 0$ (fijo), definimos

$$f_{2,t}(x) := -\left(\frac{2}{t}\right)^2 \int_0^{t/2}\int_0^{t/2} [f(x+2s_1+2s_2) - f(x+s_1+s_2)]\, ds_1\, ds_2. \quad (6.26)$$

Por definición tenemos

$$f_{2,t} \in W^1(L_p) \quad (6.27)$$

y casi dondequiera

$$f_{2,t}^1(x) := (f_{2,t})^1(x) = -\left(\frac{2}{t}\right)^2 \int_0^{t/2} \left[\frac{1}{2}\Delta_t f(x+2s) - 2\Delta_{\frac{t}{2}} f(x+s)\right] ds \quad (6.28)$$

Lema 1. *Si* $g \in W^1(L_p)$, *entonces existe una constante* C, *que sólo depende de* α *y de* p, *tal que*

$$\psi_\alpha(g,t)_p \leq C\, t^{1-\beta} ||g^1||_p. \quad (6.29)$$

Demostración:

$$\begin{aligned}\psi_\alpha(g,t)_p^p &= \frac{1}{2\pi^2}\int_0^t \left(\int_0^{2\pi} \left|\frac{\Delta_s g(x)}{s^\alpha}\right|^p dx\right) ds \\ &= \frac{1}{2\pi^2}\int_0^t\int_0^{2\pi} \left|\int_0^s \frac{g^1(x+y)\,dy}{s^\alpha}\right|^p dx\, ds.\end{aligned}$$

Supongamos que $p > 1$ y apliquemos la desigualdad de Hölder. Se obtiene

$$\begin{aligned}\psi_\alpha(f,t)_p^p &\leq \frac{1}{2\pi^2}\int_0^t\int_0^{2\pi} \frac{s^{\frac{p}{q}}}{s^{\alpha p}} \int_0^s \left|g^1(x+y)\right|^p dy\, dx\, ds \\ &= \frac{1}{2\pi^2}\int_0^t \frac{s^{\beta/q}}{s^{\alpha p}} \int_0^s\int_0^{2\pi} |g^1(x+y)|^p\, dx\, dy\, ds \\ &= \frac{1}{\pi}\int_0^t \frac{s^{p/q}}{s^{\alpha p}}\int_0^s ||g^1||_p^p\, dy\, ds\end{aligned}$$

$$= \frac{||g^1||_p^p}{\pi} \int^{t} g^{(1-\alpha)p}\, ds = \frac{t^{p(1-\beta)}}{\pi\, p(1-\beta)} ||g^1||_p^p .$$

de donde se obtiene (6.29). ■

Corolario 1.

$$W^1(L_p) \subset X_p^\alpha. \tag{6.30}$$

Lema 2. *Sea $f \in X_p^\alpha$, $t > 0$ y $f_{2,t}$ definida por* (6.26). *Entonces existe una constante C, que no depende de f, tal que*

$$||f - f_{2,t}||_p \leq C\, t^\beta \psi_\alpha(f, t)_p. \tag{6.31}$$

Demostración: Apliquemos la desigualdad de Hölder si $p > 1$ y después el teorema de Fubini, en el desarrollo.

$$||f - f_{2,t}||_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f - f_{2,t}|^p$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| \left(\frac{2}{t}\right)^2 \int_0^{t/2}\int_0^{t/2} [f(x + 2s_1 + 2s_2) - 2f(x + s_1 + s_2) + f(x)]\, ds_1\, ds_2 \right|^p dx$$

$$\leq \left(\frac{2}{t}\right)^2 \frac{1}{2\pi} \int_0^{2\pi}\int_0^{t/2}\int_0^{t/2} |\Delta^2_{s_1+s_2} f(x)|^p\, ds_1\, ds_2\, dx$$

$$= \left(\frac{2}{t}\right)^2 \int_0^{t/2}\int_0^{t/2} ||\Delta^2_{s_1+s_2} f||_p^p\, ds_1\, ds_2.$$

Como

$$||\Delta^2_{s_1+s_2} f||_p \leq 2||\Delta^1_{s_1}(f)||_p ,$$

sigue

$$||f - f_{2,t}||_p^p \leq \left(\frac{2}{t}\right)^2 2^p \int_0^{t/2}\int_0^{t/2} ||\Delta^1_{s_1} f||_p^p\, ds_1\, ds_2$$

$$\leq 2^p \left(\frac{2}{t}\right)^2 \int_0^{t/2}\int_0^{t/2} \frac{t^{\alpha p}||\Delta_{s_1}^1||_p^p}{s_1^{\alpha p}}\, ds_1\, ds_2$$

$$= C\, t^{\alpha p-1}\, \psi_\alpha(f,\, t)_t^p. \blacksquare$$

Lema 3. *Sean $f \in X_p^\alpha$, $t > 0$ y $f_{2,t}$ como en (6.26). Entonces existe una constante C, que no depende de f, tal que*

$$\psi_\alpha(f - f_{2,t}) \leq C\, \psi_\alpha(f,\, t)_p. \tag{6.32}$$

Demostración: Se estimarán separadamente las integrales sobre los intervalos $[0,\, t]$ y $[t,\, \pi]$.

Para el primer intervalo aplicando la desigualdad de Hölder, si $p > 1$, obtenemos

$$\psi_\alpha(f - f_{2,t},\, t)_p^p \leq \frac{2}{\pi^2\, t^2} \int_0^t \frac{1}{s^{\alpha p}} \int_0^{2\pi}\int_0^{t/2}\int_0^{t/2} \left|\Delta_{s_1+s_2}^2\, \Delta_s\, f(x)\right|^p ds_1\, ds_2\, dx\, ds$$

$$\leq \cdots \leq \frac{2}{\pi^2\, t^2} \int_0^{\frac{t}{2}} \int_0^{t/2}\int_0^{t} \int_0^{2\pi} \frac{1}{s^{\alpha p}} \left|\sum_{j=0}^{2} \binom{2}{j} \Delta_s\, f(x + j(s_1 + s_2))\right|^p dx\, ds\, ds_1\, ds_2$$

$$\leq \frac{2\cdot 3^p}{\pi^2\, t^2} \int_0^{t/2}\int^{t/2}\int_0^{t}\int_0^{2\pi} \left|\frac{\Delta_s f(x)}{s^\alpha}\right|^p dx\, ds\, ds_1\, ds_2 = 3^p \psi_\alpha(f,\, t)_p^p.$$

Sobre el intervalo $[t,\, \pi]$, el estimado se simplifica con la ayuda del lema 2,

$$\frac{1}{2\pi^2} \int_t^\pi \left(\int_0^{2\pi} \left|\frac{\Delta_s(f_{2,t} - f)(x)}{s^\alpha}\right|^p dx\right) ds \leq\; 2^p\, ||f_{2,t} - f||_p^p\, \frac{1}{\pi} \int_t^\pi \frac{ds}{s^{\alpha p}}$$

$$\leq\; C_2\, \psi_\alpha(f,\, t)_p^p$$

De las desigualdades obtenidas, se sigue (6.32).$\blacksquare$

Lema 4. *Sea $f \in X_p^\alpha$, $t > 0$ y $f_{2,t}$ definida como en (6.26). Entonces existe una constante C, que no depende de f, tal que*

$$||f_{2,t}^1||_p \leq C\, t^{\beta-1} \psi_\alpha(f,\, t)_p. \tag{6.33}$$

Demostración: Por (6.27) y (6.28)

$$||f^1_{2,t}(x)||_p \leq \frac{C_1}{t^2} \max_{k=1,2} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{t/2} \Delta_{k\frac{t}{2}} f(x+ks)ds \right|^p dx \right)^{\frac{1}{p}}.$$

Pero, para $k = 1, 2$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{t/2} \Delta_{k\frac{t}{2}} f(x+ks)ds \right|^p dx = \left(\frac{t}{2}\right)^{p-1} \int_0^{t/2} \frac{1}{2\pi} \int_0^{2\pi} \left| \Delta_{k\frac{t}{2}} f(x+ks) \right|^p dx\, ds$$

$$\leq \left(\frac{t}{2}\right)^{p-1} \left(\frac{t}{2}\right) w^p \left(f, \frac{kt}{2}\right)_p$$

$$\leq \left(\frac{t}{2}\right)^p w^p(f, t)_p.$$

Así que

$$||f^1_{2,t}(x)||_p \leq \frac{C_1}{t^2} \left(\left(\frac{t}{2}\right)^p w^p(f, t)_p \right)^{\frac{1}{p}} = C_1 t^{-1} w(f, t)_p.$$

Y por (21), tenemos

$$||f^1_{2,t}||_p \leq C\, t^{-1}\, t^{\beta}\, \psi_\alpha(f, t)_p. \blacksquare$$

Ahora estamos en condiciones de demostrar otros teoremas fundamentales del epígrafe.

Definición 3. Sea $f \in X^\alpha_p$ y $t > 0$. Entonces

$$K_\alpha(f, t)_p := \inf_{g \in W^1(L_p)} \{||f - g||_{\alpha,p} + t||g^1||_p\}, \tag{6.34}$$

donde $||\cdot||_{\alpha,p}$ es la norma en X^α_p.

Teorema 1. *Existen constantes C_1 y C_2, que no dependen de $f \in X^\alpha_p$ tales que, para todo $t > 0$, se tiene*

$$C_1\, \psi_\alpha(f, t)_p \leq K_\alpha(f, t^{1-\beta})_p \leq C_2\, \psi_\alpha(f, t)_p. \tag{6.35}$$

Demostración: Aplicando (6.29) a $f - g$, con $g \in W^1(L_p)$, y $C > 1$ tenemos

$$\psi_\alpha(f, t)_p \ \leq\ C \left[||f - g||_{\alpha,p} + t^{1-\beta} ||g^1||_p \right].$$

Tomando $C_1 = 1/C$ y el ínfimo en $g \in W^1(L_p)$ según (6.34), se obtiene la primera desigualdad de (6.35).

Apliquemos (6.34) con $g = f_{2,t}$ definida por (6.26) y según (6.27), esta función está en $W^1(L_p)$. Tenemos

$$K_\alpha(f, t^{1-\beta}) \leq ||f - f_{2,t}||_{\alpha,p} + t^{1-\beta}||f^1_{2,t}||_p, \tag{6.36}$$

y

$$||f - f_{2,t}||_{\alpha,p} \;\leq\; [||f - f_{2,t}||_p + \psi_\alpha(f - f_{2,t})_p] \, .$$

Por (6.31), se tiene

$$||f - f_{2,t}||_p \leq C_1\, t^\beta\, \psi_\alpha(f, t)_p \leq C_2\, \psi_\alpha(f, t)_p.$$

Por (6.32)

$$\psi_\alpha(f - f_{2,t})_p \leq C_3 \psi_\alpha(f, t)_p.$$

Para acotar el segundo sumando a la derecha de (6.36), aplicamos (6.33). Sustituyendo en (6.36) y acotando t por π, se tiene la segunda desigualdad de (6.35). ■

Finalmente, para obtener teoremas inversos, necesitamos una desigualdad de tipo Bernstein, dada por el teorema siguiente.

Teorema 2. *Existe una constante C, válida para todo polinomio trigonométrico t_n de grado n, tal que*

$$||t_p'|| \leq C\, n^{1-\beta}\, \psi_\alpha\left(t_n, \frac{\pi}{2n}\right)_p . \tag{6.37}$$

Demostración: Utilizaremos a continuación las desigualdades expresadas en Timan [9], página 215:

$$||t_n'||_p \leq \left(\frac{n}{2}\right) \left(\frac{1}{2\pi} \int_0^{2\pi} |\Delta_{(\pi/2n)}\, t_n(x)|^p \, dx\right)^{\frac{1}{p}} .$$

De aquí obtenemos

$$||t_n'||_p \leq C_1\, n\, w\left(t_n, \frac{\pi}{2n}\right)_p \leq C\, n \cdot n^{-\beta}\, \psi_\alpha\left(t_n, \frac{\pi}{2n}\right)_p . \blacksquare$$

Corolario 2. *Existe una constante $C > 0$, que no depende de $n = 0, 1, \cdots$, ni de $t_n \in \mathrm{T}_n$, tal que*

$$||t_n'||_p \leq C\, n^{1-\beta} ||t_n||_{\alpha,p}. \tag{6.38}$$

Finalmente demostremos el teorema inverso que completa la caracterización de los espacios X_p^α.

Teorema 3. *Existe una constante $C > 0$, válida para toda función $f \in X_p^\alpha$ y $n = 1, 2, \ldots$, tal que*

$$\psi_\alpha\left(f, \frac{1}{n}\right)_p \leq C \frac{1}{n^{(1-\beta)}} \sum_{k=0}^{n} k^{(1-\beta)} E_k(f)_{X_p^\alpha}. \tag{6.39}$$

Demostración: Del teorema 1 tenemos

$$\psi_\alpha\left(f, \frac{1}{n}\right)_p \leq C_1 k_\alpha \left(f, \frac{1}{n^{1-\beta}}\right)_p. \tag{6.40}$$

Como $K(f, \cdot)$ es monótono, bastará hacer las acotaciones del K-funcional para $n = 2^m$, $m = 0, 1, 2, \ldots$.

Para simplificar los subíndices, denotemos $\varphi_k := t_{2^k}$, al polinomio de la mejor aproximación de f a T_{2^k} en la norma X_p^α. O sea

$$||f - \varphi_k||_{\alpha,p} = E_{2^k}(f)_{X_p^\alpha}.$$

Definamos $\psi_k := \varphi_k - \varphi_{k-1}$, $k = 1, 2, \ldots$, y $\psi_0 := \varphi_0$. Luego $\psi_k \in T_{2^k}$. Tenemos

$$||\psi_k||_{\alpha,p} \leq ||f - \varphi_k||_{\alpha,p} + ||f - \varphi_{k-1}||_{\alpha,p} \leq 2\, E_{2^{k-1}}(f)_{X_p^\alpha}. \tag{6.41}$$

Utilizando que, por construcción, es $\varphi_m = \sum_{k=0}^{m} \psi_k$, la definición de $K-$funcional que hemos dado, la desigualdad de tipo Bernstein en (6.38) y que por ser $\psi_0 \in T_{2^0} = T_1$, es $||\psi_0^1||_p = 0$, tendremos

$$\begin{aligned} K_\alpha\left(f, \frac{1}{2^{m(1-\beta)}}\right)_p &\leq\ ||f - \varphi_m||_{X_p^\alpha} + 2^{-m(1-\beta)} ||\varphi_m^1||_p \\ &\leq\ E_{2^m}(f)_{X_p^\alpha} + 2^{-m(1-\beta)} \left(\sum_{k=1}^{m} ||\psi_k^1||_p\right) \\ &\leq\ E_{2^m}(f)_{X_p^\alpha} + C_1\, 2^{-m(1-\beta)} \left(\sum_{k=1}^{m} 2^{k(1-\beta)} ||\psi_k||_{X_p^\alpha}\right) \end{aligned}$$

$$\leq C_2 2^{-m(1-\beta)} \left(\sum_{k=1}^{m} 2^{k(1-\beta)} E_{2^{k-1}}(f)_{X_p^\alpha} \right)$$
$$\leq C n^{-(1-\beta)} \left(\sum_{k=0}^{n} k^{(1-\beta)} E_k(f)_{X_p^\alpha} \right),$$

que basta sustituir en (6.40) para obtener (6.39).■

6.5 Bibliografía

1. Anastassious G. A. and Gal S. G., *Approximation Theory, Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser 2000.

2. Bustamante J. and Jiménez M. A., *The degree of best approximation in the Lipschitz norm by trigonometric polynomials*, Aportaciones Matemáticas, 25 (1999), 23–30.

3. Bustamante J., Jiménez M. A. and Linares R. *Direct and inverse theorems in generalized Lipschitz spaces*, Automation Computers Applied Mathematics, Vol. 15 (2006) No. 1

4. Butzer P. L. and Berens, H., *Semi–Groups of Operators and Approximation*, Springer–Verlag (1967).

5. DeVore R. A. and Lorentz, G. G., *Constructive Approximation*, Grundlehren der mathematischen Wissenschaften 303, Springer–Verlag (1993).

6. Jiménez M. A., *A new approach to Lipschitz spaces of periodic integrable functions*, Aportaciones Matemáticas, Serie Comunicaciones, 25 (1999), 153–157.

7. Jiménez M. A., *A tauberian theorem for a class of function spaces*, Proceeding of the International Conference on Multivariate Approximation and Interpolation with Application, Amuñecar (2001), Memorias de la Academia de Ciencias de Zaragoza. Nr 20 (2002), 77–85.

8. Jiménez M. A. and Linares R., *Another approach to generalized Lipschitz spaces*, Aportaciones Matemáticas, Serie Investigación, 18 (2004), 155–167.

9. Timan A. F., *Theory of Approximation of Functions of a Real Variable*, MacMillan, New York, 1963.

10. Tribel H., *Theory of Functions Spaces*, Vol. 1 and 2, Birkhaüser Vellag, Basel, Vol. 1 in 1983, Vol. 2 in 1992.

11. Zygmund A., *Smooth functions*, Duke Math. J., 12 (1945), 47–76.

Capítulo 7

On the completion of the product of topological measures

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Abstract: Denote by $\sigma(\mathbb{D})$ the $\sigma-$algebra generated by a class $\mathbb{D}$ of subsets of a given set. Let $(X, \mathbb{T}_X)$ and $(Y, \mathbb{T}_Y)$ be nontrivial metrizable topological spaces. Assume μ_X and μ_Y are positive $\sigma-$finite inner regular measures on $\mathbb{B}_X := \sigma(\mathbb{T}_X)$ and $\mathbb{B}_Y := \sigma(\mathbb{T}_Y)$, respectively. Hereby we prove the $\mu_X \otimes \mu_Y$ completion of the product $\sigma-$algebra $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains the $\sigma-$algebra $\mathbb{B}_{X\times Y} := \sigma(\mathbb{T}_X \otimes_\tau \mathbb{T}_Y)$ generated by the open sets of the topological product space. This result is based on the fact the supports of the measures with the induced topologies satisfy the second axiom of countability. It follows every real or complex continuous function on $(X \times Y, \mathbb{T}_X \otimes_\tau \mathbb{T}_Y)$ is $\mu_X \otimes \mu_Y-$measurable a. e. and as a consequence the extended version of Fubini theorem on iterated integrals could be applied.

Key words Fubini theorem, Haar measure, Inner regular measure, Parametric extension of operators, Product measure problem, Radon measure.

Mathematical Subject Classification Primary 28 C 15, Secondary 28 C 05, 28 C 10, 47 B 99.

7.1 Introduction

In this paper we are going to examine the so called *product measure problem* and its consequence on the application of Fubini theorem to related topics of Approximation theory. We refer the reader to [1], [2], [8], [11], [12] and [17] for the fundaments of Topology and Measure we are dealing with. Nevertheless, we will collect the notations and review the main concepts and results to be used here. The topological measure problem in hands has own interest as we will comment later. Another motivation for us comes from the following subject.

Let X and Y be nontrivial topological spaces. Suppose $X \times Y$ carries the product topology generated by the first two given spaces. Define $x^{\backsim} : C(X) \to \mathbb{R}$, by $x^{\sim}(f) = f(x)$ and $x^{\backsim} : C(X \times Y) \to C(X)$ by $x^{\sim}f(y) = f(x, y)$. The *parametric extension* $L^* : C(X \times Y) \to C(X \times Y)$ of a linear operator $L_X = L : C(X) \to C(X)$ is defined by $(L^*f)(x, y) = x^{\sim}Ly^{\sim}f$. The parametric extension of operators already appears in so old works as the one by Neder [15] in 1926. Further developments of the concept may be found in Stancu [18], Haussmann and Pottinger [9] and Mastroianni [14]. An important review of this and other related topics can be found in Gonska [6].

As observed by Cheney and Gordon [3] and later but independently by Potapov and Jiménez-Pozo [16], the parametric extensions of continuous operators are examples of linear operators on $C(X \times Y)$ which commute with each other, i. e. $L_X^* \circ L_Y^* = L_Y^* \circ L_X^*$. The proof given in [16] was based upon the classical Fubini theorem after the Riesz representation of the linear functionals $x^{\backsim}$ and $y^{\backsim}$ by means of regular measures . Since the main objective of Potapov and Jiménez-Pozo was the application of the Boolean sum of Bernstein type operators $Bm_X \oplus Bn_Y = Bm_X^* + Bn_Y^* - Bm_X^* \circ Bn_Y^*$ to the approximation of bivariate functions in $C\left([0,1]^2\right)$, the hypothesis of Fubini theorem are in their standard context (Instead Boolean sum of Bm_X and Bn_Y, the mathematical name given in [16] to $Bm_X \oplus Bn_Y$ is *angular operator*. The explanation of this term -originated by the former Soviet school- can be found in [12]). As soon as the topological spaces are arbitrary, a rigorous application of the classical Fubini theorem does not hold. In fact, we shall see below the product of the Borel $\sigma-$algebras involved in the process could be strictly included in the $\sigma-$algebra generated by the product of the topologies even for locally compact spaces, thus one cannot affirm a priory that a continuous function on the topological product space is measurable with respect to the set theoretic product space of the

$\sigma-$algebras.

In this paper we will know conditions under which one can apply the classical Fubini theorem or other related topics to continuous functions on the product of Hausdorff topological spaces and the product of measures on the Borel subsets. Let us begin by showing that the proof of the commutativity of the parametric extension given by Potapov and Jiménez can be adapted to the case of spaces of real or complex valued continuous functions with compact support defined on locally compact Hausdorff spaces mutatis mutandis.

