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On the uniqueness of the n-fold pseudo-hyperspace suspension for locally connected continua

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Let X be a metric continuum. Let n be a positive integer, we consider the hyperspace $C_n(X)$ of all nonempty closed subsets of X with at most n components and $F_1(X) = \{\{x\}: x \in X\}$. The n-fold pseudo-hyperspace suspension of X is the quotient space $C_n(X)/F_1(X)$ and it is denoted by $PHS_n(X)$. In this paper we prove that: (1) if X is a meshed continuum and Y is a continuum such that $PHS_n(X)$ is homeomorphic to $PHS_n(Y)$, then X is homeomorphic to Y, for each $n > 1$. (2) There are locally connected continua without unique hyperspace $PHS_n(X)$.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. The set of positive integers is denoted by $\mathbb{N}$. Given a continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\} \text{ and}$$
$$C(X) = C_1(X).$$

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All the hyperspaces considered are metrized by the Hausdorff metric H [13, Theorem 2.2].

Related to a continuum X, Sam B. Nadler, Jr. [20], introduced the hyperspace suspension of a continuum, $HS(X)$, as the quotient space $C(X)/F_1(X)$. Twenty five years later in [15], Sergio Macías gave a generalization of it, defining the n-fold hyperspace suspension of a continuum, $HS_n(X)$, as the quotient space $C_n(X)/F_n(X)$. In 2008, Juan C. Macías [16] introduced the n-fold pseudo-hyperspace suspension of a continuum, $PHS_n(X)$, as the quotient space $C_n(X)/F_1(X)$. Given a continuum X, let $\mathcal{H}(X)$ be any of the hyperspaces 2^X, $C_n(X)$, $F_n(X)$, $HS_n(X)$, or $PHS_n(X)$. The continuum X is said to have *unique hyperspace* $\mathcal{H}(X)$ provided that the following implication holds: if Y is a continuum and $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y.

One of the problems that has been widely studied lately on the theory of continua and their hyperspaces is to search for continua with unique hyperspace $\mathcal{H}(X)$. The problem of finding conditions for X in order that X has unique $\mathcal{H}(X)$ has been widely studied for several families of continua, especially for finite graphs, meshed continua and almost meshed locally connected continua. In [12], Alejandro Illanes proved that finite graphs have unique $C_n(X)$ and later, in [6] Rodrigo Hernández-Gutiérrez, A. Illanes and Verónica Martínez-de-la-Vega studied the uniqueness of the hyperspace $C_n(X)$ for locally connected continua and proved that meshed continua have unique $C_n(X)$. Later, adopting some of the techniques presented in [12] it was proved that finite graphs have unique $HS_n(X)$, see [7]. Later, in [8] María de J. López jointly with the second and third authors proved that framed continua have unique $HS_n(X)$. In relation to this topic, Germán Montero-Rodríguez, M. de J. López jointly with the second and third authors proved that finite graphs have unique hyperespace $F_n(X)/F_1(X)$, for each $n \geq 4$, see [19, Theorem 3.8]. Recently, in [18] it was proved that finite graphs have unique $PHS_n(X)$. Following the study of this property in the hyperspace $PHS_n(X)$, in the present work we prove that

(1) Meshed continua have unique n-fold pseudo-hyperspace suspension, for $n > 1$, see Theorem 4.8.
(2) There are almost meshed locally connected continua without unique n-fold pseudo-hyperspace suspension, see Theorem 5.3.
(3) There exists an almost meshed locally connected continuum that is not meshed with unique 2-fold pseudo-hyperspace suspension, see Example 5.4.
(4) There exist locally connected continua that are not almost meshed without unique n-fold pseudo-hyperspace suspension, see Theorem 5.5.

