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FINITE GRAPHS HAVE UNIQUE n-FOLD SYMMETRIC PRODUCT SUSPENSION

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Abstract. Let Z be a metric continuum and n be a positive integer. We consider the hyperspace $F_n(Z)$ of all nonempty closed subsets of Z with at most n points. Given $n > 1$, the *n-fold symmetric product suspension* of Z is the quotient space $F_n(Z)/F_1(Z)$, denoted by $SF_n(Z)$. In this paper we prove that if $n \geq 4$, X is a finite graph, and Y is a continuum such that $SF_n(X)$ is homeomorphic to $SF_n(Y)$, then X is homeomorphic to Y. This result answers a question posed by Alejandro Illanes.

1. Introduction

Recently, the study of the uniqueness of hyperspaces has become a relevant field in continuum theory.

A *continuum* is a nonempty compact, connected metric space. The set of positive integers is denoted by $\mathbb{N}$.

Given a continuum Z and $n \in \mathbb{N}$, we consider the following hyperspaces of Z:

$$\begin{aligned} 2^Z &= \{A \subset Z \colon A \text{ is a nonempty closed subset of } Z\}, \\ F_n(Z) &= \{A \in 2^Z \colon A \text{ has at most } n \text{ points}\}, \text{ and} \\ C_n(Z) &= \{A \in 2^Z \colon A \text{ has at most } n \text{ components}\}. \end{aligned}$$

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All these hyperspaces are metrized by the Hausdorff metric H [29, Theorem 2.2]. The hyperspace $F_n(Z)$ is called the *n-fold symmetric product of* Z.

In 1979 Sam B. Nadler, Jr. introduced the *hyperspace suspension of a continuum* Z, $HS_1^1(Z)$, in [36]. Later in 2004, S. Macías defined the *n-fold hyperspace suspension of a continuum* Z, $HS_n^n(Z)$, where $n \in \mathbb{N}$ with $n \geq 2$ (see [32]). Furthermore, in 2008, Juan C. Macías introduced the *n-fold pseudo-hyperspace suspension* of a continuum $HS_1^n(Z)$, in [30]. The study of the (n,m)-*fold hyperspace suspension* of a continuum Z, $HS_m^n(Z)$, where $n, m \in \mathbb{N}$, and $n \geq m$, is a generalization of the latter research, that has gained recent interest (see [2]).

The *n-fold symmetric product suspension of a continuum* Z was defined in 2010 by F. Barragán [3] to be the quotient space $F_n(Z)/F_1(Z)$, denoted by $SF_n(Z)$ for each $n \in \mathbb{N}$ with $n \geq 2$, is obtained from $F_n(Z)$ by identifying $F_1(Z)$ to a one-point set, with the quotient topology.

Given a continuum Z, the symbol q_Z denotes the natural projection $q_Z \colon F_n(Z) \to SF_n(Z)$, and F_Z denotes the element $q_Z(F_1(Z))$, notice that $q_Z|_{F_n(Z)-F_1(Z)} \colon F_n(Z) - F_1(Z) \to SF_n(Z) - \{F_Z\}$ is a homeomorphism. We write q_Z^* instead of $q_Z|_{F_n(Z)-F_1(Z)}$.

For a continuum Z, let $\mathcal{H}(Z)$ be any of the hyperspaces defined above. We say that a continuum Z has *unique hyperspace* $\mathcal{H}(Z)$ provided that the following implication holds: if Y is a continuum and $\mathcal{H}(Z)$ is homeomorphic to $\mathcal{H}(Y)$, then Z is homeomorphic to Y.

Problem 1.1. *For $n \in \mathbb{N}$ with $n \geq 2$ find conditions on a continuum Z, so that Z has unique hyperspace $SF_n(Z)$.*

The problem on finding conditions on a continuum Z in order that Z has unique $\mathcal{H}(Z)$ has been widely studied (see [1], [2], [4]–[6], [8]–[27], [32], [33], [35]).

Recall that a *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which are either disjoint or intersect only at one or both of their end points.

The following are well-known results regarding this subject, for the particular case of finite graphs.

(a) If X is a finite graph and $n \in \mathbb{N}$, then X has unique hyperspace $F_n(X)$ (see [4, Corollary 5.9]).

(b) If X is a finite graph different from an arc or a simple closed curve, then X has unique hyperspace $C_1(X)$ (see [1, Theorem 1] and [6, 9.1]).

(c) If X is a finite graph and $n \in \mathbb{N}$ with $n \geq 2$, then X has unique hyperspace $C_n(X)$ (see [24] and [25, Theorem 3.8]).

(d) If X is a finite graph and $n, m \in \mathbb{N}$ with $n \geq m$, then X has unique hyperspace $HS_m^n(X)$ (see [2, Theorem 3.6], [15, Theorem 3.2], and [35, Theorem 5.7]).

Related to Problem 1.1, the aim of this paper is to prove that:

(e) If X is a finite graph and $n \in \mathbb{N}$ with $n \geq 4$, then X has unique hyperspace $SF_n(X)$ (see Theorem 3.8).

2. Definitions and Preliminary Results

Given a continuum Z and a subset A of Z, $\mathrm{int}_Z(A)$, $\mathrm{cl}_Z(A)$, and $\mathrm{bd}_Z(A)$ denote the *interior*, the *closure* and *boundary* of A in Z, respectively. If A is a set, $|A|$ denotes the cardinality of A. If d_Z is the metric of Z, $\varepsilon > 0$, and $p \in Z$, the set $\{z \in Z \colon d_Z(p, z) < \varepsilon\}$ is denoted by $B_Z(p, \varepsilon)$.

Given $n \in \mathbb{N}$, an *n-cell* is a space which is homeomorphic to $[0,1]^n$. An *arc* is an 1-cell. A *simple closed curve* is a space which is homeomorphic to S^1 in the plane.

Let Z be a continuum, $z \in Z$, and β be a cardinal number. We say that z *has order less than or equal to* β, *in* Z, written $\mathrm{ord}(z, Z) \leq \beta$, whenever z has a basis of neighborhoods $\mathfrak{B}$ in Z such that the cardinality of the boundary of U in Z is less than or equal to β, for each $U \in \mathfrak{B}$. We say that z *has order equal to* β, *in* Z ($\mathrm{ord}(z, Z) = \beta$) provided that $\mathrm{ord}(z, Z) \leq \beta$ and $\mathrm{ord}(z, Z) \leq \alpha$ for any cardinal number $\alpha < \beta$. Let $E(Z) = \{z \in Z \colon \mathrm{ord}(z, Z) = 1\}$ and $R(Z) = \{z \in Z \colon \mathrm{ord}(z, Z) \geq 3\}$. The elements of $E(Z)$ (respectively, $R(Z)$) are called *end points* (respectively, *ramification points*) of Z.

Let $n, r \in \mathbb{N}$ and let $U_1, \ldots, U_r$ be subsets of a continuum Z. We denote by $\langle U_1, \ldots, U_r \rangle$ the set $\langle U_1, \ldots, U_r \rangle_{2^Z} \cap F_n(Z)$, where $\langle U_1, \ldots, U_r \rangle_{2^Z} = \{A \in 2^Z \colon A \subset U_1 \cup \cdots \cup U_r \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \in \{1, \ldots, r\}\}$. Recall that the family of all sets $\langle U_1, \ldots, U_r \rangle_{2^Z}$ is a basis for the topology in 2^Z [29, Theorem 1.2].

Given $n \in \mathbb{N}$ and a continuum Z, we consider the following subspaces of $F_n(Z)$:

$$\mathcal{E}_n(Z) = \{A \in F_n(Z) \colon A \text{ has a neighborhood in } F_n(Z) \text{ which is an } n\text{-cell}\},$$

$$R_n(Z) = \{A \in F_n(Z) \colon A \cap R(Z) \neq \emptyset\},$$

$$\text{if } n > 1, \text{ let } \mathcal{N}_n(Z) = \{A \in F_n(Z) - F_{n-1}(Z) \colon A \cap R(Z) = \emptyset\}, \text{ and}$$

$$\mathcal{G}(Z) = \{z \in Z \colon z \text{ has a neighborhood in } Z \text{ which is a finite graph}\}.$$

Recall that, as in [10], a continuum Z is said to be *almost meshed* whenever $\mathcal{G}(Z)$ is a dense subset of Z.

Given a finite graph X, a *free arc* in X is an arc J with end points x and z such that $J - \{x, z\}$ is an open subset of X. A *maximal free arc* in X is a free arc in X that is maximal with respect to the inclusion. A *cycle* in X is a simple closed curve J in X such that $J - \{a\}$ is an open set in X for some $a \in J$. Let

$$\mathcal{A}_R(X) = \{J \subset X\colon J \text{ is a cycle in } X\},$$
$$\mathcal{A}_S(X) = \{J \subset X\colon J \text{ is a maximal free arc in } X\} \cup \mathcal{A}_R(X), \text{ and}$$
$$\mathcal{A}_E(X) = \{J \subset X\colon J \text{ is a maximal free arc and } |J \cap R(X)| = 1\}.$$

The elements of $\mathcal{A}_S(X)$ are known as *edges* of X.

The word *map* means a continuous function. We shall make use of other concepts not defined here, which will be taken as in [29].