In fact, under those conditions, we only need to consider $x^{\sim}$ and $y^{\sim}$ as Radon measures (c. f. [1], [2] or [12]). Expressing $x^{\backsim} : C(X \times Y) \to C(X)$ and $y^{\backsim} : C(X \times Y) \to C(Y)$ by means of integrals with respect to the corresponding Radon measures, we arrive to the expressions

$$(L_X^* f)(x, y) = \int_X f(s, y)\, d\mu_{x^{\sim}}$$

and

$$(L_Y^* f)(x, y) = \int_Y f(x, t)\, d\mu_{y^{\sim}}.$$

The definition of the Radon product measure (see theorem 5.2, page 263, of [12]) let us express

$$\begin{aligned} L_X^* \circ L_Y^* (f; (x, y)) &= \int_Y \left[\int_X f(s, t)\, d\mu_{x^{\sim}} \right] d\mu_{y^{\sim}} = \\ &= \int_X \left[\int_Y f(s, t)\, d\mu_{y^{\sim}} \right] d\mu_{x^{\sim}} = L_Y^* \circ L_X^* (f; (x, y)). \end{aligned}$$

As we already pointed out, for arbitrary general spaces X and Y we are not able of legitimating the application of Fubini theorem with simple hypothesis of continuity. In the next sections we not only examine the reasons of this problem but also prove that under certain conditions the extended version of Fubini theorem for completed measures applies.

7.2 The product measure problem

In what follows we shall need of exact notations. Let $(X, \mathbb{T}_X)$ and $(Y, \mathbb{T}_Y)$ be nontrivial Hausdorff topological spaces. Denote by $\sigma(\mathbb{D})$ the $\sigma-$algebra generated by a class $\mathbb{D}$ of subsets of a given set, thus $\mathbb{B}_X := \sigma(\mathbb{T}_X)$ and

$\mathbb{B}_Y := \sigma(\mathbb{T}_Y)$ are the corresponding Borel $\sigma-$algebras generated by these topologies. In the product spaces we always have

$$\begin{aligned}\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y \quad : &= \sigma(\mathbb{B}_X \times \mathbb{B}_Y) = \sigma(\sigma(\mathbb{T}_X) \times \sigma(\mathbb{T}_Y)) \\ &= \sigma(\mathbb{T}_X \times \mathbb{T}_Y) \subset \sigma(\mathbb{T}_X \otimes_\tau \mathbb{T}_Y) := \mathbb{B}_{X \times Y}.\end{aligned}$$

Let us recall that X satisfies the second axiom of countability, if there exists a countable basis of the topology. If X and Y satisfy this axiom, then $\sigma(\mathbb{T}_X \times \mathbb{T}_Y) = \sigma(\mathbb{T}_X \otimes_\tau \mathbb{T}_Y)$ and there is a chain of equalities in the expressions above. But in other cases the situation can be different. In fact, it is known (see [13], page 132) that for every set Z whose cardinal is strictly larger than the cardinal of the continuous and any two $\sigma-$algebras $\mathbb{B}_1$ and $\mathbb{B}_2$ in Z, the diagonal $\Delta_{Z \times Z}$ does not belong to the set theoretic product of $\sigma-$algebras $\mathbb{B}_1 \otimes_\sigma \mathbb{B}_2 = \sigma(\mathbb{B}_1 \times \mathbb{B}_2)$. Thus, if X has a strictly larger cardinal than the cardinal of the continuous, we have $\Delta_{X \times X} \notin \mathbb{B}_X \otimes_\sigma \mathbb{B}_X$. On the other hand, since $(X, \mathbb{T}_X)$ is a Hausdorff space, the diagonal $\Delta_{X \times X}$ is closed in the product topological space $X \times X$. Thus $\Delta_{X \times X} \in \mathbb{B}_{X \times X} := \sigma(\mathbb{T}_X \otimes_\tau \mathbb{T}_X)$. For instance, we may choose the Stone-Čech compactification of the natural numbers that is separable but has a strictly larger cardinal than the one of the continuous. We have obtained that $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y \subset \mathbb{B}_{X \times Y}$ and the inclusion could be strict. However, this lack may be partially covered whenever the completion of the product measure generates a $\sigma-$algebra that contains the product of the topologies and then the extended version of Fubini theorem applies.

Assume that μ_X and μ_Y are $\sigma-$finite measures on $\mathbb{B}_X$ and $\mathbb{B}_Y$. Thus the standard product $\mu_X \otimes \mu_Y$ on the $\sigma-$algebra $\mathbb{A}$, the completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, is derived with uniqueness by Carathéodory's method from the outer measure generated by the set function $\mu_X \otimes \mu_Y(A \times B) = \mu_X(A)\mu_Y(B)$, where $A \in \mathbb{B}_X, B \in \mathbb{B}_Y, \mu_X(A) < \infty$ and $\mu_Y(B) < \infty$. It could be expected that $(X \times Y, \mathbb{T}_{X \times Y}, \mathbb{A}, \mu_X \otimes \mu_Y)$ is again a topological measure space, i. e. that $\mathbb{B}_{X \times Y} \subset \mathbb{A}$. This conjecture was known as the *product measure problem* and remained open long time. Finally, Fremlin [4] presented a counterexample. In a relatively recent paper, Fremlin and Grekas [5] presented a sufficient criterion that guarantees a positive answer to the product measure problem: "If X is quasi-dyadic, μ_X completion regular and μ_Y is $\tau-$additive, then $(X \times Y, \mathbb{T}_{X \times Y}, \mathbb{A}, \mu_X \otimes \mu_Y)$ is a topological measure space". An evolution of the ideas of this subject may be found in Grekas' work [7].

Since metric spaces are quasi-dyadic, one can apply Fremlin and Grekas theorem to the most important function spaces of Approximation Theory.

However, the terminology and the scope of this theorem become too extensive to be described here and its proof is rather complicated and difficult. Thus, one of our objectives here is to present a simple but useful variation for the particular case of metric spaces, the most common underlying topologies that support the functional spaces of Approximation theory.

7.3 A particular solution to the product measure problem

Let μ be a measure on $\mathbb{B}_X$. A Borel set E in X is $\mu-$inner regular (or inner regular for the sake of simplicity) if $\mu(E) = \sup\{\mu(K) : K \subset E, K \in \mathbb{K}\}$, where $\mathbb{K}$ stands for the class of compact subsets of X. The measure itself is called inner regular whenever all Borel sets have the corresponding properties of regularity. If μ is bounded on the compact sets, then it is called a Borel measure.

Let $(X, \mathbb{T}_X)$ be a Hausdorff topological space and μ any positive measure on its Borel subsets. To prove the existence of a maximal $\mu-$negligible open set W or equivalent, the existence of a smaller closed set F that supports the measure, it is sufficient to assume that open sets are μ-inner regular. In fact, we obviously define $F := W^C$ and W to be the union of all $\mu-$negligible open sets $(U_i)_{i\in I}$. If $K \subset W$ and $K \in \mathbb{K}$, there exists a finite subfamily $(U_i)_{i\in J}$, $J \subset I$, that still covers the compact set K. Then $\mu(K) \leq \sum_{i\in J} \mu(U_i) = 0$. Since W is μ-inner regular and K was arbitrarily chosen, it follows that also $\mu(W) = 0$. Now we are in a position to state the following

Teorema 7.3.1 *Let (X, d) be a metric space and μ a $\sigma-$finite and inner regular measure on the Borel sets of X. Let F be the support of μ. Then F, with the induced topology, satisfies the second axiom of countability.*

Proof. First suppose $F = X$. It follows any non-empty open set in X has a strictly positive measure. Since μ is $\sigma-$finite, there exists a sequence (E_n) of disjoint Borel sets of finite $\mu-$ measures, such that

$$X = \sqcup_n E_n \text{ and } \mu(X) = \sum_n \mu(E_n).$$

Using that μ is inner regular, we find sequences $(K_{m,n}) \subset \mathbb{K}$, such that

$$K_{m,n} \subset E_n \text{ and } \mu(E_n) = \mu(\cup_m K_{m,n}), \text{ for every } n = 1, 2, ...$$

Since X is a metric space, for each couple (m, n), the compact set $K_{m,n}$ is separable. Let $A_{m,n}$ be countable sets dense in $K_{m,n}$.Thus $A := \sqcup_{m,n} A_{m,n}$ is a countable set that is dense in $\sqcup_{m,n} K_{m,n}$. We claim that A is also dense in X. In fact, the existence of an open set in X disjoint with $\cup_n K_{m,n}$ should contradict the hypothesis about the non-null measures of the open sets unless this open set be empty. Thus X is separable. Since separability and the second axiom of countability are equivalent in metric spaces, the theorem yields in the case $F = X$. If $F \subsetneq X$, the compact subsets of F with the induced topology are just the compact subsets of X included in F, thus the restriction of the measure μ to F is still inner regular. Set $W := F^C$. The subset W is the maximal $\mu-$negligible open set of X. Let V be any open set in F. Then $V = U \cap F$, for some open set U in X. From $U = V \cup (U \setminus F)$, we obtain $\mu(U) = \mu(V)$. If $\mu(V) = 0$, it follows that $U \subset W$ and $V = \phi$. Then, we can apply to F in the place of X, the first part of the theorem.■

Teorema 7.3.2 *Let $(X, \mathbb{T}_X)$ and $(Y, \mathbb{T}_Y)$ be metrizable topological spaces. Let μ_X and μ_Y be $\sigma-$finite positive and inner regular measures on $\mathbb{B}_X$ and $\mathbb{B}_Y$, respectively. Then the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{B}_{XxY}$.*

Proof. Obviously, it is sufficient to prove that the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{T}_{X \times Y}$. Express X and Y as the disjoint unions $F_X \cup W_X$ and $F_Y \cup W_Y$, where F_X and F_Y are the supports of μ_X and μ_Y, respectively. Thus any open set in $\mathbb{T}_{X \times Y}$ can be expressed as a union of a relatively open set in $\mathbb{T}_{F_X \times F_Y}$ with open sets in $\mathbb{T}_{W_X \times Y}$ and $\mathbb{T}_{X \times W_Y}$. By the theorem above, F_X and F_Y, with the induced topology, satisfy the second axiom of countability. Thus $\mathbb{B}_{F_X} \otimes_\sigma \mathbb{B}_{F_Y} = \mathbb{B}_{F_X \times F_Y} \supset \mathbb{T}_{F_X \times F_Y}$. On the other hand $W_X \times Y \in \mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, $X \times W_Y \in \mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, $\mu_X \otimes \mu_Y (W_X \times Y) = 0$ and $\mu_X \otimes \mu_Y (X \times W_Y) = 0$. It follows the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{T}_{W \times Y}$.■

7.4 Applications

The first application of theorem 7.3.2 is to guarantee under its hypothesis any function in $C(X \times Y)$ is $\mu_X \otimes \mu_Y-$measurable a. e.. Thus it is possible the application of the extended Fubini theorem (see [12], for instance) on iterated integrals whenever the function is positive or integrable. Other applications are also immediate. For a reference to the next one see [10].

Teorema 7.4.1 *Let G be an abelian metrizable locally compact group for which the associated Haar measure is $\sigma-$finite . Then G satisfies the second axiom of countability.*

Proof. To apply theorem 7.3.2, observe the Haar measure does not vanish on the non-empty open sets of G, while its inner regular property follows from the Riesz representation theorem and the hypothesis of $\sigma-$finiteness (see [12] for instance).■

If X and Y are locally compact, let L_X and L_Y be positive linear functionals on the spaces of continuos complex functions with compact supports $C_{\mathcal{K}}(X)$ and $C_{\mathcal{K}}(Y)$, respectively. By the Riesz representation theorem for positive linear functionals there exist measures μ_X and μ_Y defined on $\mathbb{B}_X$ and $\mathbb{B}_Y$, respectively, that represent the functionals by integrals. Moreover, μ_X and μ_Y can be chosen with certain regular properties and they are unique under such regularity assumptions. These measures as well as the corresponding functionals are equally called Radon measures. As we have already pointed out, there exists a technique to define the product Radon measure L_{XxY} and then the Radon associated measure $\mu_{X \times Y}$ is defined on $\mathbb{B}_{X \times Y}$. It can be proved that $\mu_{X \times Y}(A \times B) = \mu_X(A)\,\mu_Y(B)$ for every Borel sets $A \subset X$ and $B \subset Y$. Then the restriction $\mu_{X \times Y}/\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, coincides with $\mu_X \otimes \mu_Y$ whenever μ_X and μ_Y are $\sigma-$finite. But more is true:

Teorema 7.4.2 *Let $(X, \mathbb{T}_X)$ and $(Y, \mathbb{T}_Y)$ be metrizable locally compact topological spaces. Let μ_X and μ_Y be $\sigma-$finite positive Radon measures on $\mathbb{B}_X$ and $\mathbb{B}_Y$, respectively. Then the completion of the Radon measure $\mu_{X \times Y}$ defined on $\mathbb{B}_{X \times Y}$ and the product measure $\mu_X \otimes \mu_Y$ defined on $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, are equal.*

Proof. The restriction $\mu_{X \times Y}/\mathbb{B}_X \otimes \mathbb{B}_Y$, coincides with $\mu_X \otimes \mu_Y$ and from theorem 7.3.2, the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{B}_{X \times Y}$.■

7.5 References

1. Bourbaki, N. *Éléments de Mathématique, Livre III Topologie Général*, Chapitres 1-4, Hermann, Paris1971.

2. Bourbaki, N. *Éléments de Mathématique, Livre VI Integration*, Chapitres 1-4, 5, 7-8 & 9 in Actualités Scientifiques et Industrielles , Hermann, nr. 1175, 1244, 1306 & 1343, Paris1965, 1967, 1968 & 1969, respectively.

3. Cheney E. W. & Gordon, W. J. *Bivariate and multivariate interpolation with noncommutative projectors* in Linear Spaces and Approx. (ed. by P. L. Butzer & B. Sz. Nagy), Basel: Birkhäuser,1978, 381-387.

4. Fremlin, D. H. *Products of Radon measures: a counterexample* Canad. Math. Bull. 19 (1976), 285-289.

5. Fremlin, D. H. & Grekas, S. *Products of completion regular measures* Fund. Math. 147, No.1 (1995), 27-37.

6. Gonska, H. H. *Quantitative Approximation in* $C(X)$, Dept. of Math. Duisburg Univ. Nr.94, 1986.

7. Grekas, S. *On products of topological measure spaces* in Pap, E. (ed.) Handbook on Measure Theory , Vol. I and II, Amsterdam, (2002), 745-764.

8. Halmos, P. R., *Measure Theory*, Springer New York. 1974.

9. Haussmann, W. & Pottinger, P. *On the construction and convergence of multivariate interpolation operators*, J. Approx. Theory 19, 1977, 205-221.

10. Hewitt, E. & Ross, K. A. *Abstract Harmonic Analysis.* Springer Verlag, 1963

11. Hinrichsen D. et al, *Topología General*, Aportaciones Matemáticas, Textos 22, México D. F. , 2003.

12. Jiménez-Pozo, M. A. *Medida, Integración y Funcionales* Editorial Pueblo y Educación, Habana,1989.

13. König, H. *Measure and Integration*, Springer, Berlin Heidelberg1997.

14. Mastronniani, G. *Sull' approssimazione di funzioni continue mediante operatori lineari*, Calcolo 14,1977, 343-368.

15. Neder, L. *Interpolationsformein für Funktionen mehrerer Argumente,* Scandinavisk Aktuarietidskrift 9, 1926, 59-69.

16. Potapov, M. K. & Jiménez-Pozo, M. A., *Sobre la aproximación mediante ángulos en espacios de funciones continuas*, Cienc. Mat. Univ Habana, 2, 1981, 101-110.

17. Rudin, W. *Real and Complex Analysis* McGraw Hill, New York, 1966.

18. Stancu, D. D. *The remainder of certain linear approximation formulas in two variables*, SIAM J. Numer. Anal. Ser. B 1, 1964, 137-163.

Capítulo 8

A note on skew-orthogonal polynomials

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Abstract: In this note we introduce a general approach to the skew-orthogonal polynomials. We propose an anologue of the Gram-Schmidt method in the skew-orthogonal case for any real vector space.

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Keywords: Skew-orthogonal polynomials, Skew-orthogonalization process

8.1 Introduction

The skew-orthogonal polynomials were introduced in 1970 by Dyson in his phenomenal work about correlations between the eigenvalues of a random matrix (see [4]), and later developed by Mehta (the reader should consult the book [7] for more ditails) and others. Recently also these polynomials were studied by Adler, Horozov and P.Van Moerbeke in [1], related with the close connection which exists between them and some integrable systems (more exactly Pfaff lattices). On the contrary to what happens with the usual orthogonal polynomials, the theory for these skew-orthogonal polynomials is in its childhood. To the best of our knowledge, a method to build these polynomials was presented for the first time in [1]. In this note, we propose a general and simple setting for these polynomials that turn out differently than was proposed by Adler, Horozov and P.Van Moerbeke. Our method emphasizes the appearance of non-uniqueness in the construction of these polynomials. Our approach is very similar to the classic way to expose the orthogonal polynomials which can be found, for example, in [2] and [3]. Finally, for all real vector space the notion of skew-orthogonal systems of vectors is introduced and a skew orthogonal version of the Gram-Schmidt method is presented.

8.2 General approach for skew-orthogonal polynomials

Let $M = (c_{i,j})$ be an infinite matrix of real numbers, where $i, j = 0, 1, 2, \ldots$ In this section E always denotes the vector space of polynomials with real coefficients. We define on $E \times E$ a functional Ω. First, we do

$$\Omega\left(x^i, x^j\right) = c_{i,j} \qquad i, j = 0, 1, 2, \ldots \tag{1}$$

and extend this to all $E \times E$ as follows: let $P(x) = \sum_{k=0}^{n} p_k x^k$, $Q = \sum_{i=0}^{m} q_i x^i$ two real polynomials, then we define

$$\Omega(P, Q) = \sum_{k=0}^{n}\sum_{i=0}^{m} p_k q_i \Omega\left(x^k, x^i\right) = \sum_{k=0}^{n}\sum_{i=0}^{m} p_k q_i c_{i,j}. \tag{2}$$

It is easy to show that $\Omega(.,.)$ is bilinear and it coincide with the usual Cauchy product of two polynomials with respect to the matrix M. Let us choose M, such that $c_{i,j} = -c_{j,i}$ (it follows that $c_{i,i} = 0$ for any i), then under this condition we have $\Omega(P, Q) = -\Omega(Q, P)$, that is, Ω turns out to be an anti-symmetric bilinear functional.

Definition 8.2.1 *Let $\Omega(.,.)$ be an anti-symmetric bilinear functional defined as above. We say that the sequence $\{P_n\}_{n\geq 0}$ of polynomials is skew-orthogonal if $\deg P_n \leq n$ and a) $\Omega(P_{2n}, P_{2m}) = 0 = \Omega(P_{2n+1}, P_{2m+1})$ for $n, m = 0, 1, \ldots$, b) $\Omega(P_{2n}, P_{2m+1}) = \gamma_n \delta_{n,m}$ for $n, m = 0, 1, \ldots$, here $\{\gamma_n\}_{n\geq 0}$ is a sequence of real non-zero numbers.*

It should be observed that if P_{2n} and P_{2n+1} are skew-orthogonal, then P_{2n} and $P_{2n+1} + bP_{2n}$ for all $b \in \mathbb{R}$ are also skew-orthogonal.

Obviuosly a sequence formed by real polynomials $\{P_n\}_{n\geq 0}$ which satisfies the following conditions:

$$\Omega\left(P_{2n}, x^k\right) = 0 = \Omega\left(P_{2n+1}, x^k\right) \qquad \text{for } k = 0, 1, \ldots, 2n-1 \text{ and } n \geq 1, \tag{3}$$

$$\Omega\left(P_{2n}, P_{2n+1}\right) = \gamma_n \qquad \text{for every } n \geq 0, \tag{4}$$

is skew-orthogonal. Note that $\Omega(P_{2n}, Q) = \Omega(P_{2n+1}, Q) = 0$ for every polynomial Q with degree at most $2n-1$ (see (3)).

The matrix M is called admissible if $c_{i,j} = -c_{j,i}$ for any i, j and moreover

$$H_{2n}(M) = \begin{vmatrix} 0 & c_{1,0} & c_{2,0} & \cdots & c_{2n-2,0} & c_{2n-1,0} \\ c_{0,1} & 0 & c_{2,1} & \cdots & c_{2n-2,1} & c_{2n-1,1} \\ c_{0,2} & c_{1,2} & 0 & \cdots & c_{2n-2,2} & c_{2n-1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-2} & c_{1,2n-2} & c_{2,2n-2} & \cdots & 0 & c_{2n-1,2n-2} \\ c_{0,2n-1} & c_{1,2n-1} & c_{2,2n-1} & \cdots & c_{2n-2,2n-1} & 0 \end{vmatrix} \neq 0,$$

and

$$I_{2n+1}(M) = \begin{vmatrix} 0 & c_{1,0} & c_{2,0} & \cdots & c_{2n-1,0} & c_{2n,0} \\ c_{0,1} & 0 & c_{2,1} & \cdots & c_{2n-1,1} & c_{2n,1} \\ c_{0,2} & c_{1,2} & 0 & \cdots & c_{2n-1,2} & c_{2n,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-1} & c_{1,2n-1} & c_{2,2n-1} & \cdots & 0 & c_{2n,2n-1} \\ c_{0,2n+1} & c_{1,2n+1} & c_{2,2n+1} & \cdots & c_{2n-1,2n+1} & c_{2n,2n+1} \end{vmatrix} \neq 0,$$

where $n = 1, 2, \ldots$.