2. Definitions

Let X be a continuum. Given a subset A of X, $\mathrm{int}_X(A)$, $\mathrm{cl}_X(A)$, and $\mathrm{bd}_X(A)$, denote the *interior*, the *closure*, and the *boundary* of A in X, respectively, and when there is no possible confusion with the underlying continuum in which A lies, we simply will use A° instead of $\mathrm{int}_X(A)$. Through this paper, we write d for the metric associated to the continuum X. Let $\varepsilon > 0$ and $p \in X$; the set $\{x \in X : d(p, x) < \varepsilon\}$ is denoted by $B_X(p, \varepsilon)$, when there is no possible confusion with the underlying continuum in which d lies, we use $B(p, \varepsilon)$ instead of $B_X(p, \varepsilon)$. The Hausdorff metric H is defined as follows: for each $A, B \in 2^X$,

$$H(A, B) = \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\},$$

where $N(\varepsilon, A) = \{x \in X : d(x, A) < \varepsilon\}$. The hyperspaces $F_n(X)$ and $C_n(X)$ are called the *n-fold symmetric product of* X and the *n-fold hyperspace* of X, respectively. The cardinality of A is denoted by $|A|$. Let $p \in X$ and β be a cardinal number. We say that *p has order less than or equal to β in X*, written $\mathrm{ord}(p, X) \leq \beta$, whenever p has a basis of neighborhoods $\mathfrak{B}$ in X such that the cardinality of $\mathrm{bd}_X(U)$ is less than or equal to β, for each $U \in \mathfrak{B}$. We say that *p has order equal to β in X* ($\mathrm{ord}(p, X) = \beta$) provided that $\mathrm{ord}(p, X) \leq \beta$ and $\mathrm{ord}(p, X) \nleq \alpha$ for any cardinal number $\alpha < \beta$. Let $E(X) = \{x \in X : \mathrm{ord}(x, X) = 1\}$, $O(X) = \{x \in$

$X\colon \operatorname{ord}(x,X)=2\}$, and $R(X)=\{x\in X\colon \operatorname{ord}(x,X)\geq 3\}$. The elements of $E(X)$ (respectively, $O(X)$ and $R(X)$) are called *end points* (respectively, *ordinary points* and *ramification points*) of X. A *map* is a continuous function.

A *finite graph* is a continuum which is a finite union of arcs such that every two of them meet at a subset of their end points.

Given a continuum X, a *free arc* in X is an arc J with end points p and q such that $J-\{p,q\}$ is an open subset of X. A *maximal free arc* in X is a free arc in X that is maximal with respect to the inclusion. A *cycle* in X is a simple closed curve J in X such that $J-\{a\}$ is an open subset of X, for some $a\in J$. Notice that if X is not a simple closed curve and J is a cycle in X, then $J\cap R(X)=\{a\}$. Let

$$\mathfrak{A}_R(X)=\{J\subset X\colon J \text{ is a cycle in } X\},$$
$$\mathfrak{A}_E(X)=\{J\subset X\colon J \text{ is a maximal free arc in } X \text{ and } |J\cap R(X)|=1\},$$
$$\mathfrak{A}_S(X)=\{J\subset X\colon J \text{ is a maximal free arc in } X\}\cup\mathfrak{A}_R(X),$$
$$\mathcal{G}(X)=\{x\in X\colon x \text{ has a neighborhood in } X \text{ which is a finite graph}\} \text{ and}$$
$$\mathcal{P}(X)=X-\mathcal{G}(X).$$

According to [6, p. 1584] a continuum X is said to be *almost meshed* whenever the set $\mathcal{G}(X)$ is dense in X. An almost meshed continuum X is *meshed* provided that X has a basis of neighborhoods $\mathcal{B}$ such that $U-\mathcal{P}(X)$ is connected, for each $U\in\mathcal{B}$.

Given a continuum X and $n\in\mathbb{N}$, the function $q_X^n\colon C_n(X)\to PHS_n(X)$ is the natural projection, and F_X^n denotes the element $q_X^n(F_1(X))$. Notice that

$$q_X^n|_{C_n(X)-F_1(X)}\colon C_n(X)-F_1(X)\to PHS_n(X)-\{F_X^n\} \text{ is a homeomorphism.} \tag{2.1}$$

Given $m\in\mathbb{N}$ and $U_1,\ldots,U_m$ subsets of X, let

$$\langle U_1,\ldots,U_m\rangle_n=\{A\in C_n(X)\colon A\subset U_1\cup\cdots\cup U_m \text{ and } A\cap U_i\neq\emptyset, \text{ for each } i\in\{1,\ldots,m\}\}.$$

By [13, Theorem 1.2], it is known that the family of all sets $\langle U_1,\ldots,U_m\rangle_n$, where each U_i is an open subset of X, forms a basis for the topology in $C_n(X)$.