From now on, when we refer to X as a finite graph, we mean that X has $E_1, \dots, E_m$ edges, with $m \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and let X be a finite graph. Given $i_1, \dots, i_m \in \mathbb{N} \cup \{0\}$ such that $i_1 + \cdots + i_m = n$, we consider $\mathcal{K}_X(i_1, \dots, i_m)$ to be the subset of $F_n(X)$ such that each member of $\mathcal{K}_X(i_1, \dots, i_m)$ has exactly i_j elements in the interior of edge E_j, for each $j \in \{1, \dots, m\}$. Given $j, l \in \{1, \dots, m\}$, define

$$\mathcal{K}_X^j = \mathcal{K}_X(i_1, \dots, i_m), \text{ provided that } i_j = n, \text{ and}$$
$$\mathcal{K}_X(j, l) = \mathcal{K}_X(i_1, \dots, i_m), \text{ provided that } j \neq l \text{ and } i_j + i_l = n.$$

Notice that $\mathcal{K}_X^j \subsetneq \langle \mathrm{int}_X(E_j) \rangle$ and $\mathrm{cl}_{F_n(X)}(\mathcal{K}_X^j) = \langle E_j \rangle$.

Lemma 2.1. *Let $n \in \mathbb{N}$. If X is a finite graph, then*

(a) $\mathcal{K}_X(i_1, \dots, i_m)$ *is arcwise connected.*

(b) $\mathcal{K}_X(i_1, \dots, i_m) \cap \mathcal{K}_X(l_1, \dots, l_m) = \emptyset$ *if and only if there is $j \in \{1, \dots, m\}$ such that $i_j \neq l_j$.*

(c) $\mathcal{K}_X(i_1, \dots, i_m)$ *is an open subset of $\mathcal{E}_n(X)$.*

Proof. (a) Let $A, B \in \mathcal{K}_X(i_1, \dots, i_m)$ with $A \neq B$. Let $M = \{j \in \{1, \dots, m\}\colon i_j \neq 0\}$. Fix $j \in M$. Let $A_j = A \cap \mathrm{int}_X(E_j)$ and $B_j = B \cap \mathrm{int}_X(E_j)$. Let L be an interval in the real line which is homeomorphic to $\mathrm{int}_X(E_j)$. We identify $\mathrm{int}_X(E_j)$ with L. Thus, we may suppose that $A_j = \{a_1, \dots, a_{i_j}\}$ and $B_j = \{b_1, \dots, b_{i_j}\}$, where $a_1 < \cdots < a_{i_j}$ and $b_1 < \cdots < b_{i_j}$. Let $\mu_j\colon [0,1] \to \langle \mathrm{int}_X(E_j) \rangle$ be defined as $\mu_j(t) = \{tb_1 + (1-t)a_1, \dots, tb_{i_j} + (1-t)a_{i_j}\}$, for each $t \in [0,1]$, which is continuous, $\mu_j(0) = A_j$ and $\mu_j(1) = B_j$. Moreover, $|\mu_j(t)| = i_j$, for each $t \in [0,1]$. Let $\mu\colon [0,1] \to \mathcal{K}_X(i_1, \dots, i_m)$ be defined as $\mu(t) = \bigcup\{\mu_j(t)\colon j \in M\}$, for each $t \in [0,1]$, which is continuous. Thus, $\mu([0,1])$ is a locally connected continuum. Since $A, B \in \mu([0,1])$, there exists an arc in $\mu([0,1])$ with end points A, B.

(b) The proof is straightforward.

(c) Let $A \in \mathcal{K}_X(i_1,\ldots,i_m)$. Notice that $A \in \mathcal{N}_n(X)$. Assume that $A = \{a_1,\ldots,a_n\}$. Let $J_1,\ldots,J_n$ be pairwise disjoint arcs of X such that $J_i \cap R(X) = \emptyset$ and $a_i \in \mathrm{int}_X(J_i)$, for each $i \in \{1,\ldots,n\}$. Thus, $A \in \langle \mathrm{int}_X(J_1),\ldots,\mathrm{int}_X(J_n)\rangle$. Since $\langle J_1,\ldots,J_n\rangle$ is homeomorphic to $J_1 \times \cdots \times J_n$, we have that $\langle \mathrm{int}_X(J_1),\ldots, \mathrm{int}_X(J_n)\rangle$ is an open subset of $\mathcal{E}_n(X)$. If $B \in \langle \mathrm{int}_X(J_1),\ldots,\mathrm{int}_X(J_n)\rangle - \{A\}$, then $|B| = n$. Let $j \in \{1,\ldots,m\}$. Since $|A \cap \mathrm{int}_X(E_j)| = i_j$, we may assume that $J_1,\ldots,J_{i_j} \subset \mathrm{int}_X(E_j)$. Hence, $B \cap \mathrm{int}_X(J_1) \subset \mathrm{int}_X(E_j),\ldots, B \cap \mathrm{int}_X(J_{i_j}) \subset \mathrm{int}_X(E_j)$. This implies that $|B \cap \mathrm{int}_X(E_j)| = i_j$. So, $B \in \mathcal{K}_X(i_1,\ldots,i_m)$. Thus, $\langle \mathrm{int}_X(J_1),\ldots,\mathrm{int}_X(J_n)\rangle \subset \mathcal{K}_X(i_1,\ldots,i_m)$. Therefore, $\mathcal{K}_X(i_1,\ldots,i_m)$ is an open subset of $\mathcal{E}_n(X)$. $\square$

Lemma 2.2. *Let $n \in \mathbb{N}$ with $n \geq 4$. If X is a finite graph, then the components of $\mathcal{E}_n(X)$ are the sets $\mathcal{K}_X(i_1,\ldots,i_m)$.*

PROOF. Notice that each $\mathcal{K}_X(i_1,\ldots,i_m)$ is a subset of $\mathcal{N}_n(X)$. Let $A \in \mathcal{N}_n(X)$. For each $k \in \{1,\ldots,m\}$, let $l_k = |A \cap \mathrm{int}_X(E_k)|$. Since $|A| = n$, then $l_1 + \cdots + l_m = n$. Thus, $A \in \mathcal{K}_X(l_1,\ldots,l_m)$. Therefore, the union of the sets $\mathcal{K}_X(i_1,\ldots,i_m)$ is $\mathcal{N}_n(X)$. Corollary 4.4 of [4] implies that $\mathcal{N}_n(X) = \mathcal{E}_n(X)$. Therefore Lemma 2.1 implies that the sets $\mathcal{K}_X(i_1,\ldots,i_m)$ are the components of $\mathcal{E}_n(X)$. $\square$

Lemma 2.3. *Let X be a finite graph and $n \in \mathbb{N}$ with $n \geq 4$.*

(a) *If $A \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X(i_1,\ldots,i_m))$, then $|A \cap \mathrm{int}_X(E_j)| \leq i_j$, for each $j \in \{1,\ldots,m\}$.*

(b) *The only component of $\mathcal{E}_n(X)$ contained in $\mathrm{cl}_{F_n(X)}(\mathcal{K}_X(i_1,\ldots,i_m))$ is $\mathcal{K}_X(i_1,\ldots,i_m)$.*

PROOF. (a) Let $A \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X(i_1,\ldots,i_m))$. Suppose that there exists $j \in \{1,\ldots,m\}$ such that $|A \cap \mathrm{int}_X(E_j)| > i_j$. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{K}_X(i_1,\ldots, i_m)$ which converges to A. Notice that $|A \cap E_j| \geq |A \cap \mathrm{int}_X(E_j)| > i_j$. On the other hand, the sequence $\{A_k \cap E_j\}_{k=1}^{\infty}$ converges to $A \cap E_j$ and $|A_k \cap E_j| = i_j$ which is a contradiction.

(b) By Lemma 2.2, $\mathcal{K}_X(i_1,\ldots,i_m)$ is a component of $\mathcal{E}_n(X)$ contained in $\mathrm{cl}_{F_n(X)}(\mathcal{K}_X(i_1,\ldots,i_m))$. Suppose that there exists $\mathcal{K}_X(l_1,\ldots,l_m)$ component of $\mathcal{E}_n(X)$ distinct of $\mathcal{K}_X(i_1,\ldots,i_m)$ contained in $\mathrm{cl}_{F_n(X)}(\mathcal{K}_X(i_1,\ldots,i_m))$. By Lemma 2.1(b), there exists $j_0 \in \{1,\ldots,m\}$ such that $i_{j_0} \neq l_{j_0}$.

Case 1. $i_{j_0} < l_{j_0}$.

Let $A \in \mathcal{K}_X(l_1,\ldots,l_m)$. Thus, $l_{j_0} = |A \cap \mathrm{int}_X(E_{j_0})|$. Since $A \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X(i_1, \ldots,i_m))$, by (a), $|A \cap \mathrm{int}_X(E_{j_0})| \leq i_{j_0}$. Thus, $l_{j_0} \leq i_{j_0}$ which is a contradiction.

Case 2. $i_{j_0} > l_{j_0}$.

This implies that there exists $j_1 \in \{1, \ldots, m\}$ such that $i_{j_1} < l_{j_1}$. In a similar way as the Case 1, we have a contradiction.

Thus (b) holds. □

Given $n \in \mathbb{N}$ with $n \geq 2$ and a continuum Z, let

$$\mathcal{SE}_n(Z) = \{A \in SF_n(Z) \colon A \text{ has a neighborhood in } SF_n(Z) \text{ which is an } n\text{-cell}\}.$$

Remark. Let $n \in \mathbb{N}$. If Z is a continuum, then $\mathcal{E}_n(Z)$ is an open subset of $F_n(Z)$ and $\mathcal{SE}_n(Z)$ is an open subset of $SF_n(Z)$.