If the anti-symmetric bilinear functional $\Omega(.,.)$ is defined by a matrix M which is admissible, we say that $\Omega(.,.)$ is skew-determined.

The following fact is well-known: if the matrix M is admissible, then

$$K_{2n+1} = \begin{vmatrix} 0 & c_{1,0} & c_{2,0} & \cdots & c_{2n-1,0} & c_{2n+1,0} \\ c_{0,1} & 0 & c_{2,1} & \cdots & c_{2n-1,1} & c_{2n+1,1} \\ c_{0,2} & c_{1,2} & 0 & \cdots & c_{2n-1,2} & c_{2n+1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-1} & c_{1,2n-1} & c_{2,2n-1} & \cdots & 0 & c_{2n+1,2n-1} \\ c_{0,2n+1} & c_{1,2n+1} & c_{2,2n+1} & \cdots & c_{2n-1,2n+1} & 0 \end{vmatrix} = 0.$$

We are now in a position to prove the main result of this section which can be summarized in the following statement

Teorema 8.2.1 *Let us assume that $\Omega(.,.)$ is skew-determined, then if we take $P_0(x) = \frac{\gamma_1}{c_{0,1}}$, $P_1(x) = x + b_0$ and*

$$P_{2n}(x) = \frac{D_{2n}}{H_{2n}} \begin{vmatrix} 0 & c_{1,0} & c_{2,0} & \cdots & c_{2n-1,0} & c_{2n,0} \\ c_{0,1} & 0 & c_{2,1} & \cdots & c_{2n-1,1} & c_{2n,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-2} & c_{1,2n-2} & c_{2,2n-2} & \cdots & c_{2n-1,2n-2} & c_{2n,2n-2} \\ c_{0,2n-1} & c_{1,2n-1} & c_{2,2n-1} & \cdots & 0 & c_{2n,2n-1} \\ 1 & x & x^2 & \cdots & x^{2n-1} & x^{2n} \end{vmatrix}, \tag{5}$$

$$P_{2n+1}(x) = F_{2n} \begin{vmatrix} 0 & c_{1,0} & \cdots & c_{2n-1,0} & c_{2n,0} & c_{2n+1,0} \\ c_{0,1} & 0 & \cdots & c_{2n-1,1} & c_{2n,1} & c_{2n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-1} & c_{1,2n-1} & \cdots & 0 & c_{2n,2n-1} & c_{2n+1,2n-1} \\ 0 & 0 & \cdots & 0 & 1 & -b_n \\ 1 & x & \cdots & x^{2n-1} & x^{2n} & x^{2n+1} \end{vmatrix}, \tag{6}$$

where $n = 1, 2, \ldots$ and b_n is an arbitrary constant for all n, lastly $F_{2n} = \frac{1}{H_{2n}}$ and $D_{2n} = \frac{\gamma_n . H_{2n}}{I_{2n+1}}$. Then $\{P_n\}_{n \geq 0}$ is skew-orthogonal.

Proof We must check (3) and (4). Now, from (5) it follows that

$$\Omega\left(P_{2n}(x), x^k\right) = \frac{D_{2n}}{H_{2n}} \begin{vmatrix} 0 & c_{1,0} & \cdots & c_{2n-1,0} & c_{2n,0} \\ c_{0,1} & 0 & \cdots & c_{2n-1,1} & c_{2n,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-2} & c_{1,2n-2} & \cdots & c_{2n-1,2n-2} & c_{2n,2n-2} \\ c_{0,2n-1} & c_{1,2n-1} & \cdots & 0 & c_{2n,2n-1} \\ c_{0,k} & c_{1,k} & \cdots & c_{2n-1,k} & c_{2n,k} \end{vmatrix},$$

for all k. Thus, we see that two rows are identical for $k = 0, 1, \ldots, 2n-1$, so $\Omega\left(P_{2n}(x), x^k\right) = 0$ for any $k = 0, 1, \ldots, 2n-1$. A similar argument is used to prove that $\Omega\left(P_{2n+1}(x), x^k\right) = 0$ if $0 \leq k \leq 2n-1$. Now we observe that $\Omega(P_{2n}, P_{2n+1})$ is equal to

$$F_{2n}\begin{vmatrix} 0 & c_{1,0} & c_{2,0} & \cdots & c_{2n-1,0} & c_{2n,0} & c_{2n+1,0} \\ c_{0,1} & 0 & c_{2,1} & \cdots & c_{2n-1,1} & c_{2n,1} & c_{2n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,2n-1} & c_{1,2n-1} & c_{2,2n-1} & \cdots & 0 & c_{2n,2n-1} & c_{2n+1,2n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -b_n \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Omega\left(P_{2n}, x^{2n+1}\right) \end{vmatrix},$$

here we have used the fact that also $\Omega\left(P_{2n}, x^{2n}\right) = 0$ (since $\Omega\left(P_{2n}, x^{k}\right) = 0$ for all $k = 0, 1, \ldots, 2n-1$ and $\Omega\left(P_{2n}, P_{2n}\right) = 0$). Then, we have $\gamma_n = \Omega\left(P_{2n}, x^{2n+1}\right) = \frac{D_{2n} I_{2n+1}}{H_{2n}}$.■

Observe that, from the latter equality at the end of the proof of Theorem 2 we can calculate D_{2n}. As in the classic case we have

Proposición 8.2.1 *Let $\{P_n\}_{n\geq 0}$ be a skew-orthogonal system of polynomials associated to $\Omega(.,.)$ which is skew-determined, then the P_n are linearly independent.*

Proof Let $\{P_{n_1}, P_{n_2}, \ldots, P_{n_m}\}$ be an arbitrary finite subset of $\{P_n\}_{n\geq 0}$. If $b_{n_1}, b_{n_2}, \ldots, b_{n_m}$ are such that

$$b_{n_1} P_{n_1}(x) + b_{n_2} P_{n_2}(x) + \ldots + b_{n_m} P_{n_m}(x) = 0 \qquad \forall\, x, \qquad (5)$$

then, if for some n_i it holds that $n_i = 2k_i$ we have

$$0 = \Omega\left(b_{n_1} P_{n_1} + b_{n_2} P_{n_2} + \ldots + b_{n_m} P_{n_m}, P_{2k_i+1}\right)$$

$$= b_{n_i} \Omega\left(P_{n_i}, P_{2k_i+1}\right) = b_{n_i} \gamma_{k_i}$$

hence $b_{n_i} = 0$. It may happen that $P_{2k_i+1} \in \{P_{n_1}, P_{n_2}, \ldots, P_{n_m}\}$, but note that this possibility is not excluded. In this form, we prove that all the even coefficients in (5) are zero. When n_i is odd, the proof is similar.■

Remark 8.2.1 *It is natural to propose a reproducing kernel in the context of the skew-orthogonal polynomials. Indeed, let us assume that* $\deg P_k = k$ *for $k = 0, 1, \ldots, 2n+1$, then we define*

$$K_n(x, y) = \sum_{i=0}^{n} \frac{\left(P_{2i}(x) P_{2i+1}(y) - P_{2i}(y) P_{2i+1}(x)\right)}{\gamma_i},$$

it is easy to see that $\Omega_x(K_n(x,y), P_{2k}(x)) = P_{2k}(y)$ *and*

$$\Omega_x(K_n(x,y), P_{2k+1}(x)) = P_{2k+1}(y)$$

if $k \leq n$. *For instance, let us see the first equality*

$$\begin{aligned}
\Omega_x(K_n(x,y), P_{2k}(x)) &= \sum_{i=0}^{n} \frac{P_{2i+1}(y)\,\Omega_x(P_{2i}(x), P_{2k}(x))}{\gamma_i} \\
&\quad - \sum_{i=0}^{n} \frac{P_{2i}(y)\,\Omega_x(P_{2i+1}(x), P_{2k}(x))}{\gamma_i} \\
&= -\frac{P_{2k}(y)\,\Omega_x(P_{2k+1}(x), P_{2k}(x))}{\gamma_k} \\
&= P_{2k}(y),
\end{aligned}$$

now, it follows that for any polynomial q *of degree* $\leq 2n+1$, $\Omega_x(K_n(x,y), q(x)) = q(y)$. *These properties justify the name of reproducing kernel of order* n *for* $K_n(x,y)$.

8.3 Gram-Schmidt skew-orthogonalization

Up to now we have dealt only with skew-orthogonal systems whose vectors are polynomials, however this section is dedicated to exhibit an anologue for the skew-orthogonal case of the Gram-Schmidt method in any real vector space.

In the sequel let F be a real vector space and let $\{u_n\}_{n\geq 0}$ be a system (which is not assumed to be linearly independent) of vectors. Suppose that over F is defined a bilinear form $\Omega(.,.)$ such that $\Omega(x,y) = -\Omega(y,x)$. In this case $\Omega(.,.)$ is called skew-determined. We define for every n

$$\begin{aligned}
e_{2n} &= a_n u_{2n} + \textstyle\sum_{i=0}^{2n-1} \alpha_i^n e_i, \\
e_{2n+1} &= u_{2n+1} + b_n u_{2n} + \textstyle\sum_{i=0}^{2n-1} \beta_i^n e_i,
\end{aligned} \tag{6}$$

the coefficients $a_n \neq 0$, b_n, $\{\alpha_i^n\}_{i=0}^{2n-1}$ and $\{\beta_i^n\}_{i=0}^{2n-1}$ must be chosen such that

$$\begin{aligned}
&\Omega(e_{2n}, e_{2m}) = \Omega(e_{2n+1}, e_{2m+1}) = 0 \qquad && n,m = 0,1,\ldots, \\
&\Omega(e_{2n}, e_{2m+1}) = \gamma_n \delta_{n,m} && n,m = 0,1,\ldots,
\end{aligned} \tag{7}$$

where $\{\gamma_n\}_{n\geq 0}$ is a sequence of non-zero real numbers. A system of vectors $\{e_n\}_{n\geq 0}$ which satisfies (7) is called skew-orthogonal system.

For $n=0$ we shall clearly have $e_0 = \left(\frac{\gamma_0}{\Omega(u_0,u_1)}\right) u_0$ and $e_1 = u_1 + b_0 u_0$ being b_0 an arbitrary constant. Here it is assumed that $\Omega(u_0,u_1)\neq 0$. To obtain the coefficients a_n, b_n, $\{\alpha_i^n\}_{i=0}^{2n-1}$ and $\{\beta_i^n\}_{i=0}^{2n-1}$ in the step n $(n\geq 1)$ we assume that $e_0, e_1, \ldots, e_{2n-2}, e_{2n-1}$ were already calculated. We define

$$\begin{aligned} l_n &= \Omega(u_{2n}, u_{2n+1}) - \sum_{i=0}^{n-1} \frac{\Omega(u_{2n}, e_{2i+1})\,\Omega(e_{2i}, u_{2n+1})}{\gamma_i} \\ &\quad + \sum_{i=0}^{n-1} \frac{\Omega(u_{2n}, e_{2i})\,\Omega(e_{2i+1}, u_{2n+1})}{\gamma_i}, \end{aligned}$$

in the sequel we need also to assume that l_n is different from zero.

In order to find e_{2n} and e_{2n+1} we impose the conditions $\Omega(e_{2n}, e_s) = \Omega(e_{2n+1}, e_s) = 0$ for $s = 0, 1, \ldots, 2n-1$ and $\Omega(e_{2n}, e_{2n+1}) = \gamma_n$. With this motivation we substitute (6) in these equations. Now, this gives

$$\alpha_{2k}^n = -\frac{a_n \Omega(u_{2n}, e_{2k+1})}{\gamma_k}, \qquad \alpha_{2k+1}^n = \frac{a_n \Omega(u_{2n}, e_{2k})}{\gamma_k}, \tag{8}$$

$$\beta_{2k}^n = -\frac{(\Omega(u_{2n+1}, e_{2k+1}) + b_n \Omega(u_{2n}, e_{2k+1}))}{\gamma_k}, \tag{9}$$

and

$$\beta_{2k+1}^n = \frac{(\Omega(u_{2n+1}, e_{2k}) + b_n \Omega(u_{2n}, e_{2k}))}{\gamma_k}, \tag{10}$$

whenever $2k+1 \leq 2n-1$. Hence, from (6), (8) and (9) we now obtain

$$e_{2n} = a_n u_{2n} + a_n \sum_{i=0}^{n-1} \frac{(-\Omega(u_{2n}, e_{2i+1})\, e_{2i} + \Omega(u_{2n}, e_{2i})\, e_{2i+1})}{\gamma_i}, \tag{11}$$

and also

$$\begin{aligned} e_{2n+1} &= f_{2n+1} + \sum_{i=0}^{n-1} \frac{-[\Omega(u_{2n+1}, e_{2i+1}) + b_n \Omega(u_{2n}, e_{2i+1})]}{\gamma_i} e_{2i} \qquad (8.1) \\ &\quad + \sum_{i=0}^{n-1} \frac{[\Omega(u_{2n+1}, e_{2i}) + b_n \Omega(u_{2n}, e_{2i})]}{\gamma_i} e_{2i+1}, \end{aligned}$$

where $f_{2n+1} = u_{2n+1} + b_n u_{2n}$. On the other hand the equation $\Omega(e_{2n}, e_{2n+1}) = \gamma_n$ allow us to calculate a_n. We remember that a_n must be different from zero. Now, rewriting $\Omega(e_{2n}, e_{2n+1})$ as

$$\begin{aligned}
\Omega(e_{2n}, e_{2n+1}) &= \Omega(e_{2n}, u_{2n+1} + b_n u_{2n}) \\
&= \Omega\left(e_{2n}, u_{2n+1} + \frac{b_n}{a_n}\left(a_n u_{2n} \pm \sum_{i=0}^{2n-1} \alpha_i^n e_i\right)\right) \\
&= \Omega\left(e_{2n}, u_{2n+1} + \frac{b_n}{a_n}\left(e_{2n} - \sum_{i=0}^{2n-1} \alpha_i^n e_i\right)\right) \\
&= \Omega(e_{2n}, u_{2n+1}) \\
&= a_n \Omega(u_{2n}, u_{2n+1}) - a_n \sum_{i=0}^{n-1} \frac{\Omega(u_{2n}, e_{2i+1}) \Omega(e_{2i}, u_{2n+1})}{\gamma_i} \\
&\quad + a_n \sum_{i=0}^{n-1} \frac{\Omega(u_{2n}, e_{2i}) \Omega(e_{2i+1}, u_{2n+1})}{\gamma_i} \\
&= a_n l_n,
\end{aligned}$$

under our assumption $l_n \neq 0$, it follows immediately that $a_n = \frac{\gamma_n}{l_n}$. The reader can quickly check that the equation $\Omega(e_{2n+1}, e_{2n+1}) = 0$ says to us that b_n is arbitrary.

The method described will be called a skew-orthogonalization process. We note that all the preceding arguments in this section apply equally well whether we take the following recurrent scheme

$$\begin{gathered}
e_{2n} = u_{2n} + \sum_{i=0}^{2n-1} \alpha_i^n e_i, \\
e_{2n+1} = a_n u_{2n+1} + b_n e_{2n} + \sum_{i=0}^{2n-1} \beta_i^n e_i,
\end{gathered}$$

instead of (6).

Now we give an example of how work the skew-orthogonalization process in a concrete real vector space. Let $l^2(\mathbb{Z}^+)$ denote the collection of all real functions φ on $\mathbb{Z}^+$, such that, $\sum_{n=0}^{\infty} |\varphi(n)|^2$ is finite. Then $l^2(\mathbb{Z}^+)$ is a real linear space with an inner product defined of the following form $(f, g) = \sum_{n=0}^{\infty} f(n)\, g(n)$. Define T on $l^2(\mathbb{Z}^+)$ such that $(Tf)(0) = f(1)$ and if $n \geq 1$ $(Tf)(n) = f(n+1) - f(n-1)$ for $f \in l^2(\mathbb{Z}^+)$. It is clear that T is linear and $\Omega(f, g) = (Tf, g) = -(Tg, f) = -\Omega(g, f)$. Let $\{u_n\}_{n \geq 0}$ be a subset of $l^2(\mathbb{Z}^+)$ where each u_n is defined as $u_n(m) = \delta_{nm}$ for all $m \in \mathbb{Z}^+$. Observe first that $\Omega(u_i, u_j) = \delta_{i(j+1)} - \delta_{i(j-1)}$. In fact,

$$\begin{aligned}\Omega\left(u_i,u_j\right) &= \sum_{n=0}^{\infty}\left(u_i\left(n+1\right)-u_i\left(n-1\right)\right)u_j\left(n\right)=u_i\left(j+1\right)-u_i\left(j-1\right)\\ &= \delta_{i(j+1)}-\delta_{i(j-1)}.\end{aligned}$$

As a consequence of this fact we have $l_n=-1$ for any n. On the other hand

$$\begin{aligned}\alpha^n_{2(n-1)} &= -\frac{a_n\Omega\left(u_{2n},e_{2n-1}\right)}{\gamma_{(n-1)}}=-\frac{a_n}{\gamma_{(n-1)}},\quad \alpha^n_{2k}=0 \text{ for } k=0,\ldots,n-2,\\ \alpha^n_{2k+1} &= \frac{a_n\Omega\left(u_{2n},e_{2k}\right)}{\gamma_k}=0 \text{ for } k=0,\ldots,n-2,\end{aligned}$$

it implies that $e_{2n}=a_nu_{2n}-\frac{a_n}{\gamma_{(n-1)}}e_{2(n-1)}$. We next derive the general form of e_{2n+1}. Note that

$$\begin{aligned}\beta^n_{2(n-1)} &= -\frac{b_n}{\gamma_{n-1}},\qquad \beta^n_{2k}=0 \text{ for } k=0,\ldots,n-2,\\ \beta^n_{2k+1} &= 0 \text{ for } k=0,\ldots,n-1,\end{aligned}$$

hence $e_{2n+1}=u_{2n+1}+b_nu_{2n}-\frac{b_n}{\gamma_{n-1}}e_{2(n-1)}$. Finally, in view of $\Omega(e_{2n},e_{2n+1})$ $=\gamma_n$ then $a_n=-\gamma_n$.

8.4 Acknowledgement

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8.5 References

1. M. Adler, E.Horozov and P. van Moerbeke, The Pfaff lattice and skew-orthogonal polynomials, IMRN (Inter. Math. Res. Notices) 1999, No 11.

2. N. I. Akhiezer, The classical moment problem, Oliver and Boyd, Edinburgh, 1965.

3. C. Brezinski, Padé-type approximation and general orthogonal polynomials, International Series of Numerical Mathematics. V 50, 1980.

4. F. J. Dyson, A clase of matrix ensembles, J. Math. Phys. 13, (1972) 90-97.

5. R. Felipe and L. Villafuerte, Skew-orthogonal polynomials of a discrete variable, Mathematica Pannonica 17 (2006), 139-150.

6. S. Ghosh, Generalized Christoffel-Darboux formula for skew-orthogonal polynomials and random matrix theory. J. Phys. A: Math. Gen. 39 (2006) 8775-8782.

7. M. L. Mehta, Random Matrices, 2da ed. Academic Press, Boston, 1991.

Capítulo 9

Nonlinear Problems with Exponential Nonlinearities

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Abstract: Elliptic and parabolic scalar equations involving exponential nonlinearity are studied. Results concerning multiplicity of solutions as functions of a nonlinear eigenvalue in the steady case, the same as estimations for blow-up time for the unsteady case are given. Numerical trials in the unitary ball are presented.

9.1 Introduction

Elliptic and parabolic partial differential equations with exponential nonlinearity have a long story since Liouville's pioneer work, to Gelfand's work, until today. This report is concerned with scalar differential equations of both steady and unsteady type

$$\begin{aligned} -\Delta u &= \lambda e^u \quad \text{in}\,\Omega \\ u &= 0 \quad \text{on}\,\partial\Omega\ , \end{aligned} \tag{9.1}$$

[1]Mathematics Subject Classification: primary 35J40, 35J60.

and

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \Delta v + \lambda e^v \quad \text{in}\, \Omega \times]0,T[\qquad (9.2)\\
v &= 0 \quad \text{on}\, \partial\Omega \times]0,T[\ ,\\
v(0,x) &= v_0(x) \quad \text{in}\, \Omega\ ;
\end{aligned}$$

they are known as the *Gelfand Problem* (steady and unsteady), because of Gelfand's work [6], where he was the first to study and establish multiplicity results in the radial case.

Concerning the steady problem (9.1), we give a stability result, a (simple) characterization of the bending solutions, and some particular results for the case when Ω is the unit ball.

Concerning the unsteady problem (9.2), the existence of local or global solutions is studied. In the case when λ is in the spectrum of (9.1), we give a criteria for the development of blow-up. In the case when λ is not in the spectrum of (9.1), when blow-up is ensured to develop, some new estimations on the blow-up time are given for a general bounded domain Ω.

At the end, some numerical experiments for Ω the unit ball are shown.