A topological manifold M (possibly with boundary) of dimension $n<\infty$ is a metrizable topological space M such that each point x in M admits an open neighborhood U and a homeomorphism $\kappa: U\longrightarrow\kappa(U)$ onto an open subset of the Euclidean half-space $\mathbb{R}^n_+=\{(x_1,\ldots,x_n)\in\mathbb{R}^n\colon x_1\geq 0\}$. The points x in M that correspond to points $\kappa(x)$ in the hyperplane $\{(x_1,\ldots,x_n)\in\mathbb{R}^n_+\colon x_1=0\}$ form the manifold boundary of M. The manifold interior of M is defined as the complement of the manifold boundary on M, as in [14, p. 7].

We use the following notations: $\dim[X]$ stands for the dimension of X, and $\dim_p[X]$ stands for the dimension of X at the point $p\in X$, as in [22, p. 5].

Given a continuum X and $n\in\mathbb{N}$, let

$$\mathcal{L}_n(X)=\{A\in C_n(X): A \text{ has a neighborhood in } C_n(X) \text{ which is a } 2n\text{-cell}\},$$
$$\partial\mathcal{L}_n(X)=\{A\in C_n(X): A \text{ has a neighborhood } \mathcal{N} \text{ in } C_n(X) \text{ such that}$$
$$\mathcal{N} \text{ is a } 2n\text{-cell and } A \text{ belongs to the manifold boundary of } \mathcal{N}\},$$
$$\mathcal{D}_n(X)=\{A\in C_n(X): A\notin\mathcal{L}_n(X) \text{ and } A \text{ has a basis of neighborhoods}$$
$$\mathcal{A} \text{ in } C_n(X) \text{ such that for each } \mathcal{U}\in\mathcal{A}, \dim[\mathcal{U}]=2n$$
$$\text{and } \mathcal{U}\cap\mathcal{L}_n(X) \text{ is arcwise connected}\},$$

$$\mathcal{PHL}_n(X) = \{B \in PHS_n(X) : B \text{ has a neighborhood in } PHS_n(X) \text{ which is a } 2n\text{-cell}\},$$
$$\partial\mathcal{PHL}_n(X) = \{B \in PHS_n(X) : B \text{ has a neighborhood } \mathcal{N} \text{ in } PHS_n(X) \text{ such that}$$
$$\mathcal{N} \text{ is a } 2n\text{-cell and } B \text{ belongs to the manifold boundary of } \mathcal{N}\},$$
$$\mathcal{PHD}_n(X) = \{B \in PHS_n(X) : B \notin \mathcal{PHL}_n(X) \text{ and } B \text{ has a basis of neighborhoods}$$
$$\mathcal{B} \text{ in } PHS_n(X) \text{ such that for each } \mathcal{V} \in \mathcal{B}, \dim[\mathcal{V}] = 2n$$
$$\text{and } \mathcal{V} \cap \mathcal{PHL}_n(X) \text{ is arcwise connected}\}, \text{ and}$$
$$\mathcal{PHE}_n(X) = \{B \in PHS_n(X) : \dim_B[PHS_n(X)] = 2n\}.$$

By (2.1), we have the following remark.

Remark 2.1. Let X be a continuum and $n \in \mathbb{N}$. Then

(*a*) $q_X^n(\mathcal{L}_n(X) - F_1(X)) = \mathcal{PHL}_n(X) - \{F_X^n\}$,
(*b*) $q_X^n(\partial\mathcal{L}_n(X) - F_1(X)) = \partial\mathcal{PHL}_n(X) - \{F_X^n\}$ and
(*c*) $q_X^n(\mathcal{D}_n(X) - F_1(X)) = \mathcal{PHD}_n(X) - \{F_X^n\}$.