Lemma 2.4. *Let $n \in \mathbb{N}$, with $n \geq 4$. If Y is an almost meshed locally connected continuum, then no neighborhood of F_Y in $SF_n(Y)$ can be embedded in $\mathbb{R}^n$.*

Proof. Let $\mathcal{U}$ be an open subset of $SF_n(Y)$ such that $F_Y \in \mathcal{U}$. We have that, $\mathcal{U} - \{F_Y\}$ is an open subset of $SF_n(Y)$, and since the quotient q_Y^* is continuous, $(q_Y^*)^{-1}(\mathcal{U} - \{F_Y\})$ is an open subset of $F_n(Y)$. Let $\mathcal{V} = (q_Y^*)^{-1}(\mathcal{U} - \{F_Y\})$. Let $\{a\} \in F_1(Y)$. Since q_Y is continuous, there exists $\delta > 0$ such that $q_Y(B_{F_n(Y)}(\{a\}, \delta)) \subset \mathcal{U}$. Since Y is connected, the cardinality of $B_Y(a, \delta)$ is not finite. Let $b \in B_Y(a, \delta) - \{a\}$. So, $\{a, b\} \in B_{F_n(Y)}(\{a\}, \delta)$. Therefore, $q_Y^*(\{a, b\}) \in \mathcal{U}$. Furthermore, $q_Y^*(\{a, b\}) \in \mathcal{U} - \{F_Y\}$. Hence, $\{a, b\} \in \mathcal{V}$.

Since Y is an almost meshed locally connected continuum, by [22, Theorem 3.1], $\mathcal{E}_n(Y)$ is a dense subset of $F_n(Y)$. Furthermore, $\{a, b\} \in F_{n-1}(Y)$ and $\mathcal{V}$ is a neighborhood of $\{a, b\}$ in $F_n(Y) - F_1(Y)$. Hence, by [22, Theorem 3.5], we have that $\mathcal{V}$ cannot be embedded in $\mathbb{R}^n$. Thus, $q_Y^*(\mathcal{V}) = \mathcal{U} - \{F_Y\}$ cannot be embedded in $\mathbb{R}^n$. Since $\mathcal{U} - \{F_Y\} \subset \mathcal{U}$, then $\mathcal{U}$ cannot be embedded in $\mathbb{R}^n$. □

Lemma 2.5. *Let $n \in \mathbb{N}$ with $n \geq 4$.*

(a) *If Y is an almost meshed locally connected continuum, then $q_Y^*(\mathcal{E}_n(Y)) = \mathcal{SE}_n(Y)$.*
(b) *If X and Y are almost meshed locally connected continua and $h \colon SF_n(X) \to SF_n(Y)$ is a homeomorphism, then $h(q_X^*(A)) \neq F_Y$, for each $A \in \mathcal{E}_n(X)$.*

Proof. (a) By [22, Theorem 3.8(b)], $\mathcal{E}_n(Y) \subset F_n(Y) - F_1(Y)$. Let $A \in \mathcal{E}_n(Y)$ and let $\mathcal{U}$ be a neighborhood of A in $F_n(Y)$ such that $\mathcal{U}$ is an n-cell. By [13, proposition 1], we may suppose that $\mathcal{U} \subset F_n(Y) - F_1(Y)$. Since q_Y^* is a homeomorphism, $q_Y^*(\mathcal{U})$ is a neighborhood of $q_Y^*(A)$ in $SF_n(Y)$ which is an n-cell. Thus, $q_Y^*(A) \in \mathcal{SE}_n(Y)$. Therefore, $q_Y^*(\mathcal{E}_n(Y)) \subset \mathcal{SE}_n(Y)$.

On the other hand, by Lemma 2.4, $F_Y \notin \mathcal{SE}_n(Y)$. Thus, $\mathcal{SE}_n(Y) \subset SF_n(Y) - \{F_Y\}$. Since q_Y^* is a homeomorphism, in a similar way as before, $(q_Y^*)^{-1}(\mathcal{SE}_n(Y)) \subset \mathcal{E}_n(Y)$.

(b) The proof follows from Lemma 2.4 and (a). □

Remark. Let $n \in \mathbb{N}$ with $n \geq 4$. If X is a finite graph, then the components of $\mathcal{SE}_n(X)$ are the sets $q_X^*(\mathcal{K}_X(i_1, \ldots, i_m))$.

Lemma 2.6. *Let $n \in \mathbb{N}$ with $n \geq 4$. If X and Y are finite graphs and $h : SF_n(X) \to SF_n(Y)$ is a homeomorphism, then $\mathcal{E}_n(X) = (q_X^*)^{-1}(h^{-1}(q_Y^*(\mathcal{E}_n(Y)))$.*

PROOF. By Lemma 2.5(a), we obtain that

$$q_Y^*(\mathcal{E}_n(Y)) = \mathcal{SE}_n(Y) = h(\mathcal{SE}_n(X)) = h(q_X^*(\mathcal{E}_n(X))).$$

□

Lemma 2.7. *Let $n \in \mathbb{N}$ and Z be a continuum. Then $\mathcal{E}_n(Z)$ is a dense subset of $F_n(Z)$ if and only if $\mathcal{SE}_n(Z)$ is a dense subset of $SF_n(Z)$.*

PROOF. Notice that $q_Z^*\colon F_n(Z) - F_1(Z) \to SF_n(Z) - \{F_Z\}$ is a homeomorphism and $q_Z^*(\mathcal{E}_n(Z) - F_1(Z)) = \mathcal{SE}_n(Z) - \{F_Z\}$. Thus, $\mathcal{E}_n(Z) - F_1(Z)$ is a dense subset of $F_n(Z) - F_1(Z)$ if and only if $\mathcal{SE}_n(Z) - \{F_Z\}$ is a dense subset of $SF_n(Z) - \{F_Z\}$. Then the lemma follows from the fact that $F_n(Z) - F_1(Z)$ is dense in $F_n(Z)$ and $SF_n(Z) - \{F_Z\}$ is dense in $SF_n(Z)$. □

Lemma 2.8. *Let $n \in \mathbb{N}$, with $n \geq 4$. Let Z, Y be continua such that $SF_n(Z)$ is homeomorphic to $SF_n(Y)$. Then Z is an almost meshed locally connected continuum if and only if Y is an almost meshed locally connected continuum.*

PROOF. Let $h\colon SF_n(Z) \to SF_n(Y)$ be a homeomorphism and assume that Z is an almost meshed locally connected continuum. By [3, Theorem 5.2], Y is a locally connected continuum. By [22, Theorem 3.1], we have that $\mathcal{E}_n(Z)$ is a dense subset of $F_n(Z)$ and hence, by Lemma 2.7, $\mathcal{SE}_n(Z)$ is a dense subset of $SF_n(Z)$. Thus, $h(\mathcal{SE}_n(Z))$ is a dense subset of $SF_n(Y)$. Since $h(\mathcal{SE}_n(Z)) = \mathcal{SE}_n(Y)$, by Lemma 2.7, we deduce that $\mathcal{E}_n(Y)$ is a dense subset of $F_n(Y)$. Therefore, by [22, Theorem 3.1], Y is an almost meshed locally connected continuum. □

3. Main results

Theorem 3.1. *Let $n \in \mathbb{N}$ with $n \geq 4$. If X and Y are continua such that $SF_n(X)$ is homeomorphic to $SF_n(Y)$, then X is a finite graph if and only if Y is a finite graph.*

PROOF. Suppose that X is a finite graph. Recall that $\mathcal{E}_n(X)$ is a dense subset of $F_n(X)$ with a finite number of components [4, Theorem 3.4]. By Lemma 2.7, we have that $\mathcal{SE}_n(X)$ is a dense subset of $SF_n(X)$. By Lemma 2.5(a), we have that

$\mathcal{SE}_n(X)$ has a finite number of components. Thus, $\mathcal{SE}_n(Y)$ is a dense subset of $SF_n(Y)$ with a finite number of components. On the other hand, by Lemma 2.8, Y is an almost meshed locally connected continuum. By Lemma 2.7 and Lemma 2.5(a), we deduce that $\mathcal{E}_n(Y)$ is a dense subset of $F_n(Y)$ and $\mathcal{E}_n(Y)$ has a finite number of components. By [4, Theorem 3.4], Y is a finite graph. $\square$

Let $m \in \mathbb{N}$ with $m \geq 3$. A θ_m-graph is the union of m arcs $E_1, \ldots, E_m$ sharing the same end points u and v and satisfying $E_i \cap E_j = \{u, v\}$, for $i \neq j$ (see [35, p. 221]).

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n \geq 2$. If X is a finite graph with $R(X) \neq \emptyset$, then* $\bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\}$

$$= \begin{cases} \{F_X\}, & \textit{if } X \textit{ is not a } \theta_m\textit{-graph,} \\ \{F_X, q_X(\{u,v\})\}, & \textit{if } X \textit{ is a } \theta_m\textit{-graph.} \end{cases}$$

Proof. Let $E_j \in \mathcal{A}_S(X)$ and $p \in \mathrm{int}_X(E_j)$. Since $\{p\}$ can be approximated by elements in $\mathcal{K}_X^j$, we have that $\{p\} \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X^j)$. Thus, $F_X \in \mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j))$. Hence, $F_X \in \bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\}$. Furthermore, if X is a θ_m-graph, then $u, v \in E_j$, for each $j \in \{1, \ldots, m\}$. Since $n \geq 2$, we may approximate $\{u, v\}$ by elements of $\mathcal{K}_X^j$. This implies that $\{u, v\} \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X^j)$. Thus, $q_X(\{u, v\}) \in \mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j))$. Therefore, $q_X(\{u, v\}) \in \bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\}$.