9.2 The Steady Gelfand Problem

We are interested in problems like (9.1) and (9.2). It is well known that the relation between the two problems is that if (9.2) has a solution $v(\cdot,\cdot)$ defined for all $t > 0$ (usually known as global solutions), then $v(t,\cdot) \to u(\cdot)$ as $t \nearrow \infty$ in the appropriate functional norm, where $u(\cdot)$ is a solution of (9.1).

Let us notice that (9.1) (in the unsteady case (9.2)) corresponds to a particular case of the quasi-linear problem

$$\begin{aligned}
-\Delta u &= \lambda f(u) \quad \text{in}\, \Omega, \qquad (9.3)\\
u &= 0 \quad \text{on}\, \partial\Omega
\end{aligned}$$

with f convex increasing, such that $f(0) > 0$. Although not completely accepted in the literature, Keller baptized these type of problems *Positone Problems*, and the main motivation behind them is: the solutions (steady or unsteady) represents the temperature distribution of a certain reaction in a region Ω; that will be our motivation. Also, let us notice that the problem has two unknowns: the nonlinear eigenvalue λ and the nonlinear eigenfunction u.

A little history about the problem (steady and unsteady case):

1852 J. Liouville studies the problem in connection with differential geometry (exact quadrature formulas for the whole $\mathbb{R}^2$)

1882 Lord Kelvin proposes radial equations concerning the gravitational stability of a star

1930's Y. Zeldovich and D. Franz-Kamenetskii study the problem concerning the propagation of flames (in fact the chemistry of the reaction)

1959 I.M. Gelfand presents his classical work *Some problems in the theory of quasilinear equations*; Amer. Math. Soc. Trans., vol. 29 (1963), pp. 295-381 (the second date corresponds to the publication of this work in western countries)

1969 H. Fujita proves that the spectrum of the steady problem is bounded from above

1973 D.D. Joseph and T.S. Lundgren study the steady problem in radial geometry

1974 H. Keller and J. Keener establish an order in the solutions (previously, there is a pioneering work by Keller from 1969)

1975 C. Bandle studies the steady problem in connection with Isoperimetric Inequalities

1983 A. Lacey studies the blow-up, in particular bounds for the blow-up time

1985 J. Dold introduces a new kernel for the study of self-similar solutions

and on and on and on.......; the bibliography provides a good source of references.

From a mathematical point of view, a first question to be done is: are there solutions for (9.1)? Let us remark that (9.1) is a classical bifurcation problem, this is, it depends on the value of the parameter λ if there can be no solutions, one solution or plenty of solutions, where for a fixed λ it is understood that the solution is the function u.

In difference with other type of quasi linear elliptic problems, not many existence results can be pointed for (9.1). In fact, classical bifurcation theory does not work (Krasnoselskii and Crandall-Ravinowitz results), as the nonlinearity $f(s) = e^s$ does not satisfy the basic hypothesis $f(0) = 0$. Another restriction for the use of existence results as Mountain Pass Theorem, Fixed Point, or Degree Theory, is because of the strong nonlinearity (not polynomial), then it is difficult to get a priori bounds on the solutions which for example ensures continuity of the Nemitskii's operator associated with the integral equation.

Despite the previous problems, some interesting tools can be used to ensure the existence of at least a branch of solutions: Implicit Function Theorem from $(\lambda, u) = (0, 0)$ (then, the variable u becames $u(\lambda) = u_\lambda$), and Monotone Methods by the use of sub and super solutions (in general only valid for monotone operator, and strongly based on the Maximum Principle); see [2] for a review of results.

Still concerning the existence of solutions for (9.1), some interesting results to mention are:

- the spectrum is bounded from above; in fact $\lambda \leq \lambda_1 e^{-1}$, where λ_1 is the first eigenvalue of Laplace's operator in Ω with Dirichlet homogeneous boundary condition (let us observe that it is not known if the bound is optimal)

- if λ is in the spectrum of the problem, then there exists a solution $\underline{u}$ such that for any other solution u (associated to the same eigenvalue) it is held that

$$\underline{u} \ \leq \ u \ ,$$

and given three solutions, let us say $\underline{u}$, u and $\hat{u}$, it is not possible that

$$\underline{u} \ \leq \ u \ \leq \ \hat{u};$$

$\underline{u}$ is called *the minimal solution*, and let us remark that the branch of minimal solutions always exists (precisely by the use of the Implicit Function Theorem from $(0, 0)$).

It is also proved that for $(\lambda, \underline{u}_\lambda)$ a minimal solution of (9.1), then the first eigenvalue of

$$\begin{aligned} -\Delta\psi &= \eta\, e^{\underline{u}_\lambda}\, \psi \quad \Omega \\ \psi &= 0 \quad \partial\Omega, \end{aligned} \tag{9.4}$$

is such that $\eta_1 > \lambda$, while if (λ, u_λ) is nonminimal, then the first eigenvalue of

$$\begin{aligned} -\Delta\psi &= \xi\, e^{u}\, \psi \quad \Omega \\ \psi &= 0 \quad \partial\Omega, \end{aligned} \tag{9.5}$$

is such that $\xi_1 < \lambda$.

Then, what happens at λ^* the bending point? (or where the minimal branch ends); well, the answer is simple: at λ^*, it is held that

$$\eta_1 = \lambda^* = \xi_1 \ .$$

Bandle's work [1] gives a detailed proof of previous results and also provides of other interesting results that will be used (and tested) later in the numerics.

We are now in conditions to give some new results, with some ideas concerning the proof.

Teorema 9.2.1 *Let (λ, u_λ) a nonminimal solution of* (9.1).

If $(\lambda, \underline{u}_\lambda)$ is a minimal solution of (9.1) *such that*

$$\eta_1 \int_\Omega f'(u_\lambda) w \psi_1 \, dx \;\; \leq \;\; \lambda \int_\Omega f'(\underline{u}_\lambda) w \psi_1 \, dx \tag{9.6}$$

$\forall\, w \geq 0$, with η_1 and ψ_1 the first eigenvalue and eigenfunction of

$$\begin{aligned} -\Delta \psi &= \eta f'(u_\lambda) \psi \quad \Omega \\ \psi &= 0 \quad \partial\Omega \,, \end{aligned} \tag{9.7}$$

then (λ, u_λ) is a bending point to the left.

If $(\lambda, \underline{u}_\lambda)$ is a minimal solution of (9.1) such that

$$\eta_1 \int_\Omega f'(\underline{u}_\lambda) w \psi_1 \, dx \;\; \geq \;\; \lambda \int_\Omega f'(u_\lambda) w \psi_1 \, dx \tag{9.8}$$

$\forall\, w \geq 0$, with η_1 and ψ_1 the first eigenvalue and eigenfunction of

$$\begin{aligned} -\Delta \psi &= \eta f'(\underline{u}_\lambda) \psi \quad \Omega \\ \psi &= 0 \quad \partial\Omega \,, \end{aligned} \tag{9.9}$$

then (λ, u_λ) is a bending point to the right.

Proof. It is based on the fact that any solution of (9.1) corresponds to a zero of the bifurcation equation

$$F(\lambda, u) \;\; = \;\; \Delta u + \lambda f(u) \; .$$

Probably the radial case, this is $\Omega = B(0,1) \subset I\!R^N$ (with $N \geq 3$), is the case that has called the attention concerning (9.1) the most, because of the multiplicity of solutions (see [2] and [7] for multiplicity results).

Teorema 9.2.2 *Let $\Omega = B(0,1) \subset I\!R^N$, with $N \geq 3$. For any radial solution u of* (9.1) *it is held that*

$$\int_0^1 |u'|^2 r^{N-1} dr \leq \frac{N^2}{N-2} . \tag{9.10}$$

Also, it is held that

$$u'(1) \in [-N - \sqrt{N^2 - 2\lambda}, -N + \sqrt{N^2 - 2\lambda}] . \tag{9.11}$$

Teorema 9.2.3 *Let $\Omega = B(0,1) \subset I\!R^N$, with $N \geq 3$. For any any radial solution u of* (9.1), *it is held that*

$$\int_0^1 u\, r^{N-1} dr \leq \frac{2(N-1)}{N}. \tag{9.12}$$

Proofs. Both results are strongly based on the used of appropriate test functions and certain inequalities.

Let us notice that Theorem 9.2.3 is strongly linked with Lord Kelvin's motivation: the gravitational collapse of a star. If the mass of the star is bigger than the right side of (9.12), then the star should not sustain itself.

9.3 The Unsteady Problem

Concerning the unsteady problem, there are local existence theorems based on nonlinear semigroup theory. The local character on time is because of the strong nonlinear reaction. Clearly (9.2) can develop blow-up in finite time. Also, existence results based on monotone methods by the use of super and sub solutions can be mentioned (again strongly based on parabolic differential inequalities, see [12]).

Of course the main area of research concerning (9.2) is the develop of blow-up as a function of the initial condition v_0 (see [9]), and also the blow-up profile; less studied has been the blow-up set.

For the unsteady problem (9.2), the following important results can be stated:

- there is an exact solution, given by the only time dependent function

$$v(t) = -ln(e^{-x_0} - \lambda(t - t_0))$$

in the case λ is in the spectrum of the steady problem

- for λ not in the spectrum of the steady problem there is always blow-up (not very clear at what rate and in what time)

- for λ in the spectrum of the steady problem and an initial condition v_0, the evolving solution can converge to a steady solution or develop blow-up.

As was previously mentioned, $(\lambda^*, u_{\lambda^*})$ represents the bending solution, or where the minimal branch ends. We can now state the following

Teorema 9.3.1 *Let $\lambda > \lambda^*$ and $u_0 = 0$ in (9.2). Let also $(\lambda^*, u_{\lambda^*})$ be the bending solution for (9.1), and ψ the first eigenfunction of (9.4).*

Then, (9.2) blows-up in a finite time T that satisfies

$$T \leq \frac{e^{h_0}}{\lambda - \lambda^*} , \tag{9.13}$$

where $h_0 = \int_\Omega \psi \, u_{\lambda^*} \, dx.$

Teorema 9.3.2 *Let $\Omega \subset I\!R^N$ open and bounded with regular boundary, $N \geq 3$. Let λ at the interior of the spectrum of (9.1), and $\underline{u}$ the minimal solution of (9.1) for the eigenvalue λ.*

Given $v_0 \geq \underline{u}$ a non-negative initial value for (9.2) such that

$$\alpha = \int_\Omega (v_0 - \underline{u})\psi_1 \, dx \; > \; 2\frac{\lambda_1}{e\lambda} , \tag{9.14}$$

then (9.2) blows-up at a finite time $T_{blow-up}$ such that

$$\frac{1}{\lambda_1} \ln\left(\frac{\alpha}{\alpha - \frac{2\lambda_1}{e\lambda}} \right) \geq T_{blow-up} , \tag{9.15}$$

where λ_1 and ψ_1 corresponds to the first eigenvalue and eigenfunction of Laplace's operator in Ω with Dirichlet homogeneous boundary condition.

Proofs. They are based in estimations and inequalities.

9.4 Numerical Results

In this section, numerical results for both problem (9.1) and (9.2) are given with Ω the unit ball; let us remember that because of the Gidas-Ni-Niremberg Theorem, in the unit ball (9.1) is reduced to the integration of the nonlinear differential equation

$$\begin{aligned} -(r^{N-1}u')' &= \lambda e^u r^{N-1} \quad \text{in} \,]0,1[\\ u'(0) &= 0 \\ u(1) &= 0. \end{aligned} \tag{9.16}$$

Table 9.1: Parameters for radial steady problem.

Dimension (n)	$u(0)$ (bending)	λ (bending)	μ_1 (bending)
3	1.607360	1.661006	1.661008
4	1.860788	2.407386	2.407398
5	2.162290	3.226877	3.226897
10	8.573174	8.003538	8.003639

The first result are the minimal branch of solutions for different values of N (the dimension) given in figure 9.1. Also, table 9.1 summarizes the main values, in particular is interesting the last two columns that compare the nonlinear bending value λ^* and the first eigenvalue of the linearized problem. Figure 9.2 corresponds to the numerical integration of some parts of the bifurcation branch for dimension three. Let us notice that the bifurcation branch must be connected, but because of highly numerical instability, only part of it could be reconstructed. Figures 9.3 and 9.4 show the minimal and a nonminimal solution for the same eigenvalue.

Still in the unit ball, but now concerning the unsteady problem, figures 9.5 and 9.6 show different behaviour of the initial value problem for λ at the interior of the spectrum of the steady problem. Figure 9.7 shows the develop of blow-up for λ not in the spectrum of the steady problem, and in particular *"far"* from the bending value λ^*.

In the case of the steady problem, the numerical routine was an implementation of Newton's Method for the seek of zeroes with a Finite Element integration of the problem. In the case of the unsteady problem, a Runge-Kutta method was used on the integration. The degree of accuracy to stop iterations was a Cauchy criterion for the sequence, with tolerance of 10^{-10} and 200 point where put in the mesh (uniform and not uniform distributed).

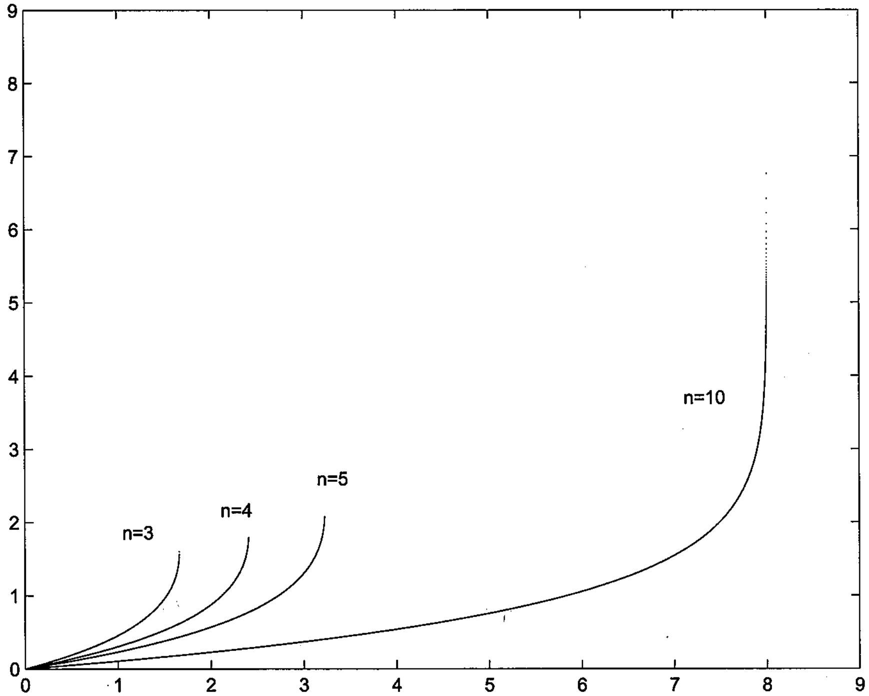

Figure 9.1: Bifurcation Diagram for $n = 3$ (radial case).

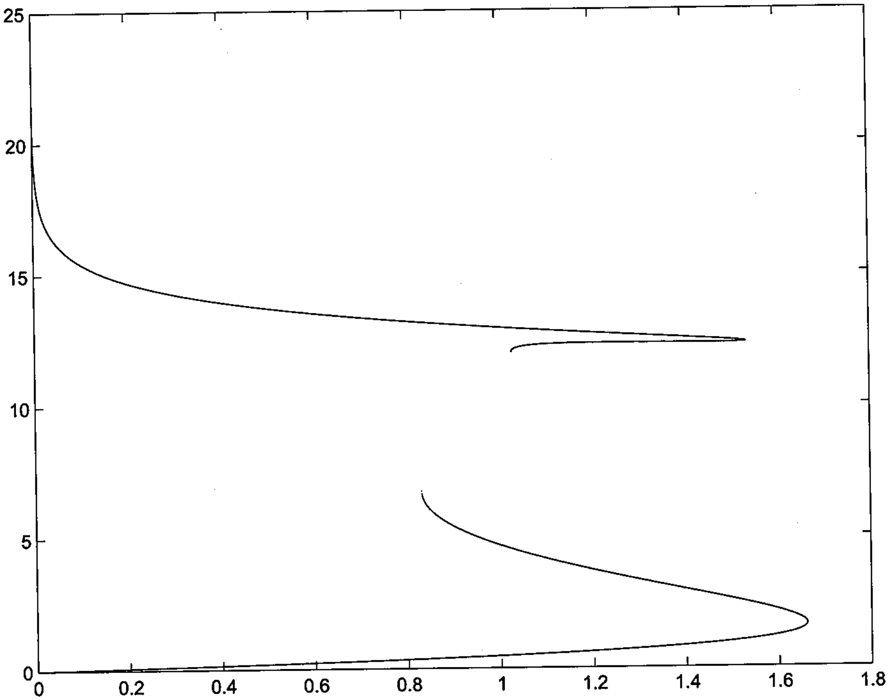

Figure 9.2: Bifurcation Diagram for $n = 3$ (radial case).

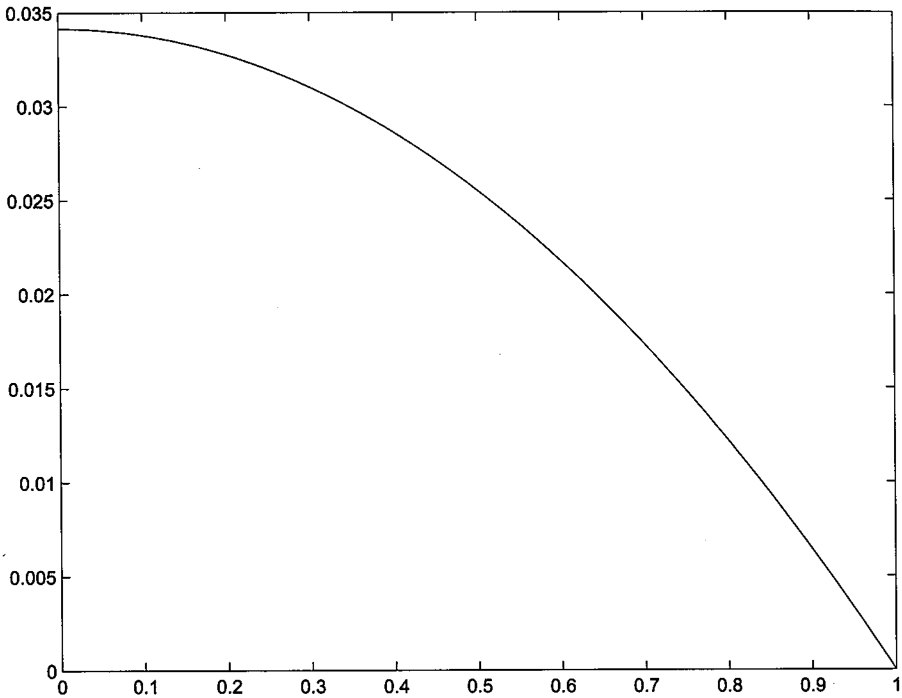

Figure 9.3: Minimal steady solution for $n = 3$ and $\lambda = 0.1$.

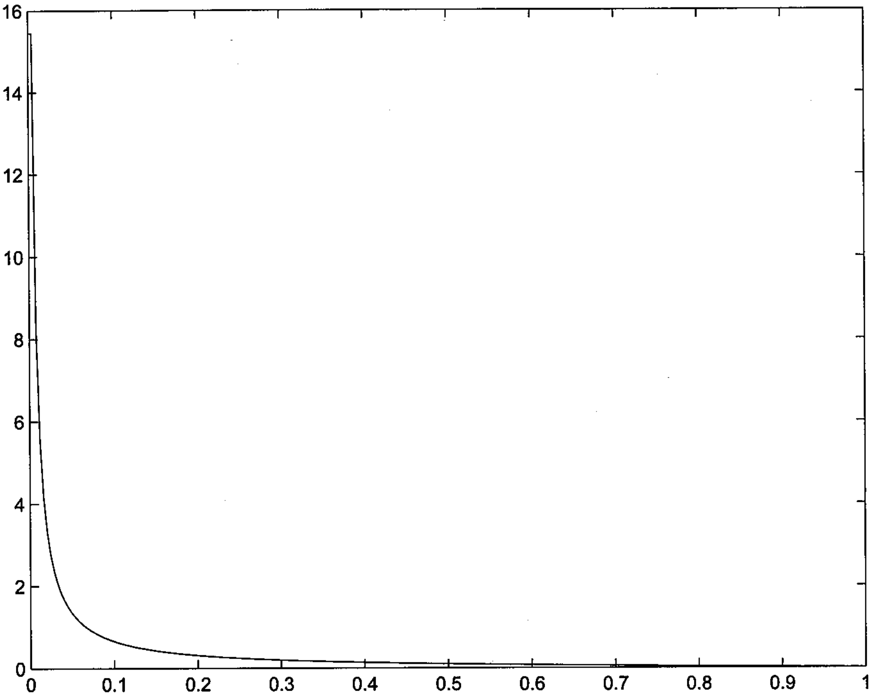

Figure 9.4: Nonminimal steady solution for $n = 3$ and $\lambda 0.1$.