3. Preliminary results

Lemma 3.1. *Let X be a locally connected continuum and $J, K \in \mathfrak{A}_S(X)$. Then*

(*a*) *$J^\circ \cap R(X) = \emptyset$,*
(*b*) *$\mathrm{bd}_X(K) \subset R(X)$ and*
(*c*) *if $J^\circ \cap K \neq \emptyset$, then $J = K$.*

Proof. (*a*) Take $p \in J^\circ$. Let U be an open subset of X such that $p \in U$. Then, there exists an arc L in J such that $p \in \mathrm{int}_J(L) \subset L \subset U \cap J^\circ$. Then $\mathrm{int}_J(L)$ is an open connected subset of X. Moreover, $\mathrm{bd}_X(\mathrm{int}_J(L)) \subset L - \mathrm{int}_J(L)$ and $L - \mathrm{int}_J(L)$ has at most 2 elements. Thus, $p \notin R(X)$. Consequently, $J^\circ \cap R(X) = \emptyset$.

(*b*) If $R(X) = \emptyset$, by [21, 8.40], we have that X is an arc or a simple closed curve and the result follows. Suppose that $R(X) \neq \emptyset$. Let $p \in \mathrm{bd}_X(K)$ and $\mathfrak{B}$ be a basis of neighborhoods of p in X.

Case 1. K is a cycle.
Let $q \in X - K$ and L be an arc in X with end points p and q. Since $K - \{p\}$ is an open subset of X, we have that $K \cap L = \{p\}$. Let $r = d(p, q)$ and $U \in \mathfrak{B}$ be such that $U \subset B(p, r)$ and $K \not\subset U$. Notice that $\mathrm{bd}_X(U)$ has at least 3 elements. This implies that $p \notin E(X) \cup O(X)$. Therefore, $p \in R(X)$.

Case 2. K is an arc.
Notice that p is an end point of K. Let a be the other end point of K. Let $s = \min\{\frac{\mathrm{diam}(K)}{2}, \frac{d(a,p)}{2}\}$ and let W be an open connected subset of X such that $p \in W \subset B(p, s)$. By [21, 8.26], W is arcwise connected. Let $q \in W - K$ and L be an arc in W with end points p and q. Notice that $K \not\subset L$ and $a \notin L$. Since $K - \{a, p\}$ is an open subset of X, we have that $K \cap L \subset \{a, p\}$. Hence, $K \cap L = \{p\}$. Suppose that there exists $\delta > 0$ such that $B(p, \delta) \subset K \cup L$. Let C_p be the component of $B(p, \delta)$ such that $p \in C_p$ and $L_p = \mathrm{cl}_X(C_p)$. Hence, L_p is an arc. Since X is locally connected, C_p is an open subset of X. Let l, k be the end points of L_p, where $l \in L$ and $k \in K$. Notice that $K \cup L_p - \{a, l\} = C_p \cup (K - \{a, p\})$. Thus, $K \cup L_p$ is a free arc. This contradicts the maximality of K. Therefore, for any $\varepsilon > 0$, $B(p, \varepsilon) \not\subset K \cup L$. This implies that there exists an arc M such that $(K \cup L) \cap M = \{p\}$. Let z be the other end point of M and $r = \min\{d(p, a), d(p, q), d(p, z)\}$. Thus, there exists $V \in \mathfrak{B}$ such that $V \subset B(p, r)$. Notice that $\mathrm{bd}_X(V)$ has at least 3 elements. This implies that $p \notin E(X) \cup O(X)$. Therefore, $p \in R(X)$.

(c) Given $p \in J^\circ \cap K$, by (a), we know that $p \notin R(X)$. Using (b), we have that $p \in K^\circ$. Hence, $J^\circ \cap K^\circ = J^\circ \cap K$. Consequently, $J^\circ \cap K$ is a nonempty open and closed subset of the connected set J°. Thus, $J^\circ = J^\circ \cap K$ and $J \subset K$. By the maximality of J, we have that $J = K$. $\square$

In [17], Verónica Martínez-de-la-Vega computed the dimension of the n-fold hyperspace for a finite graph G with the following formula

$$\dim_A[C_n(G)] = 2n + \sum_{p \in A \cap R(G)} (\mathrm{ord}(p, G) - 2), \text{ where } A \in C_n(G). \tag{3.1}$$

Lemma 3.2. *[6, Theorem 4] Let X be a locally connected continuum, $n \in \mathbb{N}$ and $A \in C_n(X)$. Then the following conditions are equivalent.*

(a) $\dim_A[C_n(X)]$ is finite,
(b) there exists a finite graph G contained in X such that $A \subset \mathrm{int}_X(G)$,
(c) $A \cap \mathcal{P}(X) = \emptyset$.