Suppose that there exists $\chi \in \bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\} - \{F_X\}$. Let $A \in F_n(X) - F_1(X)$ be such that $q_X(A) = \chi$. Since X is a finite graph and $R(X) \neq \emptyset$, we know that $|\mathcal{A}_S(X)| \geq 2$ and $|\bigcap \mathcal{A}_S(X)| \leq 2$. Given $E_j \in \mathcal{A}_S(X)$, there exists a sequence $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{K}_X^j$ such that $\{q_X(A_i)\}_{i=1}^{\infty}$ converges to χ. In fact, $\{A_i\}_{i=1}^{\infty}$ converges to A. Thus, $A \subset E_j$. Therefore,

$$A \subset \bigcap \mathcal{A}_S(X). \tag{3.1}$$

Since $|A| \geq 2$, by (3.1), we have that $|\bigcap \mathcal{A}_S(X)| = 2$, so $|A| = 2$.

If X is not a θ_m-graph, we have that $|\bigcap \mathcal{A}_S(X)| < 2$. By (3.1), we deduce that $|A| < 2$, which is a contradiction. In conclusion, χ does not exist. Thus, $\bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\} = \{F_X\}$.

If X is a θ_m-graph, then $\bigcap \mathcal{A}_S(X) = \{u, v\}$. Since $A \subset \bigcap \mathcal{A}_S(X)$ and since $A \notin F_1(X)$, then $A = \{u, v\}$. In fact, $q_X(\{u, v\}) = \chi$. Therefore, $\chi \in \{F_X, q_X(\{u, v\})\}$. Thus,

$$\bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\} \subset \{F_X, q_X(\{u, v\})\}.$$

□

If $\mathcal{M}$ is a manifold, its *manifold boundary* is denoted by $\partial\mathcal{M}$.

Theorem 3.3. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let Z be a continuum.*

(a) *If $SF_n(Z)$ is homeomorphic to $SF_n([0,1])$, then Z is an arc.*
(b) *If $SF_n(Z)$ is homeomorphic to $SF_n(S^1)$, then Z is a simple closed curve.*

Proof. Let X be equal to $[0,1]$ or S^1 and let $h : SF_n(X) \to SF_n(Z)$ be a homeomorphism. By [3, Theorem 5.2], we have that Z is a locally connected continuum.

Case 1. $n \in \{2,3\}$.

Allow us to prove that $R(Z) = \emptyset$. In order to do this, let $a_1 \in R(Z)$. Let $a_2 \in Z - \{a_1\}$. If $q_Z(\{a_1,a_2\}) = h(F_X)$, then $q_Z(\{a_1,a\}) \neq h(F_X)$, for each $a \in Z - \{a_1,a_2\}$. Therefore, we may suppose that $q_Z(\{a_1,a_2\}) \neq h(F_X)$. Let $A = \{a_1,a_2\}$. Notice that $h^{-1}(q_Z(A)) \in SF_n(X) - \{F_X, h^{-1}(F_Z)\}$. Thus, $(q_X^*)^{-1}(h^{-1}(q_Z(A))) \in F_n(X) - F_1(X)$. Since $R(X) = \emptyset$, according to [4, Lemma 5.1], we have that $(q_X^*)^{-1}(h^{-1}(q_Z(A))) \in \mathcal{E}_n(X)$. Hence, there exists a neighborhood $\mathcal{U}$ of $(q_X^*)^{-1}(h^{-1}(q_Z(A)))$ in $F_n(X) - F_1(X)$ which is an n-cell. Thus, $q_X(\mathcal{U})$ is an n-cell that is a neighborhood of $h^{-1}(q_Z(A))$ in $SF_n(X) - \{F_X\}$. Furthermore, there exists a neighborhood $\mathcal{U}'$ of $h^{-1}(q_Z(A))$ which is an n-cell such that $\mathcal{U}' \subset q_X(\mathcal{U}) \cap (SF_n(X) - \{F_X, h^{-1}(F_Z)\})$. So, $h(\mathcal{U}')$ is an n-cell that is a neighborhood of $q_Z(A)$ in $SF_n(Z) - \{h(F_X), F_Z\}$. Hence, $(q_Z^*)^{-1}(h(\mathcal{U}'))$ is an n-cell that is a neighborhood of A in $F_n(Z) - F_1(Z)$. Then $A \in \mathcal{E}_n(Z)$. According to [4, Lemma 3.1], $A \notin \mathcal{E}_n(Z)$. Thus, we have a contradiction.

Therefore, Z is either an arc or a simple closed curve.

(Case $n = 2$) Since $SF_2(S^1)$ is homeomorphic to the real projective plane $\mathbb{RP}^2$ and $SF_2([0,1])$ is homeomorphic to $[0,1]^2$ ([3, Examples 3.1 and 3.3]), we deduce that Z is homeomorphic to X.

(Case $n = 3$) Since $SF_2(S^1)$ can be embedded in $SF_3(S^1)$, then $SF_3(S^1)$ cannot be embedded in $\mathbb{R}^3$. On the other hand, a model for $F_3([0,1])$ is the unit sphere (see [27, Section 3]), where $F_1([0,1])$ is a diameter. Hence, $SF_3([0,1])$ can be embedded in $\mathbb{R}^3$. Therefore, Z is homeomorphic to X.

Case 2. $n \geq 4$.

By Theorem 3.1, Z is a finite graph.

Since $|\mathcal{A}_S(X)| = 1$, by Lemma 2.2, we have that $\mathcal{E}_n(X)$ has only one component. By Lemma 2.6, $\mathcal{E}_n(Z)$ is connected. Hence, $|\mathcal{A}_S(Z)| = 1$. Therefore, Z is an arc or a simple closed curve.

Claim 1. If $B \in \mathcal{E}_n(S^1)$ and $\mathcal{M}$ is a neighborhood of B in $F_n(S^1)$ which is an n-cell, then B is in the manifold interior of $\mathcal{M}$.

Proof of Claim 1. Since $R(S^1) = \emptyset$, by [4, Corollary 4.4], we have that $B \in F_n(S^1) - F_{n-1}(S^1)$. By [22, Theorem 2.5], there exist pairwise disjoint open and connected subsets $V_1, \ldots, V_n$ of S^1 such that $B \in \langle V_1, \ldots, V_n\rangle \subset \mathrm{int}_{F_n(S^1)}(\mathcal{M})$. Notice that V_i is homeomorphic to $(0,1)$, for each $i \in \{1, \ldots, n\}$. So, $\langle V_1, \ldots, V_n\rangle$ is homeomorphic to $(0,1)^n$. Hence, B is in the manifold interior of $\mathcal{M}$.

Claim 2. If $A \in \mathcal{E}_n([0,1])$ and $A \cap \{0,1\} \neq \emptyset$, then there is a neighborhood $\mathcal{M}$ of A in $F_n([0,1])$ which is an n-cell such that $A \in \partial\mathcal{M}$.

Proof of Claim 2. We may assume that $0 \in A$. Since $A \in \mathcal{E}_n([0,1])$, $A \in F_n([0,1]) - F_{n-1}([0,1])$. Thus, $A = \{0, a_2, \ldots, a_n\}$, for some $a_2, \ldots, a_n$ in $[0,1]$. Let $\alpha_1, \ldots, \alpha_n$ in $[0,1]$ be pairwise disjoint arcs such that $0 \in \mathrm{int}_{[0,1]}(\alpha_1), a_2 \in \mathrm{int}_{[0,1]}(\alpha_2), \ldots, a_n \in \mathrm{int}_{[0,1]}(\alpha_n)$. Let $\mathcal{M} = \langle \alpha_1, \ldots, \alpha_n\rangle$. Hence, $A \in \mathrm{int}_{F_n([0,1])}(\mathcal{M})$. Notice that $\mathcal{M}$ is an n-cell which is a neighborhood of A in $F_n([0,1])$. Since $0 \in \partial\alpha_1$, we have that $A \in \partial\mathcal{M}$.

From Claim 1 and Claim 2, it follows that $\mathcal{E}_n(S^1)$ is not homeomorphic to $\mathcal{E}_n([0,1])$. By Lemma 2.6, we have that $\mathcal{E}_n(X)$ is homeomorphic to $\mathcal{E}_n(Z)$. Since Z is an arc or a simple closed curve, we conclude that X is homeomorphic to Z. $\square$

From now on when Y is a finite graph, we mean that Y has $E'_1, \ldots, E'_{m'}$ edges, with $m' \in \mathbb{N}$.

Theorem 3.4. *Let X, Y be finite graphs, let $n \in \mathbb{N}$ with $n \geq 4$ and let $h\colon SF_n(X) \to SF_n(Y)$ be a homeomorphism.*

(a) *For each $E_j \in \mathcal{A}_S(X)$, we have that $h(q_X(\mathcal{K}_X^j)) = q_Y(\mathcal{K}_Y^{j_h})$, for some $E'_{j_h} \in \mathcal{A}_S(Y)$.*

(b) *The association $E_j \mapsto E'_{j_h}$ is a bijection between $\mathcal{A}_S(X)$ and $\mathcal{A}_S(Y)$.*

Proof. (a) By Lemma 2.2 and Lemma 2.6, we obtain that $h(q_X(\mathcal{K}_X^j)) = q_Y(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$, for some component $\mathcal{K}_Y(i_1, \ldots, i_{m'})$ of $\mathcal{E}_n(Y)$. Let $M' = \{j \in \{1, \ldots, m'\}\colon i_j \neq 0\}$ and $r' = |M'|$. For convenience, let $M' = \{j_1, \ldots, j_{r'}\}$. Notice that $\mathcal{K}_Y(i_1, \ldots, i_{m'}) \subset \langle \mathrm{int}_Y(E'_{j_1}), \ldots, \mathrm{int}_Y(E'_{j_{r'}})\rangle$.