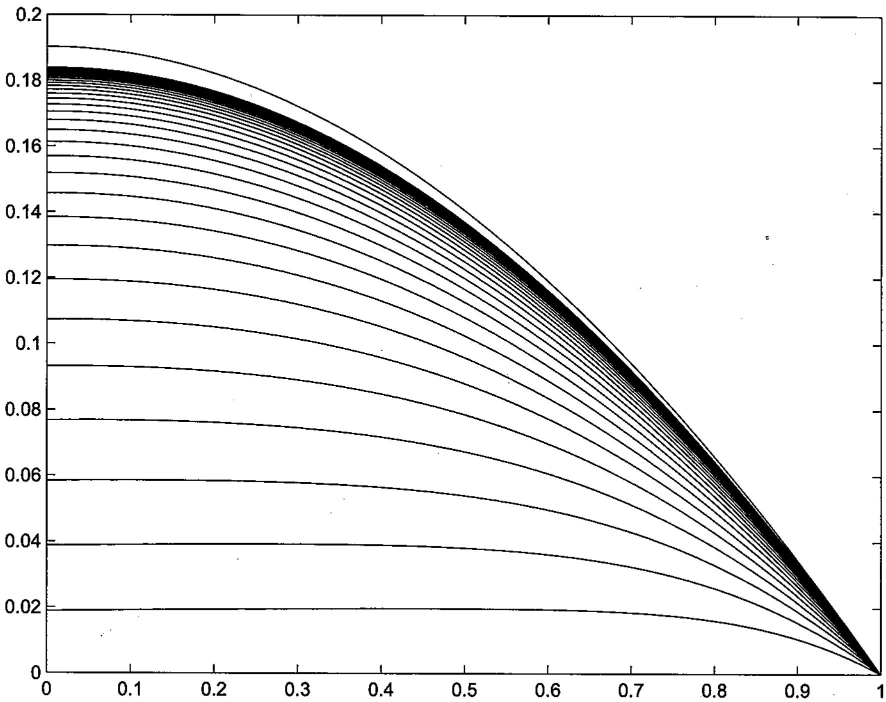

Figure 9.5: Unsteady solution for $n = 3$ and $\lambda = 0.5$; top curve corresponds to the minimal steady solution.

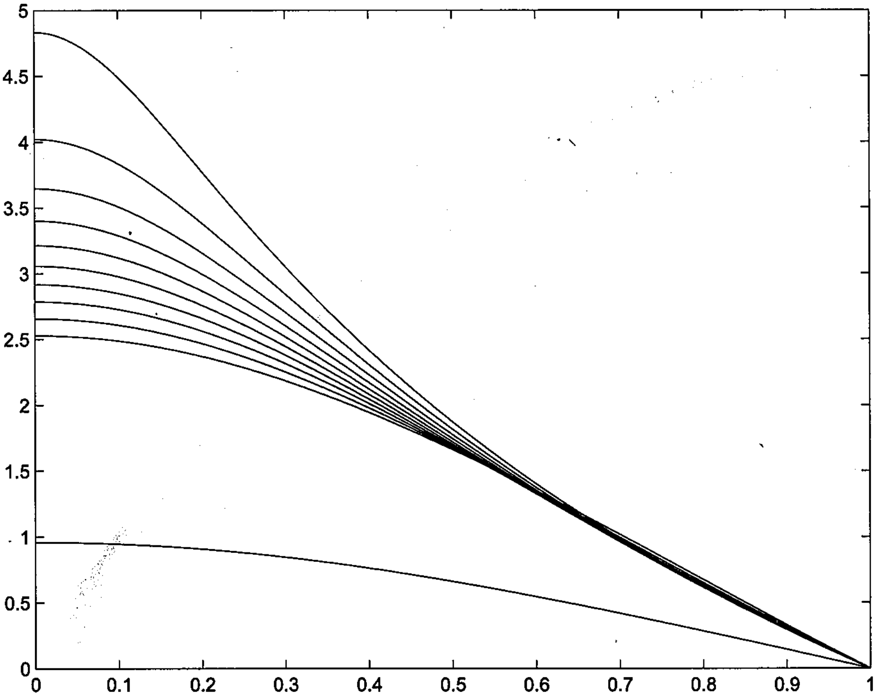

Figure 9.6: Unsteady solution for $n = 3$ and $\lambda = 1.5$; bottom curve corresponds to the minimal steady solution.

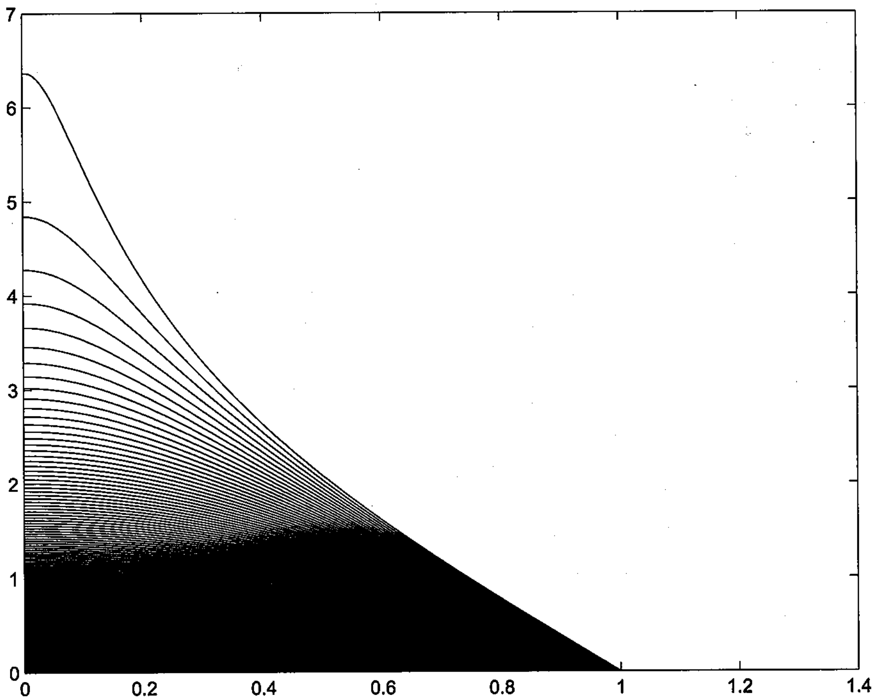

Figure 9.7: Unsteady solution for $n = 3$, $\lambda = 4$ and $|\lambda - \lambda^*| = 2.3390$; blow-up time 0.1432644....

9.5 References

[1] C. Bandle: *Existence theorems, qualitative results and a priori bounds for a class of nonlinear Dirichlet problems*; Arch. Rat. Mech. and Anal., vol. 58 (1975), pp. 219-238.

[2] J. Bebernes, D. Eberly: *Mathematical Problems from Combustion Theory*; Applied Mathematical Sciences vol. 83, Springer-Verlag (1989).

[3] H. Brezis, J.L. Vázquez: *Blow-up solutions of some nonlinear elliptic problems*; Rev. Matemática Univ. Complutense de Madrid, vol. 10 (1997), pp. 443-469.

[4] J. Dold: *Analysis of the early stage of thermal runaway*; Quart. J. Mech. Appl. Math., vol. 38 (1985), pp. 361-387.

[5] H. Fujita: *On the nonlinear equations* $\Delta u + e^u = 0$ *and* $\frac{\partial v}{\partial t} = \Delta v + e^v$; Bull. Amer. Math. Soc., vol. 75 (1969), 132-135.

[6] I.M. Gelfand: *Some problems in the theory of quasilinear equations*; Amer. Math. Soc. Trans., vol. 29 (1963), pp. 295-381.

[7] D.D. Joseph, T.S. Lundgrem: *Quasilinear Dirichlet problems driven by positive sources*; Arch. Rat. Mech. and Analysis, vol. 49 (1973), pp. 241-269.

[8] J.P. Keener, H.B. Keller: *Positive solutions of convex nonlinear eigenvalue problems*; J. Diff. Eq., vol. 16 (1974), pp. 103-125.

[9] A. Kohda, T. Susuky: *Blow-up criteria for semilinear parabolic equations*; J. Math. Anal. Appl., vol. 243 (2000), pp. 127-139.

[10] A. Lacey: *Mathematical analysis of thermal runaway for spacially inhomogeneous reactions*; SIAM J. App. Math., vol. 43 (1983), pp. 1350-1366.

[11] J. Liouville: *Sur l'equation aux dérivées partielles* $\frac{\partial^2 ln\lambda}{\partial u \partial v} \pm 2\lambda a^2 = 0$; J. de Math. vol. 18 (1853), pp. 71-72.

[12] D.H. Sattinger: *Monotone Methods in nonlinear elliptic and parabolic boundary value problems*; Indiana Uni. Maths. J., vol. 21 num. 11 (1972), pp. 979-1000.

[13] T. Susuky: *Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity*; Ann. Inst. Henri Poincaré (Analyse non linéaire), vol. 9 (1992), pp. 367-398.

Capítulo 10

New concepts of active constraints and applications: A review

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Abstract: We introduce different relaxations of the well known concept of active constraints at a given point with respect to a certain linear (or even nonlinear) inequality system with an arbitrary number of constraints. We show that these concepts provide useful local information in linear optimization, for instance, conditions for a given feasible solution to be a unique, extreme point, optimal solution, or strongly unique optimal solution. In the general linear semi-infinite optimization there is a possibility of having no active constraints at a boundary point of the corresponding feasible set. This phenomenon causes serious troubles when one intends to construct the cone of feasible directions at this point. The main future of our approach consists of exploiting the (sub)differential properties of the sup-function that allows replacing infinitely many constraints with a unique convex constraint. In the case where the set of active constraints is empty, this sup-function can present a quite abnormal behavior. To face this situation, two families of relaxed active constraint sets are introduced and the relation between them is studied in detail. Under the strong Slater condition, we obtain two different formulas for the polar of the cone of the feasible directions. They are derived through an extension of Valadier's formula for subdifferential of a

sup-function. A stability result for the most relevant mapping in our context is given.

10.1 Preliminaries

In this review, based on the ideas given in [6] and [26], then developed in [19], [20] and [21], we introduce different relaxations of the well known concept of active constraints at a given point with respect to a certain linear (or even nonlinear) inequality system with an arbitrary number of constraints. We deal with inequality systems in $\mathbb{R}^n$ (n fixed), of the form $\{a_t'x \geq b_t,\ t \in T\}$, where T represents a fixed arbitrary index set and a_t and b_t are the images of $t \in T$ by means of the mappings $a : T \longmapsto \mathbb{R}^n$ and $b : T \longmapsto \mathbb{R}$, respectively. Systems of this kind arise in fields such as functional approximation [13], linear semi-infinite programming [16] and vector semi-infinite optimization (see, [12], [33], [36], [37], [38], [39]). Observe that T is required to be neither a finite set (as in [7], [8]) nor endowed with a topology as in ([2], [3], [6], [34], [11], [25]).

Let us introduce some additional notation. Given a nonempty set X of a certain Euclidean space, by $\operatorname{aff} X$, $\operatorname{span} X$, $\operatorname{conv} X$, $\operatorname{cone} X$, and $\dim X$ we denote the affine hull, the linear hull, the convex hull, the convex conical hull, and the dimension of $\operatorname{aff} X$, respectively. Moreover, we define $\operatorname{cone} \varnothing = \{0_n\}$. We denote by X^0 the positive polar of a given convex cone X and by $X^{\perp}$ the orthogonal subspace to a given linear subspace X. From the topological side, $\operatorname{rint}X$, $\operatorname{int}X$, $\operatorname{cl}X$, bdX and $rbdX$ represent the relative interior, the interior, the closure, the boundary and the relative boundary of X, respectively. In this part the Euclidean norm in $\mathbb{R}^p$ and distance will be denoted by $\|.\|_2$ and ρ, respectively, whereas $\|.\|$ represents the Chebyshev norm in $\mathbb{R}^p$ as well as in $C(T)$. With B we shall denote the open unit ball in $\mathbb{R}^p$.

We associate with a given system $\sigma = \{a_t'x \geq b_t,\ t \in T\}$ its solution set F, which is a closed convex subset of $\mathbb{R}^n$, two *moments cones* $M := cone\{a_t,\ t \in T\}$ and $N := cone\left\{\binom{a_t}{b_t},\ t \in T\right\}$, as well as its *characteristic cone*

$$K := cone\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}.$$

When at least two systems are simultaneously considered, they will be distinguished by subindexes, as their corresponding solution sets, and their moment and characteristic cones (e.g. F_k will denote the solution set of σ_k).We say that a linear system $\sigma = \{a_t'x \geq b_t,\ t \in T\}$ of inequality is *consistent (inconsistent)*, if its solution set $F \neq \varnothing$ $(F = \varnothing)$, respectively.

We need some definitions.

Definition 10.1.1 *σ satisfies the classical* Slater condition *if there exists $x \in \mathbb{R}^n$ such that $a_t'x > b_t$, for all $t \in T$.*

Definition 10.1.2 *σ satisfies the* strong Slater condition *if there exists $x \in \mathbb{R}^n$ (called a strong Slater (SS) element) and a positive scalar $\varepsilon > 0$ such that $a_t'x > b_t + \varepsilon$, for all $t \in T$.*

Definition 10.1.3 *The inequality $cx \geq d$ is a* consequent inequality *of the system $\sigma = \{a_t'x \geq b_t,\ t \in T\}$ if for every $x \in F$ holds $cx \geq d$.*

Definition 10.1.4 *Let $F \subseteq \mathbb{R}^n$. We define the cone of feasible directions at the point, $\overline{x} \in F$, as follows*

$$D(F;\overline{x}) := \{d \in \mathbb{R}^n \mid \overline{x} + \lambda d \in F, \ \textit{for a certain } \lambda > 0\}.$$

Obviously $D(F;\overline{x}) = \mathbb{R}^n$ if $\overline{x}$ is an interior point of F.

10.2 Extended active constraints in linear optimization

10.2.1 Introduction

Given an optimization problem

$$(P) \quad \operatorname{Inf} c'x \text{ such that } x \in F,$$

where $c \in \mathbb{R}^n$ and F is a nonempty closed convex set in $\mathbb{R}^n$, and given $\overline{x} \in F$, the positive polar of the cone of feasible directions at $\overline{x}$, $D(F,\overline{x})^0$ allows us to check (see Proposition 10.2.2) whether

(Q1) $\overline{x}$ is the unique feasible solution (i.e., $F = \{\overline{x}\}$),

(Q2) $\overline{x}$ is an optimal solution of (P), and

(Q3) $\overline{x}$ is a strongly unique solution of (P) (i.e., there exists $k > 0$ such that $c'x \geq c'\overline{x} + k\|x - \overline{x}\|$ for all $x \in F$).

Moreover, $D(F,\overline{x})^0$ also provides a sufficient condition for

(Q4) $\overline{x} \in \operatorname{extr} F$ (the set of extreme points of F).

In linear optimization, F is represented through a certain linear inequality system $\sigma = \{a_t'x \geq b_t, t \in T\}$, where T is an arbitrary (possibly infinite) index set, $a : T \to \mathbb{R}^n$, and $b : T \to R$, are arbitrary mappings. Then (P) is a linear programming (LP) problem if $|T| < \infty$ and a linear semi-infinite programming (LSIP) problem otherwise.

Although there exist known formulae that express $D(F,\overline{x})^0$ in terms of certain convex cones associated with σ, these expressions are frequently too complex in practical situations in order to obtain a useful description of $D(F,\overline{x})^0$ allowing us to determine whether $\overline{x}$ satisfies properties (Q1)–(Q4) or not (see the discussion in section 3). There are two alternative issues in order to check these properties.

The first approach consists of determining the largest class of linear systems for which $D(F,\overline{x})^0 = A(\overline{x})$ for all $\overline{x} \in F$, where $A(\overline{x})$ is the convex conical hull of $\{a_t, t \in T(\overline{x})\}$, with $T(\overline{x}) := \{t \in T \mid a_t'\overline{x} = b_t\}$. The last two sets are called the set of active (or binding) constraints and the set of active indices at $\overline{x}$, respectively, whereas $A(\overline{x})$ is the active cone at $\overline{x}$. The computation of $A(\overline{x})$ requires the calculation of all the zeros of the slack function at $\overline{x}$, $a_t'\overline{x} - b_t$, followed by the formation of the nonnegative linear combinations of the set of active constraints, but this is usually easier than the calculation of $D(F,\overline{x})^0$ by means of the mentioned formulae.

The second approach consists of replacing the set of active constraints by another related set being able to give a sensible answer to questions (Q1)–(Q4). This is the main purpose of this paper.

This part is organized as follows. In section 2 we review the role played by $D(F,\overline{x})^0$ and $A(\overline{x})$ in linear optimization. Section 3 introduces a family of extended constraints sets depending on a positive parameter. Finally, section 4 analyzes the existing relationships between three different sets of extended constraints and provides conditions for the convex conical hulls of these sets to coincide with $D(F,\overline{x})^0$.

Throughout the paper we shall consider a given linear optimization problem (P) with feasible set $F \subset \mathbb{R}^n$ described by means of

$$\sigma = \left\{a_t'x \geq b_t, t \in T\right\}.$$

The optimal set of (P) (possibly empty even though the problem has a finite value) will be denoted by F^*. If different systems arise in the same context, they will be distinguished by subscripts, and the same subscripts will distinguish the associated elements. For simplicity, $\lim_k$ should be interpreted as $\lim_{k\to\infty}$.

If there exists an SS point for σ, ($\overline{x} \in \mathbb{R}^n$ is a strong Slater (SS) point for σ if there exists $\varepsilon > 0$ such that $a_t'\overline{x} \geq b_t + \varepsilon$ for all $t \in T$), we say that σ is SS. Let us remind that, any Slater point is an SS point for a continuous system. Analogously, if there exists a positive lower (upper) bound for $\{\|a_t\|, t \in T\}$, then we say that σ is LB (UB, respectively). These three properties are related as follows.

Proposición 10.2.1 *The following statements are true:*
(i) If σ is SS and UB, then $\dim F = n$.
(ii) If $\dim F = n$ *and σ is LB, then σ is SS.*
(iii) If σ is LB and UB, then σ is SS if and only if $\dim F = n$.

Let us remind that we associate with σ the so-called second moment cone, $N = \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T\right\}$, and the characteristic cone, $K = N + \operatorname{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$. The well-known nonhomogeneous Farkas lemma establishes that, given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, $a'x \geq b$ for all $x \in F$ if and only if $\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl} K$. Thus, $\operatorname{cl} K$ is the same for all the linear representations of F.

10.2.2 Active constraints: A review

The following result collects those tests for questions (Q1)–(Q4) which are based upon $D(F,\overline{x})^0 := \{z \in \mathbb{R}^n \mid y'z \geq 0 \text{ for all } y \in D(F,\overline{x})\}$, where $y \in D(F,\overline{x})$ is the cone of feasible directions of F at the point $\overline{x}$.

Proposición 10.2.2 *Given $\overline{x} \in F$, the following statements hold:*
(i) $F = \{\overline{x}\}$ if and only if $0_n \in \operatorname{int} D(F,\overline{x})^0$.
(ii) $\overline{x} \in F^*$ if and only if $c \in D(F,\overline{x})^0$.
(iii) $\overline{x}$ is a strongly unique solution of (P) if and only if $c \in \operatorname{int} D(F,\overline{x})^0$.
(iv) If $\dim D(F,\overline{x})^0 = n$*, then* $\overline{x} \in \operatorname{extr} F$.

In order to apply Proposition 10.2.2 in practical situations it is necessary to express $D(F,\overline{x})^0$ in terms of the coefficients of σ. Actually, given $\overline{x} \in F$, we have

$$D(F,\overline{x})^0 = \left\{a \in \mathbb{R}^n \mid \begin{pmatrix} a \\ a'\overline{x} \end{pmatrix} \in \operatorname{cl} N\right\} = \left\{a \in \mathbb{R}^n \mid \begin{pmatrix} a \\ a'\overline{x} \end{pmatrix} \in \operatorname{cl} K\right\} \tag{10.1}$$

(see [17, Lemma 4.2]), but the complex expressions of $\operatorname{cl} N$ and $\operatorname{cl} K$ (which involve the calculation of limits of sequences of nonnegative linear combinations of certain subsets of $\mathbb{R}^{n+1}$) make (10.1) seldom useful in practice.

In general, given $\overline{x} \in F$, $D(F,\overline{x}) \subset A(\overline{x})^0$ so that $\operatorname{cl} A(\overline{x}) \subset D(F,\overline{x})^0$, but the reverse inclusion can fail. Fortunately, there exists a large class of linear semi-infinite systems for which $D(F,\overline{x})^0 = A(\overline{x})$ for all $\overline{x} \in F$. These systems are called locally Farkas–Minkowski (LFM). In [31] it has been proved that σ is LFM if and only if every linear inequality $a'x \geq b$ which is a consequence of such σ (i.e., $a'x \geq b$ for all $x \in F$), and such that $a'x = b$ is a supporting hyperplane to F, is also the consequence of a finite subsystem of σ. This property holds, in particular, if either σ is continuous and has a Slater point or $D(F,\overline{x}) = A(\overline{x})^0$ for all $\overline{x} \in F$ (the last property always holds if $|T| < \infty$). In the last case, σ is said to be locally polyhedral (LOP). This class of systems captures the most relevant properties of the ordinary linear systems (see [1]).

Corolario 10.2.1 *If σ is LFM and $\overline{x} \in F$, then the following statements hold:*

(i) $F = \{\overline{x}\}$ if and only if $0_n \in \operatorname{int} A(\overline{x})$.

(ii) $\overline{x}$ is an optimal solution of (P) if and only if $c \in A(\overline{x})$.

(iii) $\overline{x}$ is a strongly unique solution of (P) if and only if $c \in \operatorname{int} A(\overline{x})$.