Lemma 3.3. *[6, Lemma 28] Let X be a locally connected continuum and $n \geq 3$. Then $\mathcal{D}_n(X) = \{A \in C_n(X) : A$ is connected and there exists $J \in \mathfrak{A}_S(X)$ such that $A \subset \mathrm{int}_X(J)\}$.*

The proof of following result is a modification of [7, Lemma 2.3].

Lemma 3.4. *Let X be a locally connected continuum and $n \in \mathbb{N}$. If $A \in C_n(X) - F_1(X)$ and $A \cap R(X) \neq \emptyset$, then $\dim_{q_X^n(A)}[PHS_n(X)] \geq 2n + 1$.*

Proof. From (2.1), we have that $\dim_{q_X^n(A)}[PHS_n(X)] = \dim_A[C_n(X)]$. If $\dim_A[C_n(X)]$ is not finite, the result follows. Suppose that $\dim_A[C_n(X)]$ is finite. By Lemma 3.2, there exists a finite graph G such that $A \subset \mathrm{int}_X(G)$. Notice that $\dim_A[C_n(X)] = \dim_A[C_n(G)]$. Since $A \cap R(X) \neq \emptyset$ and $A \subset \mathrm{int}_X(G)$, we have that $A \cap R(G) \neq \emptyset$. Thus, by (3.1), $\dim_A[C_n(G)] \geq 2n + 1$. Therefore, the result follows. $\square$

The proof of following result is a modification of [7, Lemma 2.4].

Lemma 3.5. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$ and $n \in \mathbb{N}$. Then for each neighborhood $\mathcal{U}$ of F_X^n in $PHS_n(X)$, $\dim[\mathcal{U}] \geq 2n + 1$.*

Proof. Let $\mathcal{U}$ be an open neighborhood of F_X^n in $PHS_n(X)$ and $\mathcal{V} = (q_X^n)^{-1}(\mathcal{U})$. Then $\mathcal{V}$ is an open subset of $C_n(X)$. Fix a point $p \in R(X)$. Since $\{p\} \in \mathcal{V}$, there exists $r > 0$ such that $B_{C_n(X)}(\{p\}, r) \subset \mathcal{V}$. Let C be the component of $B(p, r)$ containing p. Since C is an open connected subset of X, by [21, 8.26], C is arcwise connected. Hence, there exists an arc A such that $p \in A \subset B(p, r)$. Notice that $A \in \mathcal{V}$. Thus, $q_X^n(A) \in \mathcal{U}$. Therefore, by Lemma 3.4, $\dim_{q_X^n(A)}[\mathcal{U}] \geq 2n + 1$. $\square$

The proof of following result is a modification of [7, Lemma 2.9 (b)].

Lemma 3.6. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$, $n \in \mathbb{N}$ with $n \geq 3$. Then $\mathcal{PHD}_n(X) = \{q_X^n(A) \in PHS_n(X) : A \in C(X) - F_1(X)$ and $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset\}$.*

Proof. Given $B \in \mathcal{PHD}_n(X)$, there exists $A \in C_n(X)$ such that $B = q_X^n(A)$. Since $R(X) \neq \emptyset$, by Lemma 3.5, $B \neq F_X^n$, thus, $A \notin F_1(X)$. Moreover, by Remark 2.1 (c), $A \in \mathcal{D}_n(X)$. By Lemma 3.3, $A \in C(X) - F_1(X)$ and $A \subset \mathrm{int}_X(J)$, for some $J \in \mathfrak{A}_S(X)$. This implies that $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset$.