We will prove that $r' = 1$. In order to do this, assume that $r' \geq 2$.

Claim. $\mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))) = \mathcal{K}_Y(i_1, \ldots, i_{m'})$.

Proof of claim. Notice that

$$\mathcal{K}_Y(i_1, \ldots, i_{m'}) \subset \mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))).$$

We will show that $\mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))) \subset \mathcal{K}_Y(i_1, \ldots, i_{m'})$.

Let $A \in \operatorname{int}_{F_n(Y)}(\operatorname{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'})))$. We claim that (i) $A \cap R(Y) = \emptyset$ and (ii) $A \in F_n(Y) - F_{n-1}(Y)$.

In order to prove (i), suppose that there exists $p \in A \cap R(Y)$. Let $r > 0$. Since $A \in \operatorname{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$, there exists $B \in \mathcal{K}_Y(i_1, \ldots, i_{m'})$ such that $H(A, B) < \frac{r}{4}$. Thus, there exists $b \in B$ such that $d_Y(p, b) < \frac{r}{4}$. Since $B \in \langle \operatorname{int}_Y(E'_{j_1}), \ldots, \operatorname{int}_Y(E'_{j_{r'}}) \rangle$, there exists $E'_s \in \{E'_{j_1}, \ldots, E'_{j_{r'}}\}$ such that $b \in \operatorname{int}_Y(E'_s)$. Since $p \in R(Y)$, there exists $E'_t \in \mathcal{A}_S(Y) - \{E'_s\}$ such that $p \in E'_t$. Let $c \in \operatorname{int}_Y(E'_t) - B$ be such that $d_Y(c, p) < \frac{r}{4}$ and $C = (B - \{b\}) \cup \{c\}$. Since $d_Y(c, b) < \frac{r}{2}$, we have that $H(C, B) < \frac{r}{2}$. Hence, $H(A, C) < r$. Moreover, $|C \cap \operatorname{int}_Y(E'_t)| = 1 + |B \cap \operatorname{int}_Y(E'_t)|$. Since $B \in \mathcal{K}_Y(i_1, \ldots, i_{m'})$, we have that $|C \cap \operatorname{int}_Y(E'_t)| = 1 + i_t$. By Lemma 2.3(a), we have that $C \notin \operatorname{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$.

Hence, $A \notin \operatorname{int}_{F_n(Y)}(\operatorname{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'})))$, which is a contradiction.

Therefore, $A \cap R(Y) = \emptyset$.

In order to prove (ii), suppose that $A \in F_{n-1}(Y)$. Since $A \cap R(Y) = \emptyset$, we have that $A \in \langle \operatorname{int}_Y(E'_{j_1}), \ldots, \operatorname{int}_Y(E'_{j_{r'}}) \rangle \cap \operatorname{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$. By Lemma 2.3(a), we have that $|A \cap \operatorname{int}_Y(E'_j)| \leq i_j$, for each $j \in M'$. Since $A \in F_{n-1}(Y)$, there exists $j_k \in M'$ such that $|A \cap \operatorname{int}_Y(E'_{j_k})| < i_{j_k}$. Since $r' \geq 2$, we may take $j_s \in M' - \{j_k\}$. Let $l = n - |A|$. Let $r > 0$. Choose pairwise distinct points $b_1, \ldots, b_l \in \operatorname{int}_Y(E'_{j_s}) - A$ such that $H(A, B) < r$, where $B = A \cup \{b_1, \ldots, b_l\}$. Notice that $|B \cap \operatorname{int}_Y(E'_j)| = |A \cap \operatorname{int}_Y(E'_j)|$, for each $j \in M' - \{j_s\}$.

Suppose that $|B \cap \operatorname{int}_Y(E'_{j_s})| \leq i_{j_s}$. Notice that

$$\begin{aligned}
|B| &= \sum_{j \in M'} |B \cap \operatorname{int}_Y(E'_j)| \\
&= \sum_{j \in M' - \{j_k, j_s\}} |B \cap \operatorname{int}_Y(E'_j)| + |B \cap \operatorname{int}_Y(E'_{j_k})| + |B \cap \operatorname{int}_Y(E'_{j_s})| \\
&= \sum_{j \in M' - \{j_k, j_s\}} |A \cap \operatorname{int}_Y(E'_j)| + |A \cap \operatorname{int}_Y(E'_{j_k})| + |B \cap \operatorname{int}_Y(E'_{j_s})|.
\end{aligned}$$

By Lemma 2.3(a) and since $|A \cap \operatorname{int}_Y(E'_{j_k})| < i_{j_k}$, it follows that

$$\begin{aligned}
|B| &\leq \sum_{j \in M' - \{j_k, j_s\}} i_j + |A \cap \operatorname{int}_Y(E'_{j_k})| + i_{j_s} \\
&< \sum_{j \in M' - \{j_k, j_s\}} i_j + i_{j_k} + i_{j_s}.
\end{aligned}$$

Since $\sum_{j \in M'} i_j = n$, we have that $|B| < n$ which is a contradiction. Thus, $|B \cap \operatorname{int}_Y(E'_{j_s})| > i_{j_s}$. By Lemma 2.3(a), $B \notin \operatorname{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$. Hence, given

$r > 0$, $B(A, r) \not\subset \mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$. Therefore, $A \notin \mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'})))$, which is a contradiction.

Therefore, $A \in F_n(Y) - F_{n-1}(Y)$.

According to [4, Corollary 4.4], $A \in \mathcal{E}_n(Y)$. Let $\mathcal{C}$ be the component of $\mathcal{E}_n(Y)$ that contains A. By Remark 2, we obtain that $\mathcal{C}$ is an open subset of $F_n(Y)$. Since $A \in \mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))$, then $\mathcal{C} \cap \mathcal{K}_Y(i_1, \ldots, i_{m'}) \neq \emptyset$. By Lemma 2.2, we have that $\mathcal{C} = \mathcal{K}_Y(i_1, \ldots, i_{m'})$. Hence, $A \in \mathcal{K}_Y(i_1, \ldots, i_{m'})$. Therefore, $\mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))) \subset \mathcal{K}_Y(i_1, \ldots, i_{m'})$. This concludes the proof of the claim.

Let $A \in F_n(X)$ be such that $h(q_X(A)) = F_Y$ and let $A' \in F_n(Y)$ be such that $h^{-1}(q_Y(A')) = F_X$. Let $\mathcal{F} = F_n(X) - (F_1(X) \cup \{A\})$ and let $\mathcal{F}' = F_n(Y) - (F_1(Y) \cup \{A'\})$. We consider the homeomorphism $g = (q_Y|_{\mathcal{F}'})^{-1} \circ h \circ (q_X|_{\mathcal{F}})$ from $\mathcal{F}$ onto $\mathcal{F}'$.

By Lemma 2.5(b), $\mathcal{K}_X^j \subset \mathcal{F}$, and thus $\mathcal{K}_Y(i_1, \ldots, i_{m'}) \subset \mathcal{F}'$. Since $\mathrm{cl}_{\mathcal{F}}(\mathcal{K}_X^j) = \langle E_j \rangle \cap \mathcal{F}$, we have that $\mathrm{int}_{\mathcal{F}}(\mathrm{cl}_{\mathcal{F}}(\mathcal{K}_X^j)) = \langle \mathrm{int}_X(E_j) \rangle \cap \mathcal{F}$. By the claim and since $\mathcal{F}'$ is an open subset of $F_n(Y)$, we obtain that $\mathrm{int}_{\mathcal{F}'}(\mathrm{cl}_{\mathcal{F}'}(\mathcal{K}_Y(i_1, \ldots, i_{m'})))$

$$\begin{aligned}
&= \mathrm{int}_{\mathcal{F}'}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'})) \cap \mathcal{F}') \\
&= \mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'})) \cap \mathcal{F}') \cap \mathcal{F}' \\
&= \mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))) \cap \mathrm{int}_{F_n(Y)}(\mathcal{F}') \cap \mathcal{F}' \\
&= \mathrm{int}_{F_n(Y)}(\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y(i_1, \ldots, i_{m'}))) \cap \mathcal{F}' \\
&= \mathcal{K}_Y(i_1, \ldots, i_{m'}) \cap \mathcal{F}' \\
&= \mathcal{K}_Y(i_1, \ldots, i_{m'}).
\end{aligned}$$

Notice that $g(\mathrm{int}_{\mathcal{F}}(\mathrm{cl}_{\mathcal{F}}(\mathcal{K}_X^j)) = \mathrm{int}_{\mathcal{F}'}(\mathrm{cl}_{\mathcal{F}'}(\mathcal{K}_Y(i_1, \ldots, i_{m'})))$. Thus,

$$g(\langle \mathrm{int}_X(E_j) \rangle \cap \mathcal{F}) = \mathcal{K}_Y(i_1, \ldots, i_{m'}).$$

Since $\mathrm{int}_{\mathcal{F}}(\mathrm{cl}_{\mathcal{F}}(\mathcal{K}_X^j)) \neq \mathcal{K}_X^j$ and $g(\mathcal{K}_X^j) = \mathcal{K}_Y(i_1, \ldots, i_{m'})$, we have a contradiction. Therefore, $r' = 1$.