(iv) If $\dim A(\overline{x}) = n$, then $\overline{x} \in \operatorname{extr} F$. The converse statement holds if σ is LOP.

Remark 10.2.1 *The optimality condition (ii) in Corollary 10.2.1, $c \in A(\overline{x})$, is known as the KKT condition in linear optimization. This is always a sufficient condition for $\overline{x} \in F$ to be an optimal solution of (P). If σ is not LFM, there exists $\overline{x} \in F$ and $z \in D(F,\overline{x})^0 \backslash A(\overline{x})$ so that $\overline{x}$ is an optimal solution for* $\operatorname{Inf} z'x$ *such that* $x \in F$, *whereas the KKT condition fails at $\overline{x}$ (by Proposition 10.2.2, part (ii)). This means that σ is LFM if and only if the KKT condition characterizes the optimality of the feasible solutions for every possible linear optimization problem with constraint system σ. Thus the LFM property can be seen as the weakest global constraint qualification in linear optimization.*

Remark 10.2.2 *The uniqueness condition (i) in Corollary 10.2.1 can be reformulated in terms of $0_n \in \operatorname{int} \operatorname{conv} \{a_t, t \in T(\overline{x})\}$. In fact, assume that $0_n \in \operatorname{int} A(\overline{x})$. If $\varepsilon \operatorname{cl} B_n \subset A(\overline{x})$ for a certain $\varepsilon > 0$, and $u \in \mathbb{R}^n$ satisfies $\|u\| = 1$, then there exists $\alpha > 0$ such that $\alpha \varepsilon u \in \operatorname{conv} \{a_t, t \in T(\overline{x})\}$. Taking, in particular, as u the elements of the canonical basis of $\mathbb{R}^n$ and* *their opposite vectors, there will exist positive scalars* μ_i *and* δ_i *such that* $\mu_i e_i, -\delta_i e_i \in \operatorname{conv}\{a_t, t \in T(\overline{x})\}$, $i = 1, \ldots, n$, *and we get*

$$\begin{aligned} 0_n \in \operatorname{int}\operatorname{conv}\{\mu_i e_i, i = 1, \ldots, n; -\delta_i e_i, i = 1, \ldots, n\} \\ \subset \operatorname{int}\operatorname{conv}\{a_t, t \in T(\overline{x})\}. \end{aligned} \tag{10.2}$$

Next, we show that the so-called Haar's condition,

$$0_n \in \operatorname{conv}\{a_t, t \in T(\overline{x})\},$$

which is obviously weaker than (10.2), is either sufficient or necessary for the uniqueness of $\overline{x}$, provided that σ belongs to certain classes of linear semi-infinite systems arising in approximation problems.

Proposición 10.2.3 *Given* $\overline{x} \in F$ *such that* $T(\overline{x}) \neq \emptyset$, *the following statements hold:*

(i) If $F = \{\overline{x}\}$ *and* σ *is either LFM or continuous, then*

$$0_n \in \operatorname{conv}\{a_t, t \in T(\overline{x})\}.$$

(ii) If $0_n \in \operatorname{conv}\{a_t, t \in T(\overline{x})\}$ *and* $\{a_t, t \in S\}$ *is linearly independent for every set* $S \subset T(\overline{x})$ *such that* $|S| \leq n$, *then* $F = \{\overline{x}\}$.

Now, let us consider the following problem which arises in functional approximation: Given a compact Hausdorff space T, three functions $f, g, h \in C(T)$, where g and h are positive on T, and a polynomial of degree less than n, $p(t)$ such that $-g(t) \leq p(t) - f(t) \leq h(t)$ for all $t \in T$, decide whether there exists another polynomial satisfying the same conditions. Writing $p(t) = \sum_{i=1}^{n} \overline{x}_i t^{i-1}$, the problem consists of determining whether $\overline{x}$ is the unique solution of the linear semi-infinite system

$$\left\{ f(t) - g(t) \leq \sum_{i=1}^{n} x_i t^{i-1} \leq f(t) + h(t), t \in T \right\}. \tag{10.3}$$

The following result is Theorem 2.1 in [24], which extends the unicity theorems of Guerra and Jiménez [23] and Jiménez and Todorov [27] for the system in (10.3).

Corolario 10.2.2 *Let* σ *be a continuous system, and let* $\overline{x} \in F$ *such that* $T(\overline{x}) \neq \emptyset$ *and* $\{a_t,\ t \in S\}$ *is linearly independent for all* $S \subset T(\overline{x})$ *such that* $|S| \leq n$. *Then the following statements are equivalent to each other:*

(i) $F = \{\overline{x}\}$.

(ii) There exists $S \subset T(\overline{x})$, *with* $|S| = n+1$, *such that* $\{a_t' x > 0,\ t \in S\}$ *is inconsistent.*

(iii) $0_n \in \operatorname{conv}\{a_t,\ t \in T(\overline{x})\}$.

Example 10.2.1 Consider the system $\sigma = \{t_1x_1+t_2x_2 \geq t_3, t \in T\}$, where

$$T = \left\{ t \in \mathbb{R}^3 \mid \left\| \begin{pmatrix} t_1 - 1 \\ t_2 \\ t_3 + 1 \end{pmatrix} \right\| \leq 1 \right\} \cup \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Obviously, σ is a continuous system with

$$\operatorname{cl} K = \operatorname{cone} \left\{ \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

so that $F = \{0_2\}$. Observe that $t \in T(0_2)$ if and only if $t_3 = (t_1, t_2)\, 0_2 = 0$ so that the isolated index in T, $(-1, 0, 0)'$, is active at 0_2. In order to obtain the remaining elements of $T(0_2)$ we replace $t_3 = 0$ in $\|(t_1 - 1, t_2, t_3 + 1)'\| \leq 1$, which yields $(t_1 - 1)^2 + t_2^2 \leq 0$, i.e., $t_1 = 1$ and $t_2 = 0$. Consequently,

$$T(0_2) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

with $\{a_t,\, t \in T(0_2)\}$ linearly dependent. It can be easily seen that the statements (i) and (iii) in Corollary 10.2.2 hold, but (ii) fails (as well as the linear independence assumption). Observe also that

$$A(0_2) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \subsetneqq D(F, 0_2)^0 = \mathbb{R}^2$$

so that σ is not LFM, and four statements in Corollary 10.2.1 fail at 0_2.

10.2.3 γ-active constraints

Given $\overline{x} \in F$ and $\gamma > 0$, we define the set of γ -active constraints at $\overline{x}$ as

$$W(\overline{x}, \gamma) := \left\{ a_t \mid t \in T \text{ and } a_t'y = b_t \text{ for a certain } y \in \overline{x} + \gamma B_n \right\}.$$

In other words, if $a_t \neq 0_n$, then $a_t \in W(\overline{x}, \gamma)$ if and only if $a_t'\overline{x} < b_t + \gamma \|a_t\|$. Obviously, $\{a_t,\, t \in T(\overline{x})\} \subset W(\overline{x}, \gamma)$. Moreover, if $\overline{x} \in \operatorname{int} F$ there will exist $\gamma_0 > 0$ sufficiently small such that $W(\overline{x}, \gamma) \setminus \{0_n\} = \emptyset$ for all γ such that $0 < \gamma < \gamma_0$.

Lemma 10.2.1 *Given $\overline{x} \in \operatorname{bd} F$, the following statements hold:*

(i) $W(\overline{x}, \gamma)$ contains at least a nonzero vector for all $\gamma > 0$.

(ii) If $T(\overline{x}) = \emptyset$, then $W(\overline{x}, \gamma)$ is an infinite set for all $\gamma > 0$.

(iii) If $|T| < \infty$, then $W(\overline{x}, \gamma) = \{a_t,\, t \in T(\overline{x})\}$ for $\gamma > 0$ sufficiently small.

Next, we show that the γ-active constraints at $\overline{x}$ allow us to check the feasibilty of given points of the ball $\overline{x}+\gamma B_n$ and of given directions at $\overline{x}$.

Lemma 10.2.2 *Let $\overline{x}\in F$ and $y\in\overline{x}+\gamma B_n$, $\gamma>0$. Then $y\in F$ if and only if $a_t'y\geq b_t$ for all $a_t\in W(\overline{x},\gamma)$.*

Lemma 10.2.3 *Let $\overline{x}\in F$ and $d\in\mathbb{R}^n$. The following statements are true:*

(i) If for a certain $\gamma>0$ we have $a_t'd\geq 0$ for all $a_t\in W(\overline{x},\gamma)$, then $d\in D(F,\overline{x})$. So $D(F,\overline{x})^0\subset \operatorname{cl}\operatorname{cone} W(\overline{x},\gamma)$ for all $\gamma>0$.

(ii) If $d\in D(F,\overline{x})$ and $|T|<\infty$, then there exists some $\gamma_0>0$ such that $a_t'd\geq 0$ for all $a_t\in W(\overline{x},\gamma)$ and all positive $\gamma<\gamma_0$. In such a case, $D(F,\overline{x})^0=\operatorname{cone} W(\overline{x},\gamma)$.

Example 10.2.2 Consider $\sigma_1=\{x_1-tx_2\geq 0, t\in T_1\}$, where $T_1=[1,+\infty[$, $\overline{x}=(1,0)'$, and $d=(-1,0)'$ It can be easily realized that

$$F_1=\left\{x\in\mathbb{R}^2\mid x_1-x_2\geq 0,-x_2\geq 0\right\}$$

so that $\overline{x}\in F_1$ and $d\in D(F_1,\overline{x})=\left\{z\in\mathbb{R}^2\mid z_2\leq 0\right\}$. Nevertheless, $a_t'd=-1$ for all $t\in T_1$. Hence, the converse of statement (i) in Lemma 10.2.3 is not true. On the other hand, since $D(F_1,\overline{x})^0=\operatorname{cone}\left\{\begin{pmatrix}0\\-1\end{pmatrix}\right\}$ and

$$W_1(\overline{x},\gamma)=\begin{cases}\left\{\begin{pmatrix}1\\-t\end{pmatrix}\mid t\geq 1\right\} & if \gamma\geq\frac{1}{\sqrt{2}},\\ \left\{\begin{pmatrix}1\\-t\end{pmatrix}\mid t>\sqrt{\gamma^{-2}-1}\right\} & if\ 0<\gamma<\frac{1}{\sqrt{2}},\end{cases}$$

we have $D(F_1,\overline{x})^0\cap\operatorname{cone} W_1(\overline{x},\gamma)=\{0_2\}$ for all $\gamma>0$, and statement (ii) fails for infinite systems. This example also shows that it is not possible to replace $\operatorname{cl}\operatorname{cone} W(\overline{x},\gamma)$ by just $\operatorname{cone} W(\overline{x},\gamma)$ in statement (i).

We consider now the LOP system obtained by aggregating the inequality $-x_2\geq 0$ to σ_1; i.e., let $\sigma_2=\{a_t'x\geq b_t, t\in T_2\}$, with $T_2=T_1\cup\{0\}$, $a_0=\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)$, and $b_0=0$. Obviously, $\overline{x}=(1,0)'\in F_2=F_1$ and $W_2(\overline{x},\gamma)=W_1(\overline{x},\gamma)\cup\{a_0\}$ so that

$$D(F_2,\overline{x})^0=D(F_1,\overline{x})^0\subsetneqq\operatorname{cone} W_2(\overline{x},\gamma).$$

Hence, statement (ii) fails even for LOP systems.

Proposición 10.2.4 *Given $\overline{x}\in F$ and $\gamma>0$, the following statements hold:*

(i) If $F = \{\overline{x}\}$*, then* $0_n \in \operatorname{int}\operatorname{cone} W(\overline{x}, \gamma)$.
(ii) If $\overline{x} \in F^*$*, then* $c \in \operatorname{cl}\operatorname{cone} W(\overline{x}, \gamma)$.
(iii) If $\overline{x}$ *is a strongly unique solution of* (P)*, then* $c \in \operatorname{int}\operatorname{cone} W(\overline{x}, \gamma)$.
(iv) If $\overline{x} \in \operatorname{extr} F$*, then* $\dim \operatorname{cone} W(\overline{x}, \gamma) = n$.

The sets of γ-active constraints are too large in order to guarantee the converse statements in Proposition 10.2.4 separately. Next, we show that all these sets together provide sufficient conditions for (Q1)–(Q4) in LP (but not in LSIP).

Proposición 10.2.5 *Let* $\overline{x} \in F$ *and* $|T| < \infty$*. The following statements hold:*
(i) If $0_n \in \operatorname{int}\operatorname{cone} W(\overline{x}, \gamma)$ *for all* $\gamma > 0$*, then* $F = \{\overline{x}\}$.
(ii) If $c \in \operatorname{cl}\operatorname{cone} W(\overline{x}, \gamma)$ *for all* $\gamma > 0$*, then* $\overline{x} \in F^*$.
(iii) If $c \in \operatorname{int}\operatorname{cone} W(\overline{x}, \gamma)$ *for all* $\gamma > 0$*, then* $\overline{x}$ *is a strongly unique solution of* (P).
(iv) If $\dim \operatorname{cone} W(\overline{x}, \gamma) = n$ *for all* $\gamma > 0$*, then* $\overline{x} \in \operatorname{extr} F$.

The next example shows that it is not possible to replace in Proposition 10.2.5 the finiteness of σ by the (weaker) LOP property.

Example 10.2.3 Let $\sigma_0 = \{x_2 \leq 1; tx_1 + x_2 \leq 1, t \in \mathbb{N}\}$, let $\sigma_1 = \sigma_0 \cup \{x_1 \leq 0\}$, and observe that σ_1 is LOP. Obviously, $0_2 \in F_1 =]-\infty, 0] \times]-\infty, 1]$ and $0_2 \notin \operatorname{extr} F_1$ even though $\dim \operatorname{cone} W_1(0_2, \gamma) = 2$ for all $\gamma > 0$. Consider also $\sigma_2 = \sigma_1 \cup \{x_1 \geq 0, x_2 \geq 0\}$, which is also LOP, with $F_2 = \{0\} \times [0, 1]$. We have $0_2 \in \operatorname{int}\operatorname{cone} W_2(0_2, \gamma)$ for all $\gamma > 0$, but $F_2 \neq \{0_2\}$. Finally, taking $c = (0, -1)'$, we have $c \in \operatorname{int}\operatorname{cone} W_2(0_2, \gamma)$ for all $\gamma > 0$, although $0_2 \notin F_2^* = \{(0, 1)'\}$.

We have seen that each set $W(\overline{x}, \gamma)$, $\gamma > 0$, is too large for our purpose of giving a sensible answer to the questions (Q1)–(Q4). An alternative choice could be $\bigcap_{\gamma>0} W(\overline{x}, \gamma)$, but this set is too small, since

$$\bigcap_{\gamma>0} W(\overline{x}, \gamma) = \{a_t, t \in T(\overline{x})\} \tag{10.4}$$

can be empty even when $\overline{x} \in \operatorname{bd} F$. In fact, if $a_t \in \bigcap_{\gamma>0} W(\overline{x}, \gamma)$, then $a_t \in W(\overline{x}, k^{-1})$ for all $k \in \mathbb{N}$. Let $y^k \in \overline{x} + k^{-1} B_n$ such that $a_t' y^k = b_t$. Then $\lim_k y^k = \overline{x}$ and $a_t' \overline{x} = b_t$; i.e., $t \in T(\overline{x})$.

10.2.4 Extended active constraints

We discuss in this section three different sets of extended active constraints at $\overline{x} \in F$ which are motivated and defined as follows:

(a) We can replace N or K in (10.1) by the simpler (and smaller) set, $D = \left\{ \binom{a_t}{b_t}, t \in T \right\}$. This suggests the following definition:

$$D(\overline{x}) := \left\{ a \in \mathbb{R}^n \mid \begin{pmatrix} a \\ a'\overline{x} \end{pmatrix} \in \operatorname{cl} D \right\}. \tag{10.5}$$

(b) We can replace in (10.4) each set of γ-active constraints, $W(\overline{x}, \gamma)$, by its closure, obtaining the set

$$W(\overline{x}) := \bigcap_{\gamma > 0} \operatorname{cl} W(\overline{x}, \gamma).$$

Since $\{W(\overline{x}, \gamma) \mid \gamma > 0\}$ is a contractive family of nonempty sets, when $\gamma \searrow 0$, it is possible to write

$$W(\overline{x}) = \lim_k W(\overline{x}, \gamma_k) \tag{10.6}$$

(limit in the Painlevé–Kuratowski sense) for any arbitrary sequence of positive real numbers, $\{\gamma_k\}_{k=1}^{\infty}$, such that $\lim_k \gamma_k = 0$.

(c) Finally, we can take the set of active constraints at $\overline{x} \in F$ of a suitable representation of F. According to Theorem 5.12 in [16], if G is the set of accumulation points of

$$\left\{ \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|^{-1} \begin{pmatrix} a_t \\ b_t \end{pmatrix} \mid \begin{pmatrix} a_t \\ b_t \end{pmatrix} \neq 0_n, t \in T \right\},$$

then $\sigma \cup \{a'x \geq b, \binom{a}{b} \in G\}$ is another representation of F which is LFM if $\dim F = n$. Therefore, we consider the set

$$G(\overline{x}) := \left\{ a \in \mathbb{R}^n \mid \begin{pmatrix} a \\ a'\overline{x} \end{pmatrix} \in D \cup G \right\}. \tag{10.7}$$

Proposición 10.2.6 *If $\overline{x} \in \operatorname{bd} F$ and σ is UB, then $W(\overline{x}) \neq \emptyset$.*

Now, we show that $D(\overline{x})$ and $W(\overline{x})$ are essentially the same set.

Lemma 10.2.4 *For any fixed $\overline{x} \in F$, one has $D(\overline{x}) \setminus \{0_n\} \subset W(\overline{x}) \subset D(\overline{x})$. Moreover, if σ is LB, then $0_n \notin D(\overline{x})$ (and so $W(\overline{x}) = D(\overline{x})$).*

Example 10.2.4 Let $n = 1$ and $\sigma = \{t^{-1}x \geq -t^{-1}, t \in \mathbb{N}\}$. Since $W(0,\gamma) = \emptyset$ for all γ such that $0 < \gamma < 1$, $W(0) = \emptyset$. Nevertheless, $0_2 \in \operatorname{cl} D$ so that $D(0) = \{0\}$

Lemma 10.2.5 *Given $\overline{x} \in F$, one has $\mathbb{R}_+[D(\overline{x})\backslash\{0_n\}] \subset \mathbb{R}_+ [G(\overline{x}) \setminus \{0_n\}]$. Further, if σ is LB and UB, then $\mathbb{R}_+ D(\overline{x}) = \mathbb{R}_+ G(\overline{x})$.*

Proposición 10.2.7 *Given $\overline{x} \in F$, the following statements hold:*
(i) $A(\overline{x}) \subset \operatorname{cone} W(\overline{x}) = \operatorname{cone} D(\overline{x}) \subset \operatorname{cone} G(\overline{x}) \subset D(F,\overline{x})^0$.
(ii) If σ is LFM, the five cones in (i) coincide.
(iii) If $\dim F = n$, *then* $\operatorname{cone} G(\overline{x}) = D(F,\overline{x})^0$.
(iv) If σ is UB and SS, then

$$\operatorname{cone} D(\overline{x}) = \operatorname{cone} G(\overline{x}) = D(F,\overline{x})^0. \tag{10.8}$$

(v) If D is closed, then $A(\overline{x}) = \operatorname{cone} W(\overline{x}) = \operatorname{cone} D(\overline{x})$.
(vi) If σ is continuous and LB, then $A(\overline{x}) = \operatorname{cone} W(\overline{x}) = \operatorname{cone} D(\overline{x}) = \operatorname{cone} G(\overline{x})$.

Now, let us consider again the pathological Example 10.2.1. It can be easily realized that $W(0_2) = D(0_2) = G(0_2) = \left\{\pm\begin{pmatrix}1\\0\end{pmatrix}\right\}$ so that

$$\begin{aligned} A(0_2) &= \operatorname{cone} W(0_2) = \operatorname{cone} D(0_2) = \operatorname{cone} G(0_2) \\ &= \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \neq \mathbb{R}^2 = D(F,0_2)^0. \end{aligned}$$

This example shows that (10.8) may fail even though σ satisfies the LB and the UB properties. The following result is an immediate consequence of Proposition 10.2.7.

Corolario 10.2.3 *Let $\overline{x} \in F$, and let $U(\overline{x})$ be one of the three sets of extended active constraints ($W(\overline{x})$, $D(\overline{x})$, or $G(\overline{x})$). Then the following statements hold:*
(i) If $0_n \in \operatorname{int} \operatorname{cone} U(\overline{x})$, *then* $F = \{\overline{x}\}$.
(ii) If $c \in \operatorname{cone} U(\overline{x})$, *then $\overline{x}$ is an optimal solution of (P).*
(iii) If $c \in \operatorname{int} \operatorname{cone} U(\overline{x})$, *then $\overline{x}$ is a strongly unique solution of (P).*
(iv) If $\dim \operatorname{cone} U(\overline{x}) = n$, *then* $\overline{x} \in \operatorname{extr} F$.

The converse statements of (i)–(iii) are true if $\operatorname{cone} U(\overline{x}) = D(F,\overline{x})^0$ (sufficient conditions are given in Proposition 10.2.7). Example 10.2.1 shows again the necessity of these assumptions. Taking $\overline{x} = 0_2$ and $c = (0,1)'$, we observe that the converse statements of (i)–(iv) fail for the three sets of extended active constraints. In fact, σ is not LFM, since $D(F, 0_2)^0 = \{0_2\}^0 = \mathbb{R}^2 \neq A(0_2)$.