(b) Since $\mathcal{K}_X^j$ is associated to $\mathcal{K}_Y^{j_h}$, then E_j is associated to E'_{j_h}. Assume that E_l is associated to E'_{l_h} and $E'_{j_h} = E'_{l_h}$. So, $\mathcal{K}_Y^{j_h} = \mathcal{K}_Y^{l_h}$. Thus, $q_Y(\mathcal{K}_Y^{j_h}) = q_Y(\mathcal{K}_Y^{l_h})$. By (a), $h(q_X(\mathcal{K}_X^j)) = h(q_X(\mathcal{K}_X^l))$, this implies that $q_X(\mathcal{K}_X^j) = q_X(\mathcal{K}_X^l)$. Consequently, $\mathcal{K}_X^j = \mathcal{K}_X^l$. Hence, $E_j = E_l$. □

The following result shows that the θ_m-graphs have unique n-fold symmetric product suspension.

Theorem 3.5. *Let $m, n \in \mathbb{N}$ with $n \geq 4$. If X is a θ_m-graph, then X has unique n-fold symmetric product suspension.*

PROOF. Let Y be a continuum and let $h\colon SF_n(X) \to SF_n(Y)$ be a homeomorphism. By Theorem 3.1, Y is a finite graph. By Theorem 3.3, $R(Y) \neq \emptyset$. From Theorem 3.2 and Theorem 3.4, we obtain that

$$\begin{aligned} 2 &= |\{F_X, q_X(\{u,v\})\}| \\ &= |h(\{F_X, q_X(\{u,v\})\})| \\ &= |h(\bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\})| \\ &= |\bigcap\{\mathrm{cl}_{SF_n(Y)}(h(q_X(\mathcal{K}_X^j))) : E_j \in \mathcal{A}_S(X)\}| \\ &= |\bigcap\{\mathrm{cl}_{SF_n(Y)}(q_Y(\mathcal{K}_Y^{j_h})) : E'_{j_h} \in \mathcal{A}_S(Y)\}|. \end{aligned}$$

By Theorem 3.2, Y is a $\theta_{m'}$-graph. By Theorem 3.4(b), X is homeomorphic to Y. □

Theorem 3.6. *Let $n \in \mathbb{N}$ with $n \geq 4$. Let X, Y be finite graphs with $R(X) \neq \emptyset$ and $R(Y) \neq \emptyset$ such that X is not a θ_m-graph, for any $m \in \mathbb{N}$, and let $h\colon SF_n(X) \to SF_n(Y)$ be a homeomorphism. Then $h(F_X) = F_Y$.*
If we also suppose that

(a) *$E_j \in \mathcal{A}_R(X)$ if and only if $E'_{j_h} \in \mathcal{A}_R(Y)$ and*
(b) *$E_j \in \mathcal{A}_E(X)$ if and only if $E'_{j_h} \in \mathcal{A}_E(Y)$,*

then X is homeomorphic to Y.

PROOF. By Theorem 3.2,

$$\bigcap\{\mathrm{cl}_{SF_n(X)}(q_X(\mathcal{K}_X^j)) : E_j \in \mathcal{A}_S(X)\} = \{F_X\}. \tag{3.2}$$

By Theorem 3.5, Y is not a $\theta_{m'}$-graph, for any $m' \in \mathbb{N}$. So, by Theorem 3.2,

$$\bigcap\{\mathrm{cl}_{SF_n(Y)}(q_Y(\mathcal{K}_Y^{j_h})) : E'_{j_h} \in \mathcal{A}_S(Y)\} = \{F_Y\}. \tag{3.3}$$

Now, by Theorem 3.4(a), (3.2) and (3.3), we have that

$$\begin{aligned} h(\{F_X\}) &= \bigcap\{\mathrm{cl}_{SF_n(Y)}(h(q_X(\mathcal{K}_X^j))) : E_j \in \mathcal{A}_S(X)\} \\ &= \bigcap\{\mathrm{cl}_{SF_n(Y)}(q_Y(\mathcal{K}_Y^{j_h})) : E'_{j_h} \in \mathcal{A}_S(Y)\} = \{F_Y\}. \end{aligned}$$

For each $E_j \in \mathcal{A}_S(X)$, let

$$\mathcal{K}_n(E_j, X) = \mathrm{cl}_{F_n(X)}(\mathcal{K}_X^j) - F_1(X), \text{ and } \mathcal{K}_n(E'_{j_h}, Y) = \mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y^{j_h}) - F_1(Y).$$

We consider the homeomorphism $g = (q_Y^*)^{-1} \circ h \circ q_X^*$.

Claim 1.

(I) $g(\mathcal{K}_n(E_j, X)) = \mathcal{K}_n(E'_{j_h}, Y)$.
(II) $|E_j \cap R(X)| = |E'_{j_h} \cap R(Y)|$.
(III) If $A \in \mathcal{K}_n(E_j, X)$, then $|A \cap R(X)| = |g(A) \cap R(Y)|$.

(IV) If $A, B \in \mathcal{K}_n(E_j, X)$ satisfy $A \cap R(X) = \{p\}$ and $B \cap R(X) = \{p\}$, then $g(A) \cap R(Y) = g(B) \cap R(Y)$.

Proof of Claim 1. (I) It follows from
$\mathrm{cl}_{F_n(Y)}(\mathcal{K}_Y^{j_h}) - F_1(Y) = \mathrm{cl}_{F_n(Y)-F_1(Y)}(\mathcal{K}_Y^{j_h}) = g(\mathrm{cl}_{F_n(X)-F_1(X)}(\mathcal{K}_X^{j})) = g(\mathrm{cl}_{F_n(X)}$ $(\mathcal{K}_X^{j}) - F_1(X)) = g(\mathcal{K}_n(E_j, X))$.

(II) It follows from (a) and (b).

(III) Let $A \in \mathcal{K}_n(E_j, X)$.

Notice that $|A \cap R(X)| = 0$ if and only if $A \in \mathrm{int}_{F_n(X)}(\langle E_j \rangle)$ if and only if $g(A) \in \mathrm{int}_{F_n(Y)}(\langle E'_{j_h} \rangle)$ if and only if $|g(A) \cap R(Y)| = 0$.

Case 1. $|E_j \cap R(X)| = 1$.

$|A \cap R(X)| = 1$ if and only if $A \in \mathrm{bd}_{F_n(X)}(\langle E_j \rangle) - F_1(X)$ if and only if $g(A) \in \mathrm{bd}_{F_n(Y)}(\langle E'_{j_h} \rangle) - F_1(Y)$ if and only if $|g(A) \cap R(Y)| = 1$.

Case 2. $|E_j \cap R(X)| = 2$.

Assume that $E_j \cap R(X) = \{p, q\}$ and $E'_{j_h} \cap R(Y) = \{p', q'\}$, where $p \neq q$ and $p' \neq q'$. Let

$$\mathcal{A}_1 = \{B \in \mathrm{bd}_{F_n(X)}(\langle E_j \rangle) - F_1(X) : B \cap R(X) = \{p\}\},$$

$$\mathcal{A}_2 = \{B \in \mathrm{bd}_{F_n(X)}(\langle E_j \rangle) - F_1(X) : B \cap R(X) = \{q\}\},$$

$$\mathcal{A}_3 = \{B \in \mathrm{bd}_{F_n(X)}(\langle E_j \rangle) - F_1(X) : B \cap R(X) = \{p, q\}\}.$$

Notice that $\mathrm{bd}_{F_n(X)}(\langle E_j \rangle) - F_1(X) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. In a similar way, we define $\mathcal{A}'_1, \mathcal{A}'_2, \mathcal{A}'_3$, and we have that $\mathrm{bd}_{F_n(Y)}(\langle E'_{j_h} \rangle) - F_1(Y) = \mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \mathcal{A}'_3$. Notice that:

(i) $\mathcal{A}_1$, $\mathcal{A}_2$ are open subsets of $\mathrm{bd}_{F_n(X)}(\langle E_j \rangle) - F_1(X)$ and $\mathcal{A}'_1, \mathcal{A}'_2$ are open subsets of $\mathrm{bd}_{F_n(Y)}(\langle E'_{j_h} \rangle) - F_1(Y)$.

(ii) $\mathcal{A}_3$ has empty interior in $\mathrm{bd}_{F_n(X)}(\langle E_j \rangle\ ) - F_1(X)$ and $\mathcal{A}'_3$ has empty interior in $\mathrm{bd}_{F_n(Y)}(\langle E'_{j_h} \rangle) - F_1(Y)$.

We know that $g(\mathcal{A}_1 \cup \mathcal{A}_2) \subset \mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \mathcal{A}'_3$. By (i) and (ii), we have that $g(\mathcal{A}_1 \cup \mathcal{A}_2) \subset \mathcal{A}'_1 \cup \mathcal{A}'_2$. Similarly, we obtain that $g^{-1}(\mathcal{A}'_1 \cup \mathcal{A}'_2) \subset \mathcal{A}_1 \cup \mathcal{A}_2$. Thus, $g(\mathcal{A}_1 \cup \mathcal{A}_2) = \mathcal{A}'_1 \cup \mathcal{A}'_2$. Therefore, $g(\mathcal{A}_3) = \mathcal{A}'_3$. This completes the proof of Claim 1(III).

(IV) Suppose that $A = \{p, a_1, \ldots, a_s\}$ and $B = \{p, b_1, \ldots, b_l\}$ with $l \leq s$.

Let $\alpha_i \colon [0, 1] \to E_j$ be a map such that $\alpha_i(0) = a_i$ and $\alpha_i(1) = b_i$, for each $i \in \{1, \ldots, l\}$. Given $i \in \{1, \ldots, l\}$, since $a_i, b_i \in \mathrm{int}_X(E_j)$, we may suppose that $\alpha_i(t) \notin R(X)$, for each $t \in [0, 1]$.

If $l < s$, for each $k \in \{l+1, \ldots, s\}$, let $\alpha_k \colon [0,1] \to E_j$ be a map such that $\alpha_k(0) = a_k$ and $\alpha_k(1) = p$. Given $k \in \{l+1, \ldots, s\}$, since $a_k \in \mathrm{int}_X(E_j)$, we may suppose that $\alpha_k(t) \notin R(X)$, for each $t \in [0,1)$.

The function $\alpha \colon [0,1] \to \mathcal{K}_n(E_j, X)$ defined by $\alpha(t) = \{p\} \cup \bigcup_{i=1}^{s} \{\alpha_i(t)\}$, for each $t \in [0,1]$, is continuous, $\alpha(0) = A$ and $\alpha(1) = B$. Notice that $p \in \alpha(t) \cap R(X)$, for each $t \in [0,1]$. Given $t \in (0,1)$, let $w \in \alpha(t) \cap R(X)$. Thus, $w = p$ or $w \in \bigcup_{i=1}^{s} \{\alpha_i(t)\}$. If $w \in \bigcup_{i=1}^{s} \{\alpha_i(t)\}$, then there exists $s_0 \in \{1, \ldots, s\}$ such that $\alpha_{s_0}(t) = w$. Thus, $\alpha_{s_0}(t) \in R(X)$ which is a contradiction. Thus, $w = p$. Therefore, $\alpha(t) \cap R(X) = \{p\}$, for each $t \in [0,1]$. By Claim 1(III), we have $|g(A) \cap R(Y)| = 1$. Let $\{p'\} = g(A) \cap R(Y)$.

Let $T = \{t \in [0,1] : p' \in g(\alpha(t))\}$. Notice that T is a closed subset of $[0,1]$ and $0 \in T$. Suppose that $T \neq [0,1]$. Let $t_0 = \inf([0,1] - T)$ and let $\{t_i\}_{i=1}^{\infty}$ be a sequence of elements in $[0,1] - T$ which converges to t_0. Let $w \in E'_{j_h} \cap R(Y) - \{p'\}$. Then $w \in g(\alpha(t_i))$, for each $i \in \mathbb{N}$. Since $\{g(\alpha(t_i))\}_{i=1}^{\infty}$ converges to $g(\alpha(t_0))$, we have $w \in g(\alpha(t_0))$. Since $t_0 \in T$, then $w, p' \in g(\alpha(t_0))$. By Claim 1(III), $|g(\alpha(t_0)) \cap R(Y)| = 1$, so we have a contradiction. Thus, $T = [0,1]$. Hence, $g(B) \cap R(Y) = \{p'\}$. Therefore, $g(A) \cap R(Y) = g(B) \cap R(Y)$. This completes the proof of Claim 1.

We are going to define a bijection between $R(X)$ and $R(Y)$.

Let $p \in R(X)$. Suppose that $p \in E_1$. Fix $A \in \mathcal{K}_n(E_1, X)$ such that $A \cap R(X) = \{p\}$, by Claim 1(I), we have that $g(A) \in \mathcal{K}_n(E'_{1_h}, Y)$. By Claim 1(III), $|g(A) \cap R(Y)| = 1$. Let $p' \in Y$ be such that $g(A) \cap R(Y) = \{p'\}$. Notice that $p' \in E'_{1_h}$.

We claim that p' does not depend on A and, in fact, it does not depend on the choice of E_1. That is, if $t \in \{2, \ldots, m\}$ is such that $p \in E_t$ and $B \in \mathcal{K}_n(E_t, X)$ is such that $B \cap R(X) = \{p\}$, then $g(B) \cap R(Y) = \{p'\}$. In order to prove this, we take E_1, E_t, A and B as described.

Using Claim 1 (I) and (III), we have that $g(B) \in \mathcal{K}_n(E'_{t_h}, Y)$ and $|g(B) \cap R(Y)| = 1$. Let $q' \in Y$ be such that $g(B) \cap R(Y) = \{q'\}$. Notice that $q' \in E'_{t_h}$.

We are going to prove that $q' = p'$.

Fix $t \in \{2, \ldots, n\}$ such that $i_1 + i_t = n$. Since $\mathcal{K}_X(1,t)$ is a component of $\mathcal{E}_n(X)$, $g(\mathcal{K}_X(1,t))$ is a component of $\mathcal{E}_n(Y)$. By [4, Lemma 4.1], there exists a component $\langle \mathrm{int}_Y(E'_{l_1}), \ldots, \mathrm{int}_Y(E'_{l_{r'}}) \rangle$ of $F_n(Y) - R_n(Y)$ such that

$$g(\mathcal{K}_X(1,t)) \subset \langle \mathrm{int}_Y(E'_{l_1}), \ldots, \mathrm{int}_Y(E'_{l_{r'}}) \rangle. \tag{3.4}$$

Claim 2. There are $w, z \in \{l_1, \ldots, l_{r'}\}$ such that $E'_w = E'_{1_h}$ and $E'_z = E'_{t_h}$.

Proof of Claim 2. Let $a \in \mathrm{int}_X(E_1)$ and $C = \{p, a\}$. Thus, $C \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X(1, t)) \cap \mathcal{K}_n(E_1, X)$. By (3.4) and Claim 1(I), we have $g(C) \in \mathrm{cl}_{F_n(Y)}(\langle \mathrm{int}_Y(E'_{l_1}), \ldots, \mathrm{int}_Y(E'_{l_{r'}})\rangle) \cap \mathcal{K}_n(E'_{1_h}, Y)$. By Claim 1(III), $|g(C) \cap R(Y)| = 1$. Since $g(C) \notin F_1(Y)$, we have that $g(C) \cap \mathrm{int}_Y(E'_{1_h}) \neq \emptyset$. If $E'_{1_h} \neq E'_j$ for each $j \in \{l_1, \ldots, l_{r'}\}$, then $g(C) \notin \mathrm{cl}_{F_n(Y)}(\langle \mathrm{int}_Y(E'_{l_1}), \ldots, \mathrm{int}_Y(E'_{l_{r'}})\rangle)$, which is a contradiction. Thus, there exists $w \in \{l_1, \ldots, l_{r'}\}$ such that $E'_w = E'_{1_h}$. In a similar way, there exists $z \in \{l_1, \ldots, l_{r'}\}$ such that $E'_z = E'_{t_h}$. This completes the proof of Claim 2.

Now, from equation (3.4) and Claim 2, the following condition is satisfied

$$g(\mathcal{K}_X(1, t)) \subset \langle \mathrm{int}_Y(E'_{1_h}), \mathrm{int}_Y(E'_{t_h}), \ldots, \mathrm{int}_Y(E'_{l_{r'}})\rangle. \tag{3.5}$$

Claim 3. $p', q' \in E^{'}_{1_h} \cap E^{'}_{t_h}$.

Proof of Claim 3. Fix $C \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X(1, t)) \cap \mathcal{K}_n(E_1, X)$ such that $C \cap R(X) = \{p\}$. Since $g(A) \cap R(Y) = \{p'\}$, by Claim 1(IV), $g(C) \cap R(Y) = \{p'\}$. By (3.5) and Claim 1(I), $g(C) \in \langle E'_{1_h}, E'_{t_h}, \ldots, E'_{l_{r'}}\rangle \cap \mathcal{K}_n(E'_{1_h}, Y)$. Since $g(C) \in \mathcal{K}_n(E'_{1_h}, Y)$ and $g(C) \cap R(Y) = \{p'\}$, we have that $g(C) - \{p'\} \subset \mathrm{int}_Y(E'_{1_h})$. Since $g(C) \in \langle E'_{1_h}, E'_{t_h}, \ldots, E'_{l_{r'}}\rangle$, we have that $g(C) \cap E'_{t_h} \neq \emptyset$. Thus, $g(C) \cap E'_{t_h} = \{p'\}$. Therefore, $p' \in E'_{t_h}$.

Similarly, if we fix $D \in \mathrm{cl}_{F_n(X)}(\mathcal{K}_X(1, t)) \cap \mathcal{K}_n(E_t, X)$ and $D \cap R(X) = \{p\}$, then $g(D) \cap E'_{1_h} = \{q'\}$. Thus, $q' \in E'_{1_h}$.

Therefore, $p', q' \in E'_{1_h} \cap E'_{t_h}$. This completes the proof of Claim 3.

Let $\mathcal{P} = \{E_j \in \mathcal{A}_S(X) : p \in E_j\}$ and $k = |\mathcal{P}|$.

Suppose that $q' \neq p'$.

Let $E_j \in \mathcal{P}$. Fix $C \in \mathcal{K}_n(E_j, X)$ such that $C \cap R(X) = \{p\}$, by Claim 1(III), $|g(C) \cap R(Y)| = 1$. Let $p* \in Y$ be such that $g(C) \cap R(Y) = \{p*\}$. Notice that $p* \in E'_{j_h}$. Exchanging E_t by E_j in Claim 2 and Claim 3, we have that $p', p* \in E'_{1_h} \cap E'_{j_h}$.