Finally, observe that, for general systems, Proposition 10.2.4 and Corollary 10.2.3 can be seen as providing necessary conditions and sufficient conditions for (Q1)–(Q4), based upon different sets of extended active constraints.

10.3 A sup-function approach in linear semi-infinite optimization

10.3.1 Introduction

Consider the *linear semi-infinite programming problem* (LSIP, in brief)

$$(P) \quad \text{Inf } c'x \ \text{ s.t. } \ a_t'x \leq b_t, \ t \in T,$$

where, for simplicity, in the constraints system of (P) represented by σ; i.e., $\sigma = \{a_t'x \leq b_t, \ t \in T\}$, the inequalities are in the opposite sign.

Respectively, the *SS* condition has changed: Assume that there exist $x^0 \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $a_t'x^0 \leq b_t - \varepsilon$ for all $t \in T$. In this case we say that x^0 is a *strong Slater point* (an *SS element)* of σ, and that σ satisfies the *SS condition.*

According to Proposition 10.2.2 (ii), for the problem (P), the *Karush-Kuhn-Tucker condition* (KKT condition, for short) is satisfied at the point $\overline{x}$ if $-c \in A(\overline{x})$. The KKT condition at $\overline{x} \in F$ implies the optimality of this point. In order to get the KKT condition as a necessary optimality condition, some constraint qualifications are needed. In [16] the following constraint qualification is considered: As we have defined before, the constraints system of (P) is said to be *locally FM* (or LFM) *at* $\overline{x} \in F$ if $D(F, \overline{x})^o = A(\overline{x})$ (see, also, [31]). In [10] this property is studied in the context of convex semi-infinite programming, and its relationship with extended versions of the Slater constraint qualification is analyzed in detail. Also in [29] different constraint qualifications for the convex SIP are comparatively studied.

In LSIP, the set of active constraints at a point of the boundary of the feasible set can be empty. In this case the inclusion $D(F, \overline{x}) \subset A(\overline{x})^o = \mathbb{R}^n$ is trivial (it does not provide any local information), and the fulfillment of the LFM property, as well as of the constraint qualifications considered in [29], is precluded.

This section deals with the possibility of having no active constraint at the boundary of the feasible set, and the main feature of our approach consists in exploiting the (sub)differential properties of the sup-function $f(x) := \sup\{a_t'x - b_t \mid t \in T\}$. To this aim, two families of quasi-active constraint sets are introduced. The sets in one of these families are always non-empty at any point of the boundary of the feasible set, and they play a crucial role in this theory. The relationship between these families is studied here and, in particular, it is shown in this section that the members of both families are mutually embedded under certain assumption relative to the boundedness for some of them, and the fulfillment of the SS condition. Also, and again under the same assumptions, two different formulas for $D(F, \overline{x})^o$ are derived through an extension of Valadier's formula for the subdifferential of the sup-function $f(x)$. Finally, a stability result is given for the most relevant mapping in our context.

10.3.2 Two sets of quasi-active constraints

Since the set $T(\overline{x})$ can be empty for a LSIP, we shall consider, throughout the section, the set of $\rho-$*active constraints* at $\overline{x} \in bd(F)$

$$T(\overline{x}; \rho) := \{t \in T \mid a_t'\overline{x} \geq b_t - \rho\},$$

with $\rho > 0$. Obviously, with $\overline{x} \in F$,

$$T(\overline{x}) = \bigcap_{\rho>0} T(\overline{x}; \rho).$$

Associated with the $\rho-$active constraints set at $\overline{x}$, we shall introduce the $\rho-$*active gradients set* at $\overline{x} \in bd(F)$

$$\begin{aligned} V(\overline{x}; \rho) &:= \{a_t \mid t \in T(\overline{x}; \rho)\} \\ &= \{a_t \mid t \in T \text{ and } a_t'\overline{x} \geq b_t - \rho\}. \end{aligned}$$

In the previous section different results are derived via some properties of the set

$$W(\overline{x}; \gamma) := \{a_t \mid t \in T, \text{ and } a_t'y = b_t \text{ for a certain } y \in \overline{x} + \gamma\mathbb{B}\},$$

where $\gamma > 0$ and $\overline{x} \in bd(F)$. Actually, $W(\overline{x}; \gamma)$ is the set of gradients of those constraints which are *active at the points of a* γ*-ball around* $\overline{x}$.

Let us recall some relevant properties of this set, established in Lemma 10.2.1 and Proposition 10.2.4.

i) If $\overline{x} \in bd(F)$, we have $W(\overline{x};\gamma) \neq \emptyset$, for every $\gamma > 0$.

ii) If $\overline{x}$ is an extreme point of F, then $\dim(W(\overline{x};\gamma)) = n$, for all $\gamma > 0$.

The boundedness of $W(\overline{x};\gamma)$ is also related with the properties of the *sup-function*

$$f(x) := \sup\{a_t'x - b_t \mid t \in T\}.$$

f is a proper closed convex function, and $F = \{x \in \mathbb{R}^n \mid f(x) \leq 0\}$. If $f(\overline{x}) < 0$ and f is continuous at $\overline{x}$, a positive scalar δ can be found such that $f(x) \leq 0$ if $x \in \overline{x} + \delta\mathbb{B}$; i.e., $\overline{x} \in \operatorname{int}(F)$. Thus, if f is continuous at $\overline{x}$ and $\overline{x} \in bd(F)$ then $f(\overline{x}) = 0$. Obviously, if $f(\overline{x}) < 0$ the point $\overline{x}$ will be an SS element of σ. The following example shows that the SS condition is, itself, not enough to preclude the possibility $f(\overline{x}) < 0$ at a point $\overline{x} \in bd(F)$.

Example 10.3.1 Consider, in R, the system $\sigma = \{tx \leq 1,\ t \in]0,\infty[\}$. Obviously, $F =]-\infty, 0]$ and $f(0) = -1$. This abnormality is a consequence of $dom(f) = F$, which provokes that f is not continuous at $\overline{x} = 0$.

Next, we add new properties of $W(\overline{x};\gamma)$ to this pair.

Lemma 10.3.1 *If $\sigma = \{a_t'x \leq b_t,\ t \in T\}$ is consistent and $W(\overline{x};\gamma_0)$ is bounded, for a certain $\gamma_0 > 0$, then*

$$\inf\{b_t \mid t \in T \text{ and } a_t \in W(\overline{x};\gamma_0)\} > -\infty.$$

Proposición 10.3.1 *Given the consistent system $\sigma = \{a_t'x \leq b_t,\ t \in T\}$, the sup-function $f(x) := \sup\{a_t'x - b_t \mid t \in T\}$, and the point $\overline{x} \in bd(F)$, we have that $\overline{x} \in \operatorname{int}(\operatorname{dom}(f))$ if and only if a positive scalar γ_0 exists such that $W(\overline{x};\gamma_0)$ is bounded.*

Now, we proceed by studying the relationship between the sets $V(\overline{x};\rho)$ and $W(\overline{x};\gamma)$. The following examples show that this pair of sets are, in general, quite different.

Example 10.3.2 Let us consider the system, in R, $\sigma = \{tx \leq 1,\ t \in]-1,1[\}$. Obviously,

$$f(x) = \sup\{tx - 1 \mid t \in]-1,1[\} = \begin{cases} -x-1, & \text{if } x \leq 0, \\ x+1, & \text{if } x > 0. \end{cases}$$

Thus, $F = [-1,1]$, and we shall take $\overline{x} = 1 \in bd(F)$.

Straightforward calculations provide

$$V(1;\rho)=\begin{cases} [1-\rho,1[, & \text{if } 0<\rho<2,\\ ]-1,1[, & \text{if } \rho\geq 2,\end{cases}$$

and

$$W(1;\gamma)=\begin{cases}]\frac{1}{1+\gamma},1[, & \text{if } 0<\gamma\leq 2,\\ ]-1,\frac{1}{1-\gamma}[\cup]\frac{1}{1+\gamma},1[, & \text{if } \gamma>2.\end{cases}$$

In this example we observe that the sets $W(1;\gamma)$ are even non-connected, and that none of them contains the origin, which excludes the possibility of being contained in any of the sets $V(1;\rho)$, when $\rho\geq 1$.

The following example shows that $V(\overline{x};\rho)$ can even be empty, and that its boundedness at $\overline{x}$ does not imply $\overline{x}\in \operatorname{int}(\operatorname{dom}(f))$, like it was the case for $W(\overline{x};\gamma)$ (see Proposition 10.3.1).

Example 10.3.3 Consider, in R, the system $\sigma=\{t^2x\leq t,\ t\in[1,\infty[\}$. Now, we have

$$f(x)=\sup\{t^2x-t\mid t\in[1,\infty[\}=\begin{cases} x-1, & \text{if } x\leq 0,\\ +\infty, & \text{if } x>0.\end{cases}$$

Thus, $F=]-\infty,0]$, and we shall take $\overline{x}=0\in bd(F)$.

Easy calculations yield

$$V(1;\rho)=\begin{cases} \emptyset, & \text{if } 0<\rho<1,\\ [1,\rho^2], & \text{if } \rho\geq 1.\end{cases}$$

Obviously $f(0)=-1$, and the boundedness of these sets is not enough to guarantee a regular behavior of the system at the considered point.

The following proposition provides conditions giving rise to a richer relationship between the sets $V(\overline{x};\rho)$ and $W(\overline{x};\gamma)$.

Proposición 10.3.2 *Let us consider a consistent system $\sigma=\{a_t'x\leq b_t,\ t\in T\}$, and a point $\overline{x}\in \operatorname{bd}(F)$. Then the following statements are true:*

(i) Defining $M:=\sup\{\|a_t\|\mid a_t\in W(\overline{x};\gamma)\}$, $\gamma>0$, and supposing that M is finite, then

$$W(\overline{x};\gamma)\subset V(\overline{x};\gamma M);$$

(ii) Assuming, in addition, that the SS condition is satisfied, $0_n \notin clW(\overline{x};\gamma)$ *for all* $\gamma > 0$;

(iii) Provided that $0_n \notin clV(\overline{x};\rho)$, *then*

$$V(\overline{x};\rho) \subset W(\overline{x};\frac{\rho}{m}),$$

where m *satisfies* $0 < m < \inf\{\|a_t\| \mid a_t \in V(\overline{x};\rho)\}$.

10.3.3 · Two alternative ways of approaching the feasible directions cone

In this section we provide two formulas for $D(F,\overline{x})^o$, involving respectively the families of relaxed active constraints sets studied in the previous section.

Proposición 10.3.3 *Consider the system* $\sigma = \{a_t'x \leq b_t,\ t \in T\}$ *satisfying the SS condition, and the point* $\overline{x} \in bd(F)$. *Assuming the existence of a positive scalar* γ_0 *such that* $W(\overline{x};\gamma_0)$ *is bounded, the following statements hold:*

(i)

$$D(F,\overline{x})^o = \mathbb{R}_+ \left\{ \bigcap_{\rho>0} cl\left(\operatorname{conv}\left(V(\overline{x};\rho)\right)\right) \right\}, \tag{10.9}$$

and $\dim(F) = n$;

(ii) A positive scalar ρ_0 *exists such that*

$$0_n \notin clV(\overline{x};\rho_0);$$

(iii) A real number $\rho_0 > 0$ *can be found such that* $V(\overline{x};\rho_0)$ *is bounded;*

(iv)

$$\begin{aligned} D(F,\overline{x})^o &= \mathbb{R}_+ \left\{ \bigcap_{\gamma>0} cl\left(\operatorname{conv}\left(W(\overline{x};\gamma)\right)\right) \right\} \\ &= \operatorname{cone} \left\{ \bigcap_{\gamma>0} cl\left(W(\overline{x};\gamma)\right) \right\} \\ &= \operatorname{cone} \left\{ \bigcap_{\rho>0} cl\left(V(\overline{x};\rho)\right) \right\}. \end{aligned}$$

The formula

$$D(F,\overline{x})^o = \text{cone}\left\{\bigcap_{\gamma>0} cl\left(W(\overline{x};\gamma)\right)\right\}$$

is also established in [19, Lemma 5.1 and Proposition 5.2 (iv)], under stronger assumptions on the coefficient vectors a_t. Also the full- dimensionality of F concluded in (i) constitutes a stronger statement than Proposition 3.1 (i) in that paper.

It is also clear that (iv) above leads, as a by-product, to a new Valadier-type formula

$$\partial f(\overline{x}) = \bigcap_{\gamma>0} cl\left(\text{conv}\left(W(\overline{x};\gamma)\right)\right).$$

In different papers ([4], [5], [15], [18], etc.) the stability of the LSIP problem has been investigated in the case that T is an arbitrary (infinite) set, with no structure (so, the functions $t \mapsto a_t$ and $t \mapsto b_t$ have no property at all). The approach followed in these papers is based on the use of an (extended) metric between the so-called *nominal problem*, (P), and the *perturbed problem*

$$(P_1) \quad \text{Inf } (c^1)'x \text{ s.t. } (a_t^1)'x \leq b_t^1,\ t \in T.$$

This metric, providing the uniform convergence topology on T, is given through the formula

$$d(P_1,P) = \max\left\{\left\|c^1 - c\right\|_\infty, \sup\nolimits_{t\in T}\left\|\begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix}\right\|_\infty\right\},$$

where $\|.\|_\infty$ represents the Chebyshev norm. The metric gives rise to the maximum (supremum) of the perturbations, componentwisely measured. The space of all the linear SIP problems with the same index set T becomes, by means of this extended metric, a metrizable space which, in fact, locally behaves as a complete metric space.

Since this section only deals with the stability of the solution set, it makes sense to consider, as *parameter space*, the set Θ of all the linear inequality systems, in $\mathbb{R}^n$, indexed by T. Thus, we shall neglect the objective function as far as it has no influence on the feasible set. In this way, if $\sigma = \{a_t'x \leq b_t,\ t \in T\}$ and $\sigma_1 = \{(a_t^1)'x \leq b_t^1, t \in T\}$, the (restricted) distance between these two systems will be given by

$$d(\sigma_1,\sigma) = \sup_{t\in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_\infty . \tag{10.10}$$

We shall consider a slight variance of the already known parameter space (Θ, d), namely (Θ, d') which is a complete metrizable space ($d'(\sigma,\sigma_1) := \max\{1, d(\sigma,\sigma_1)\}$ is a real metric). Once again, the convergence, in this space, $\sigma_r \mapsto \sigma$, describes the uniform convergence, on T, of the associated functions $(a^r_{(.)})'x - b^r_{(.)}$ (considered as functions of t, and for every fixed x) to $a'_{(.)}x - b_{(.)}$. (Here, $\sigma_r = \{(a^r_t)'x \le b^r_t,\ t\in T\}$.)

Next, we consider, in $\Theta \times \mathbb{R}^n$, the subset

$$\Xi := \{(\sigma,x) \in \Theta_c \times \mathbb{R}^n \mid x \in bd(F),\ F = \text{solution set of } \sigma\},$$

where $\Theta_c \subset \Theta$ is the subset of all the *consistent systems*, and we define the mapping $\mathcal{W} : \Xi \rightrightarrows \mathbb{R}^n$ as follows:

$$\mathcal{W}(\sigma,x) := \bigcap_{\gamma>0} cl\,(\mathrm{conv}\,(W(x;\gamma))) .$$

In Ξ we shall consider the product topology induced from $\Theta \times \mathbb{R}^n$; i.e., $\lim_r(\sigma_r, x^r) = (\sigma, x)$ if and only if $\lim_r d(\sigma_r,\sigma) = 0$ and $\lim_r x^r = x$.

Let $\mathcal{Y}$ and $\mathcal{Z}$ be topological spaces, and consider a set-valued mapping $\mathcal{S} : \mathcal{Y} \rightrightarrows \mathcal{Z}$. We say that $\mathcal{S}$ is *upper semicontinuous*, in the Berge sense, at $y \in \mathcal{Y}$ if for each open set $W \subset \mathcal{Z}$ such that $\mathcal{S}(y) \subset W$, there exists an open set $U \subset \mathcal{Y}$, containing y, such that $\mathcal{S}(y^1) \subset W$ for each $y^1 \in U$.

The following result constitutes an stability result, connected with Theorem 2.2 in [26].

Teorema 10.3.1 *Let us consider a consistent system $\sigma = \{a_t'x \le b_t,\ t\in T\}$ satisfying the SS condition, and a point $\overline{x} \in bd(F)$. Assume the existence of a positive scalar γ_0 such that $W(\overline{x};\gamma_0)$ is bounded. Then the mapping $\mathcal{W}$ is upper semicontinuous at $(\sigma,\overline{x})$.*

10.4 Application to linear semi-infinite optimization

In this section we consider the space of all linear semi-infinite optimization problems with the same index set, endowed with a suitable topology. We have provided in [21] a constructive proof of the following generic result: if we confine ourselves to the class of problems having a bounded set of coefficient vectors (those vectors appearing in the left-hand side), the set of those problems which have a strongly unique optimal solution contains an open and dense subset of the set of solvable problems in the same class.

10.4.1 Introduction

We consider the *linear parametric optimization* problem π, in $\mathbb{R}^n$, of the form

$$\pi : \quad \text{Inf } \left\{ c'x \text{ s.t. } a_t'x \geq b_t, \ t \in T \right\}.$$

π, as in the previous sections is represented by (c, a, b). The *constraint system* of π is represented, as usual, by $\sigma = \{a_t'x \geq b_t, \ t \in T\}$, which allows us to use the alternative notation $\pi = (c, \sigma)$. If we denote the optimal value of π by v, the optimal set will be

$$F^* := \left\{ x^* \in F \mid c'x \ = v \right\},$$

and the *optimal set mapping* $\mathcal{F}^* : \Pi \longrightarrow \mathbb{R}^n$, putting into correspondence to each $\pi \in \Pi$ the optimal set F^*. If $x^* \in F$ is a strongly unique solution, this fact trivially entails $F^* = \{x^*\}$. The existence of a strongly unique minimum has far reaching properties for the convergence analysis of many iterative numerical procedures. Our aim is to investigate some topological properties of the problems π, which have strongly unique solutions. So, we need to define the *parameter space*, which in our approach is the set Π of all the problems $\pi = (c, \sigma)$, whose constrains systems have the same index set T. Obviously, we can identify Π by $\mathbb{R}^n \times \left(\mathbb{R}^{n+1}\right)^T$.

From now on, Π_c represents the *consistent problems* subset ($\pi \in \Pi_c \Longleftrightarrow F \neq \varnothing \Longleftrightarrow v < +\infty$). Π_s will be the set of *solvable problems* ($\pi \in \Pi_s \Longleftrightarrow F^* \neq \varnothing \Longleftrightarrow v$ is attained) and Π_{su} is the class of problems having a *strongly unique* (optimal) *solution.* Obviously, $\Pi_{su} \subset \Pi_s \subset \Pi_c$.

An *extended distance* $\delta : \Pi \times \Pi \longrightarrow [0, \infty]$ is introduced in Π by means of

$$\delta(\pi_1, \pi) := \max \left\{ \|c_1 - c\|_\infty , \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \right\},$$

where $\|.\|_\infty$ represents the Chebyshev norm. As (Θ, d), (Π, δ) is a Hausdorff space, whose topology satisfies the first axiom of countability (i.e., the convergence is established by means of sequences, since each point has a countable base of neighborhoods), and describes the uniform convergence topology in Π.Following Greenberg and Pierskalla [22], most of the authors dealing with stability in linear SIP have considered Π equipped with the topology induced by δ as far as this extended metric provides the supremum of the (absolute) discrepancies, between π_1 and π, componentwisely measured. Nevertheless, let us observe that all the results in this section are still valid by replacing $\|.\|_\infty$ with any other norm on the vector space

$(p = n,\ n+1)$, $\|.\|$, and assigning a positive weight β_t (scaling) to the constraint associated with the index $t \in T$, provided that two positive scalars exists $\underline{\beta}$ and $\overline{\beta}$, such that $\underline{\beta} \leq \beta_t \leq \overline{\beta}$, i.e., replacing $\delta(\pi_1, \pi)$ with

$$\delta(\pi_1, \pi) := \max \left\{ \|c_1 - c\|_\infty, \sup_{t \in T} \beta_t \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \right\}.$$

Different authors have considered families of optimization problems (i.e., spaces of parameters) forming a Banach space, proving that, under suitable assumptions on the topological properties of the optimal set mapping $\mathcal{F}^*$, most (in the sense of Bair's categories) of the optimization problems have a unique solution (see, e.g., [28] and [32], and the references therein). It is well-known that, if the given problem π is a continuous linear SIP and only continuous perturbations of the coefficients of σ are allowed, then the parameter space is Banach. In [34] and [35] we have proved, for continuous linear SIP, that the set of problems with a unique primal and dual solution contains a G_δ (i.e., a countable intersection of open sets) dense subset of the class of solvable problems. This result was improved in [6], where it was proved in a constructive way (based upon a characterization of the strong uniqueness in continuous linear SIP, due to Nürnberger [30]) that the previous statement for the primal problems remains true replacing "G_δ" with "open". The main purpose of this part is to extend the last generic result to the more general class of linear SIPs π with arbitrary index set T under the only assumption that the left-hand side vectors of σ form a bounded set. This condition is not too restrictive (it holds if we divide both members of each constraint $a_t'x \geq b_t$ such that $a_t \neq 0_n$ by $\|a_t\|$), and it is preserved by arbitrary perturbations of finite size. More precisely, our main result here establishes that the set of problems with a strongly unique solution contains an open and dense subset of the class of solvable problems when we are confined to the class

$$\Lambda := \left\{ \pi \in \Pi \mid \sup_{t \in T} \|a_t\| < \infty \right\}.$$

(Observe that Λ is open and closed in (Π, δ).) Our proof is also constructive, but it requires the application of some sophisticated machinery, including the new conceptual tools developed in the previous two sections of this paper.

10.4.2 Summary

We would like to summarize some properties of the sets defined before which will be used in what follows.

Proposición 10.4.1 *In relation to the problem* $\pi \in \Pi_c$, *a point* $\widetilde{x} \in F$ *and the associated sets* M, $D(F;\widetilde{x})$, $A(\widetilde{x})$, $D(\widetilde{x})$ *and* $W(\widetilde{x};\gamma)$, *the following statements are true:*

(i) $\pi \in int\Pi_c$ *if and only if the SS condition is satisfied.*

(ii) If $\pi \in \Pi_s$, *then* $c \in clM$.

(iii) If $c \in rintM$, *then* $\pi \in \Pi_s$.

(iv) $\pi \in int\Pi_s$ *if and only if* $\pi \in (int\Pi_c) \cap \Pi_s$ *and* $c \in intM$. *In fact, condition* $c \in intM$ *is equivalent to the boundedness of the optimal set of* π, F^*.

(v) $\widetilde{x} \in F^*$ *if and only if* $c \in D(F;\widetilde{x})$.

(vi) If $\pi \in \Lambda$ *and the SS condition is satisfied, then*

$$coneD(\widetilde{x}) = D(F;\widetilde{x})^\circ.$$

(vii) $D(\widetilde{x}) \setminus \{0_n\} \subset clW(\widetilde{x};\gamma)$ *for every* $\gamma > 0$.

(viii) $\widetilde{x}$ *is a strongly unique solution of* π *if and only if*

$$c \in intD(F;\widetilde{x}).$$

(ix) If $\widetilde{x} \in extF$, *then* $\dim W(\widetilde{x};\gamma) = n$ *for every* $\gamma > 0$.

(x) Assume that $\pi \in \Lambda$, *that the SS condition is satisfied and* $c \neq 0_n$. *Then*

$$\pi \in int\,(\Pi_{su} \cap \Lambda)$$

if and only if there exists $\widetilde{x} \in F$ *and a set* $C \subset D(\widetilde{x})$, $|C| = n$, *such that* $c \in coneC$ *and for every set* C *satisfying such a condition, we have that all the possible subsets of* n *elements of* $C \cup \{c\}$ *are linearly independent.*

Corolario 10.4.1 *In relation with* Π, *the following statements hold:*

(i) $\Pi_{su} = \varnothing$ *if and only if* $|T| < n$.

(ii) $\Pi_{su} \cap \Lambda$ *is open in* $\Pi_s \cap \Lambda$ *if and only if* $|T| \geq n+1 \geq 3$ *fails.*

10.4.3 The main result

Now, we are in a position to present the main result in this section and to mention that its proof, given in [21], is constructive.

Proposición 10.4.2 *If* $|T| \geq n$, *then the open set* $int\,(\Pi_{su} \cap \Lambda)$ *is dense in* $\Pi_s \cap \Lambda$.

Remark 10.4.1 *Let us examine the assumptions in Proposition 10.4.2.*

If $|T| < n$, we have $\Lambda = \Pi$ and $\Pi_{su} = \varnothing$ (by Corollary 10.4.1 (i)), whereas $\Pi_s \neq \varnothing$ by Proposition 10.4.1 (iii). So, the assumption $|T| \geq n$ is not superfluous.

If $\infty > |T| \geq n$, $\Lambda = \Pi$ and Λ can be removed from the statement.

If T is infinite, we shall prove that it is not possible to eliminate Λ (or replace Λ by its complementary set $\Pi \diagdown \Lambda$) in the Proposition 10.4.2. In fact, Example 1 in [14] shows the existence of a system $\sigma := \{a_t' x \geq b_t,\ t \in T\}$ (with unbounded coefficients) and $\rho > 0$ such that $F_1 = clB$ for all $\sigma^1 := \{(a_t^1)' x \geq b_t^1,\ t \in T\}$ such that

$$\sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_\infty < \rho.$$

Let $\pi = (c, \sigma) \notin \Lambda$ with $c \neq 0_n$. If $\pi_1 = (c^1, \sigma_1) \notin \Lambda$ satisfies $\delta(\pi_1, \pi) < \min\{\rho, \|c\|_\infty\}$, then we have $\pi_1 \in \Pi_s \diagdown \Pi_{su}$. Hence, Π_{su} does not intersect a certain open subset of Π_s.

The next example illustrates the constructive character of the proof of Proposition 10.4.2.

Example 10.4.1 Let us consider the linear SIP, in $\mathbb{R}^3$,

$$\begin{aligned} \pi \quad &: \quad \operatorname{Inf}(-x_3) \\ \text{s.t.} \ -2tx_1 - x_3 \ &\geq \ -t^2 - 2|t|,\ t \in T := [-1, 1]. \end{aligned}$$

Obviously, $\pi \in \Pi_s \cap \Lambda$. *In fact,* $F = \{x \in \mathbb{R}^3 \mid x_3 \leq g(x_1)\}$, *where*

$$g(x_1) = \begin{cases} x_1/2 + \dfrac{9}{16}, \ \text{for } x_1 < -\dfrac{5}{4}, \\ -(x_1+1)^2, \ \text{for } -\dfrac{5}{4} \leq x_1 \leq -1, \\ 0, \ \text{for } -1 < x_1 < 1, \\ -(x_1-1)^2, \ \text{for } 1 \leq x_1 \leq \dfrac{5}{4}, \\ -x_1/2 + \dfrac{9}{16}, \ \text{for } \dfrac{5}{4} < x_1, \end{cases}$$

so that

$$F^* = \{\widetilde{x} \in \mathbb{R}^3 \mid \widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, 0)',\ \widetilde{x}_1 \in [-1, 1],\ \widetilde{x}_2 \in \mathbb{R}\},$$

and $\pi \notin int(\Pi_s \cap \Lambda)$ *by Proposition 10.4.1 (iv).*

Certainly we have, for instance, $c = \frac{1}{2}(a_{-1} + a_1) \in M$, *but* $\dim M = 2$. *We follow the constructive process described in the proof of Proposition 10.4.2, but the number of steps is reduced to two, given the particular structure of* π.

We start by adding a_0 *to the pair* $\{a_{-1}, a_1\}$, *and observe that the set*

$$\left\{a_0 + (0, \eta, 0)', a_{-1}, a_1\right\},$$

with $\eta \neq 0$, *is linearly independent.*

Note that the point $(0, 0, -1)'$, *is an SS point of* π, *and the problem built in the first step is* $\pi_1 = \left(c^1, a^1, b^1\right)$ *with* $b^1 = b$, $c^1 = (0, \eta/3, -1)'$ *and* a^1 *defined as follows:*

$$a_t^1 := \begin{cases} (0, \eta/3, -1)', \ t = 0, \\ \quad a_t, \ t \in T \setminus \{0\}. \end{cases}$$

Since $c^1 = \frac{1}{3}\left(a_0^1 + a_{-1}^1 + a_1^1\right)$, *we have* $c^1 \in intM_1$. *In fact, the new problem*

$$\begin{aligned} \pi_1 \quad &: \quad \mathrm{Inf}((\eta/3)\, x_2 - x_3) \\ \text{s.t. } \eta x_2 - x_3 \ &\geq \ 0, \ t = 0, \\ -2tx_1 - x_3 \ &\geq \ -t^2 - 2\,|t|\,, \ T \setminus \{0\}, \end{aligned}$$

also admits $(0, 0, -1)'$, *as an SS point, and* $\delta(\pi_1, \pi) < \varepsilon/3$, *provided that* $|\eta| < \varepsilon/3$. *For the sake of simplicity we take* $\eta > 0$.

Let us check that the optimal value of π_1 *is zero. The block of constraints associated with* $T \setminus \{0\}$ *conjointly entail* $x_3 \leq 0$. *In addition, the objective function can be expressed as* $\frac{1}{3}(\eta x_2 - x_3) + \frac{2}{3}(-x_3)$ *and both terms are non-negative at any feasible point of* π_1. *So, the optimal value of this problem is zero, and it is attained if* $x_2^* = x_3^* = 0$. *The constraints associated with the indices in* $T \setminus \{0\}$ *also yield* $x_1^* \in [-1, 1]$ *and* $F_1^* = [-1, 1] \times \{0\} \times \{0\}$.

We start Step2 from the constructive proof, by choosing $\widehat{x} = (1, 0, 0)' \in F_1^* \cap extF_1$. *We know that the associated* $W_1(\widehat{x}; \gamma)$ *is full-dimensional, despite the fact that the only binding constraint at* $\widehat{x}$ *is the one corresponding to* $t = 0$.

The way in which we combine Steps 2 and 3 from the proof, consists of modifying, firstly, two constraints in π_1 *in order to get three active constraints at* $\widehat{x}$ *with linearly independent associated coefficient vectors. To this aim, we choose* $t_0 \in (0, 1)$ *such that* $(t_0)^2 + 4t_0 < \varepsilon/2$ *and, then, we replace both constraints in* π_1 *associated with* $-t_0$ *and* t_0 *by new constraints*

$-2tx_1 - x_3 \geq -2t$, *which are obviously binding at $\widehat{x}$. Finally, the independent term of the remaining constraints is changed to $-t^2 - 2|t| - \eta$. In this manner we have built the problem*

$$\pi_2 \quad : \quad \mathrm{Inf}((\eta/3)\, x_2 - x_3)$$

$$\begin{aligned} \text{s.t. } \eta x_2 - x_3 &\geq 0,\ t = 0, \\ -2tx_1 - x_3 &\geq -2t,\ t \in \{-t_0, t_0\}. \\ -2tx_1 - x_3 &\geq -t^2 - 2|t| - \eta,\ t \in T \diagdown \{0, -t_0, t_0\}. \end{aligned}$$

It is clear that $c^2 \in int A_2(\widehat{x}) \subset int D(F_2, \widehat{x})^\circ$ (in fact,

$$c^2 = \frac{1}{3}\left(a^2_{-t_0} + a^2_0 + a^2_{t_0}\right),$$

with $\left\{a^2_{-t_0}, a^2_0, a^2_{t_0}\right\}$ linearly independent), and Proposition 10.4.1 (viii) applies to conclude that $\widehat{x}$ is a strongly unique optimal solution for π_2. It is also evident that $(0,0,-1)'$ is an SS point for this problem and that $\delta(\pi_1, \pi_2) < \varepsilon/2$. Hence, π_2 is the parameter we were looking for, since $\pi_2 \in int(\Pi_{su} \cap \Lambda)$ and $\delta(\pi, \pi_2) < \delta(\pi_1, \pi_2) + \delta(\pi_1, \pi) < \varepsilon$.

10.4.4 Stability analysis

Proposition 10.4.2 shows that an arbitrary small perturbation of the data of a given linear SIP, π, with a bounded left-hand side of vectors provides a new problem of the same class, $\widetilde{\pi}$, such that the problems in a certain neighborhood of $\widetilde{\pi}$ have strongly unique solutions. It remains to determine the stability properties of the approximating problem $\widetilde{\pi}$ and the existing relationships between the values and the optimal sets of both problems.

Let us recall the well known definition of Hadamard well-posedness of optimization problems, see [9].

Definition 10.4.1 $\pi \in \Pi_s$ *is called strongly* Hadamard well-posed *if F^* is a singleton (i.e., $F^* = \{x^*\}$) and for every sequence $\{\pi_k\}_{k=1}^{\infty} \subset \Pi_s$ converging to π in (Π, δ) and every sequence $\left\{x^k\right\}_{k=1}^{\infty}$ such that $x^k \in F_k^*$, $k = 1, 2, \ldots$, one has $\lim_k x^k = x^*$.*

The following result shows that the problem $\widetilde{\pi}$ in Proposition 10.4.2 constitutes a kind of regularized problem associated with the problem π, from which the procedure has started.

Proposición 10.4.3

(i) Assume that $\widetilde{\pi} \in int\,(\Pi_{su} \cap \Lambda)$. *Then, at the parameter* $\widetilde{\pi}$, *the feasible set mapping* $\mathcal{F}$ *is continuous, the value function* v *is a (real-valued) continuous function and the optimal set valued mapping* $\mathcal{F}^*$ *is a function (valued in* $\mathbb{R}^n$, *provided that* $\mathcal{F}^*$ *is restricted to the set of parameters having a unique optimal solution) which is also continuous at* $\widetilde{\pi}$. *Moreover,* $\widetilde{\pi}$ *is strongly Hadamard well-posed.*

(ii) If π *(the problem in* $\Pi_s \cap \Lambda$ *from which the process described in the proof of Proposition 10.4.2 starts, providing* $\widetilde{\pi}$*) satisfies the SS condition and* F^* *is bounded,*

then the following statements hold:

(a) There will exist a pair of positive scalars ε *and* ρ *such that* $\delta\,(\pi, \widetilde{\pi}) < \varepsilon$ *yields the Lipschitz inequality* $|v - \widetilde{v}| \leq \rho\delta\,(\pi, \widetilde{\pi})$.

(b) $\mathcal{F}^*$ *is usc at* π, *which is equivalent to the closedness of this mapping at* π. *In other words, if* $\{\widetilde{\pi}_k\}_{k=1}^{\infty} \subset int\,(\Pi_{su} \cap \Lambda)$ *converges to* π *in* (Π, δ) *and we have a sequence* $\left\{\widetilde{x}^k\right\}_{k=1}^{\infty}$ *such that* $\widetilde{x}^k \in \widetilde{F}_k^*$, $k = 1, 2, ...,$ *and* $\lim_k \widetilde{x}^k = x^*$, *then we can assert that* x^* *is optimal for* π.

At the end, we would like to mention that the results obtained in this paper are of a special importance in the vector semi-infinite optimization, where we have used many times the obtained knowledge of the behavior of the feasible set (see, [12], [38] and [39]).

Bibliografía

[1] E.J. Anderson, M.A. Goberna, and M.A. López, Locally polyhedral linear semi-infinite systems, Linear Algebra Appl., 103 (1998), pp. 95–119.

[2] B. Brosowski, Parametric Semi-Infinite Optimization, Verlag Peter Lang, Frankfurt am Main, 1982.

[3] B. Brosowski, Parametric semi-infinite linear programming I. Continuity of the feasible set and the optimal value, Math Programming Study, 21 (1984), pp. 18-42.

[4] M.J. Cánovas, M.A. López, J. Para, M.I. Todorov, Well posedness in linear semi-infinite programming. SIAM J. on Optimization, Volume 10, Number 1 (1999), pp. 82-98.

[5] M.J. Cánovas, M.A. López, J. Para, M.I. Todorov, Solving strategies and well-posedness in linear semi-infinite optimization, Annals of Operations Research, Vol. 101 (2001), pp. 171-190.

[6] G.Christov, M.I.Todorov,. Semi-infinite optimization. Existence and uniqueness of the solutions. Mathematica Balkanica, New series Vol 2, 1989, Fask2-3, pp. 182-191.

[7] J.W. Daniel, On perturbation in systems of linear inequalities, SIAM J. Numer. Anal., 10 (1973), pp. 299-307.

[8] J.W. Daniel, Remarks on perturbation in linear inequalities, SIAM J. Numer. Anal., 12 (1975), pp. 770-772.

[9] A.L. Dontchev, T. Zolezzi, Well-Posed Optimization Problems, Berlin: Springer-Verlag, 1991.

[10] M.D. Fajardo, M.A. López, Locally Farkas-Minkowski systems in convex semi-infinite programming, J. Optim. Theory Appl., 103 (1999), pp. 313-335.

[11] T. Fischer, Contributions to semi-infinite linear optimization, in Methoden und Verfaharen der Mathematischen Physik, Band 27, Verlag Peter Lang, Berlin, 1987, pp. 175-199.

[12] P. Georgiev, M.I. Todorov, S. Helbig, T. Tanaka, Well-defined efficient points in the vector optimization, Proceedings of the 5th International Conference on Optimization: Techniques and Applications, Vol. 2 (2001), Editor D. Li, Hong Kong, pp. 657-663.

[13] K. Glashoff, S.-A. Gustafson, Linear Optimization and Approximation, Springer Verlag, Berlin, 1983.

[14] M.A. Goberna, M. Larriqueta, V.N. Vera de Serio, On the stability of the boundary of the feasible set in linear optimization, Set-valued Analysis, Vol. 11 (2003), pp. 203-223.

[15] M.A. Goberna, M.A. López, A note on topological stability of linear semi-infinite inequality systems, J. Optim. Theory Appl., 89 (1996), pp. 227-236.

[16] M.A. Goberna, M.A. López, Linear Semi-Infinite Optimization, Wiley, Chichester, 1998.

[17] M.A. Goberna, M.A. López, M.I. Todorov, Unicity in linear optimization, J. Optim. Theory Appl., 86 (1995), pp. 37–56.

[18] M.A. Goberna, M.A. López, M.I. Todorov, Stability theory for linear inequality systems, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 730-743.

[19] M.A. Goberna, M.A. López, M.I. Todorov, Extended active constraints in linear optimization with applications, SIAM J. Optim. 14 (2003), pp. 608-619.

[20] M.A. Goberna, M.A. López, M.I. Todorov, A sup-function aproach to linear semi-infinite optimization, J. Math. Sciences, Vol. 116, 4 (2003), pp. 3359-3368.

[21] M.A. Goberna, M.A. López, M.I. Todorov, A generic results in linear semi-infinite optimization, Appl. Math. Optim., 48 (2003), pp. 181–193.

[22] H.J. Greenberg, W.P. Pierskalla, Stability Theorems for infinitely constrained mathematical programs, JOTA, Vol. 16, Nos. 5/6, 1975, pp. 409-428.

[23] F. Guerra, M.A. Jiménez, On feasible sets defined through Chebyshev approximation, Math. Methods Oper. Res., 47 (1998), pp. 255–264.

[24] A. Hassouni and W. Oettli, On regularity and optimality in nonlinear semi-infinite programming, in Semi-Infinite Programming. Recent Advances, M.A. Goberna and M.A. López, eds., Kluwer, Dordrecht, The Netherlands, (2001), pp. 59–74.

[25] S. Helbig, Stability in disjunctive optimization I: Continuity of the feasible set, Optimization 21 (1990), pp. 855-869.

[26] S. Helbig, M.I. Todorov, Unicity results for general linear semi-infinite optimization problems using a new concept of active constraints, Appl. Math. Optim., 38 (1998), pp. 21–43.

[27] M.A. Jimenez, M.I. Todorov, Unicity the solutions of infinite linear inequality systems, Compt. Rend. Acad. Bulg. Sci., Tome 54, No. 6, (2001), pp. 17-20.

[28] P.S. Kenderov, Most of the optimization problems have unique solution, in Parametric Optimization and Approximation (B. Brosowski and F. Deutsch, eds.), Birkhäuser, Basel, Vol. 72 (1984), pp. 203-216.

[29] W. Li, C. Nahak, I. Singer, Constraint qualifications for semi-infinite systems of convex inequalities, SIAM J. Optim. 11 (2000), pp. 31-52.

[30] G. Nürnberger, Unicity in semi-infinite optimization, in Parametric Optimization and Approximation (B. Brosowski and F. Deutsch, eds.), Birkhäuser, Basel, Vol. 72 (1984), pp. 231-246.

[31] R. Puente, V. Vera de Serio, Locally Farkas-Minkowski linear inequality systems, TOP, 7 (1999), pp. 103–121.

[32] J.P. Revalski, N.V. Zhivkov, Well-posed constrained optimization problems in metric spaces, J. Optim. Theory Appl., 76 (1993), pp. 145–163.

[33] Y. Sawaragi, H. Nakayama, T. Tanino, Theory of Multiobjective Optimization, Math.in Sci. and Eng., Vol. 176, Acad. Press Inc., 1985.

[34] M.I. Todorov, Generic existence and uniqueness of the solution set to linear semi-infinite optimization problems, Numer. Funct. Anal. Optim., 8 (1985-86) pp. 27-39.

[35] M.I. Todorov, Uniqueness of the saddle points for most of the linear semi-infinite optimization problems, Numer. Funct. Anal. and Optim., 10(3&4), (1989), pp. 367-382.

[36] M.I. Todorov, Existence of the solutions of the linear vector semi-infinite optimization problems, Mathematica Balkanica, New series Vol. 4, (1990), Fasc. 4, pp. 390-395.

[37] M.I. Todorov, Linear vector optimization. Properties of the efficient sets. Serdica-Bulgaricae mathematicae publicationes. 18 (1992), pp. 179-185.

[38] M.I. Todorov, Well-posedness in the linear vector semi infinite optimization, Multiple Criteria Decision Making. Proceedings of the tenth international conference: "Expand and Enrich the Domains of Thinking and Applications", P.L. Yu and etc. Eds., Springer-Verlag, (1994), pp. 141-150.

[39] M.I. Todorov, Kuratowski convergence of the efficient sets in the parametric linear vector semi-infinite optimization, European Journal of Operational Research 94 (1996), pp. 610-617.

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