In a similar way, we may prove $p*, q' \in E'_{j_h} \cap E'_{t_h}$. Hence,

$$p', q' \in E'_{j_h}, \text{ for each } E_j \in \mathcal{P}. \tag{3.6}$$

Since Y is not a $\theta_{m'}$-graph, there exists $E'_{\lambda_h} \in \mathcal{A}_S(Y)$ such that $p' \in E'_{\lambda_h}$ and $q' \notin E'_{\lambda_h}$, or $p' \notin E'_{\lambda_h}$ and $q' \in E'_{\lambda_h}$. It is enough to consider the case when $p' \in E'_{\lambda_h}$ and $q' \notin E'_{\lambda_h}$. The other case, also conduces to contradiction.

Let $B' \in \mathcal{K}_n(E'_{\lambda_h}, Y)$ be such that $B' \cap R(Y) = \{p'\}$. By Claim 1(III), $|g^{-1}(B') \cap R(X)| = 1$. Let $q_1 \in X$ be such that $g^{-1}(B') \cap R(X) = \{q_1\}$. Notice that $q_1 \in E_\lambda$.

Remember that $g(A) \in \mathcal{K}_n(E'_{1_h}, Y)$ and $g(A) \cap R(Y) = \{p'\}$. Moreover, $A \cap R(X) = \{p\}$ and $p \in E_1$.

Since Y satisfies symmetric conditions as X, with similar arguments to Claim 2 and Claim 3, we obtain that $q_1, p \in E_\lambda \cap E_1$. Thus, $E_\lambda \in \mathcal{P}$. By (3.6), we have that $p', q' \in E'_{\lambda_h}$, which is a contradiction.

Therefore, $q' = p'$.

Let $\mathcal{P}' = \{E'_j \in \mathcal{A}_S(Y) : p' \in E'_j\}$. Therefore, $E_j \in \mathcal{P}$ if and only if $E'_{j_h} \in \mathcal{P}'$. Thus, $|\mathcal{P}| = |\mathcal{P}'|$. Using this fact and hypothesis (a), (b), $\mathrm{ord}(p, X) = \mathrm{ord}(p', Y)$.

Since p' does not depend on the choice of $E_j \in \mathcal{P}$ nor the choice of $B \in \mathcal{K}_n(E_j, X)$, we denote it by p_h. In this way we have a function $\varphi\colon R(X) \to R(Y)$ given by $\varphi(p) = p_h$. Notice that φ satisfies the following property: if $p \in R(X)$ and $A \in \mathcal{K}_n(E_j, X)$ are such that $A \cap R(X) = \{p\}$, then $g(A) \in \mathcal{K}_n(E'_{j_h}, Y)$ satisfies that $g(A) \cap R(Y) = \{p_h\}$.

According to Theorem 3.4, X and Y satisfy symmetric conditions. Hence, analogously as φ was defined, we may construct a function $\phi : R(Y) \to R(X)$ given by $\phi(q) = q_{h^{-1}}$ with the following property: if $q \in R(Y)$ and $B \in \mathcal{K}_n(E'_{j_h}, Y)$ are such that $B \cap R(Y) = \{q\}$, then $g^{-1}(B) \in \mathcal{K}_n(E_j, X)$ satisfies that $g^{-1}(B) \cap R(X) = \{q_{h^{-1}}\}$.

By the properties that satisfy φ and ϕ, we obtain that ϕ is the inverse of φ. Therefore, φ is a bijection from $R(X)$ onto $R(Y)$.

Now we can extend φ to a homeomorphism between X and Y. Take $E_j \in \mathcal{A}_S(X)$. If $|E_j \cap R(X)| = 2$, then E_j is an arc. Let p and q be the end points of E_j. Then $\{p, q\} = E_j \cap R(X)$. By Claim 1(II), E'_{j_h} is an arc with end points $\varphi(p)$ and $\varphi(q)$. We may consider a homeomorphism $\varphi_j\colon E_j \to E'_{j_h}$ such that $\varphi_j(p) = \varphi(p)$ and $\varphi_j(q) = \varphi(q)$. In the case that $|E_j \cap R(X)| = 1$, assuming that $E_j \cap R(X) = \{w\}$, we know that $E'_{j_h} \cap R(Y) = \{\varphi(w)\}$. By (a) and (b), there exists a homeomorphism $\varphi_j : E_j \to E'_{j_h}$ such that $\varphi_j(w) = \varphi(w)$.

Let $\psi : X \to Y$ be the function defined as $\psi(x) = \varphi_j(x)$, if $x \in E_j$. By [7, Theorem 9.4, p. 83], ψ is a homeomorphism between X and Y. $\square$

Theorem 3.7. *Let $n \in \mathbb{N}$ with $n \geq 4$. If X is a finite graph such that $R(X) \neq \emptyset$ and X is not a θ_m-graph, for any $m \in \mathbb{N}$, then X has unique n-fold symmetric product suspension.*

Proof. Let Y be a continuum and let $h\colon SF_n(X) \to SF_n(Y)$ be a homeomorphism. Since X is a finite graph, by Theorem 3.1, Y is a finite graph. By Theorem 3.3 and Theorem 3.5, we obtain that $R(Y) \neq \emptyset$ and Y is not a $\theta_{m'}$-graph, for any $m' \in \mathbb{N}$.

Let $E_j \in \mathcal{A}_S(X)$, by Theorem 3.4, we consider $E'_{j_h} \in \mathcal{A}_S(Y)$ such that $h(q_X(\mathcal{K}^j_X)) = q_Y(\mathcal{K}^{j_h}_Y)$. Thus, $h(q_X(\langle E_j\rangle) - \{F_X\}) = q_Y(\langle E'_{j_h}\rangle) - \{F_Y\}$. By Theorem 3.6, we have that $h(q_X(\langle E_j\rangle)) = q_Y(\langle E'_{j_h}\rangle)$. Since $q_X(\langle E_j\rangle)$ and $q_Y(\langle E'_{j_h}\rangle)$ are homeomorphic to $SF_n(E_j)$ and $SF_n(E'_{j_h})$, respectively, then $SF_n(E_j)$ is homeomorphic to $SF_n(E'_{j_h})$. By Theorem 3.3, we have that E_j is a cycle if and only if E'_{j_h} is a cycle and E_j in an arc if and only if E'_{j_h} is an arc.

Let $E_j \in \mathcal{A}_E(X)$. Then E'_{j_h} is an arc. Suppose that $E'_{j_h} \notin \mathcal{A}_E(Y)$. Let $g : F_n(X) - F_1(X) \to F_n(Y) - F_1(Y)$ be defined as $g(A) = (q^*_Y)^{-1} \circ h \circ q^*_X(A)$. Since $E_j \in \mathcal{A}_E(X)$, we have that $|E_j \cap R(X)| = 1$. Let $p \in X$ be such that $E_j \cap R(X) = \{p\}$. Let $a \in X$ be the other end point of E_j. Take pairwise distinct points $a_2, \ldots, a_n \in \mathrm{int}_X(E_j) - \{a\}$. Let $A = \{a, a_2, \ldots, a_n\}$. Notice that $A \in \mathcal{K}^j_X$. By Theorem 3.4(a), $g(A) \in \mathcal{K}^{j_h}_Y$. Using Claim 2 of Theorem 3.3, there exists a neighborhood $\mathcal{M}$ of A in $F_n(E_j)$ which is an n-cell such that $A \in \partial\mathcal{M}$. By [13, Proposition 1], we may assume that $\mathcal{M} \subset \mathcal{K}^j_X$. Thus, $g(\mathcal{M})$ is a neighborhood of $g(A)$ which is an n-cell such that $g(A) \in \partial g(\mathcal{M})$. Since $g(A) \in \mathcal{K}^{j_h}_Y$, we have that $|g(A)| = n$. By [22, Theorem 2.5], there exist pairwise disjoint open and connected subsets $V_1, \ldots, V_n$ of E'_{j_h} such that $g(A) \in \langle V_1, \ldots, V_n\rangle \subset \mathrm{int}_{F_n(E'_{j_h})}(g(\mathcal{M}))$. Since $E'_{h_j} \notin \mathcal{A}_E(Y)$ and $g(A) \in \mathcal{K}^{j_h}_Y$, we have that $g(A)$ does not have end points of E'_{j_h}. Moreover, we may assume that $V_i \cap R(Y) = \emptyset$, for each $i \in \{1, \ldots, n\}$. Thus, V_i is homeomorphic to $(0,1)$. Therefore, $\langle V_1, \ldots, V_n\rangle$ is homeomorphic to $(0,1)^n$, which is a contradiction since $g(A) \in \partial g(\mathcal{M})$. Therefore, $E'_{j_h} \in \mathcal{A}_E(Y)$.

By symmetry, if $E'_{j_h} \in \mathcal{A}_E(Y)$, then $E_j \in \mathcal{A}_E(X)$.

We have proved (a) and (b) of Theorem 3.6. Therefore, X is homeomorphic to Y. $\square$

So, as a consequence of Theorem 3.3, Theorem 3.5 and Theorem 3.7, we have our main result.

Theorem 3.8. *If X is a finite graph and $n \in \mathbb{N}$ with $n \geq 4$, then X has unique n-fold symmetric product suspension.*

Related to Problem 1.1, it is reasonable to ask whether this holds for $n = 2$ or $n = 3$.

Question 3.9. *Let $n \in \{2, 3\}$. If X and Y are finite graphs, $R(X) \neq \emptyset$, $R(Y) \neq \emptyset$, and $SF_n(X)$ is homeomorphic to $SF_n(Y)$, is X homeomorphic to Y?*

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