On the other hand, to prove the opposite inclusion, let $A \in C(X) - F_1(X)$ be such that $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset$. In order to prove that $q^n_X(A) \in \mathcal{PHD}_n(X)$, by Remark 2.1 (c), it will be enough to prove that $A \in \mathcal{D}_n(X)$. By Lemma 3.2, there exists a finite graph G contained in X such that $A \subset \mathrm{int}_X(G)$. Since $A \cap R(X) = \emptyset$, we have that $A \cap R(G) = \emptyset$. Thus, there exists a free arc L in G such that $A \subset \mathrm{int}_G(L)$. Since $A \subset \mathrm{int}_X(G)$, $A \subset \mathrm{int}_X(L)$ so we may assume that $L \subset \mathrm{int}_X(G)$. This implies that L is a free arc in X. By [6, Lemma 10], there exists $J \in \mathfrak{A}_S(X)$ such that $L \subset J$. Therefore, by Lemma 3.3, $A \in \mathcal{D}_n(X)$. $\square$

The proof of following result is a modification of [7, Lemma 2.10 (a) and (d)].

Lemma 3.7. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$ and $n \in \mathbb{N}$.*

(a) *For $n \geq 3$, the components of $\mathcal{PHD}_n(X)$ are the sets $q^n_X(\langle J^\circ\rangle_1) - \{F^n_X\}$, where $J \in \mathfrak{A}_S(X)$.*
(b) *The components of $\mathcal{PHE}_n(X)$ are the sets $q^n_X(\langle J^\circ_1, \ldots, J^\circ_m\rangle_n) - \{F^n_X\}$, where $J_1, \ldots, J_m \in \mathfrak{A}_S(X)$ and $m \leq n$.*

Proof. (a) By Lemma 3.6, $\mathcal{PHD}_n(X) = \bigcup\{q^n_X(\langle J^\circ\rangle_1) - \{F^n_X\} : J \in \mathfrak{A}_S(X)\}$. It is easy to see that the sets $q^n_X(\langle J^\circ\rangle_1) - \{F^n_X\}$ are arcwise connected and, therefore, connected. Moreover, the sets $q^n_X(\langle J^\circ\rangle_1) - \{F^n_X\}$ are open in $\mathcal{PHD}_n(X)$ and pairwise disjoint. We conclude that they are the components of $\mathcal{PHD}_n(X)$.

(b) By Lemma 3.5, $F^n_X \notin \mathcal{PHE}_n(X)$. Given $B \in \mathcal{PHE}_n(X)$, there exists $A \in C_n(X)$ such that $B = q^n_X(A)$. Notice that $\dim_A[C_n(X)] = \dim_B[PHS_n(X)] = 2n$. By [6, Lemma 11], there exist $J_1, \ldots, J_m \in \mathfrak{A}_S(X)$, with $m \leq n$, such that $A \in \langle J^\circ_1, \ldots, J^\circ_m\rangle_n$. This implies that $\mathcal{PHE}_n(X) \subset \bigcup\{q^n_X(\langle J^\circ_1, \ldots, J^\circ_m\rangle_n) - \{F^n_X\} : J_1, \ldots, J_m \in \mathfrak{A}_S(X)\}$. To prove the other inclusion, let $A \in \langle J^\circ_1, \ldots, J^\circ_m\rangle_n - F_1(X)$. Thus, $A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset$. By Lemma 3.2, there exists a finite graph G contained in X such that $A \subset \mathrm{int}_X(G)$. Since $A \cap R(X) = \emptyset$, we have that $A \cap R(G) = \emptyset$. Hence, by (3.1), $\dim_A[C_n(G)] = 2n$. Since $\dim_{q^n_X(A)}[PHS_n(X)] = \dim_A[C_n(X)] = \dim_A[C_n(G)]$, $q^n_X(A) \in \mathcal{PHE}_n(X)$. Therefore, $\mathcal{PHE}_n(X) = \bigcup\{q^n_X(\langle J^\circ_1, \ldots, J^\circ_m\rangle_n) - \{F^n_X\} : J_1, \ldots, J_m \in \mathfrak{A}_S(X)\}$. The rest of the proof is similar to the proof of (a). $\square$

Let X be a locally connected continuum such that $R(X) \neq \emptyset$. Given $J \in \mathfrak{A}_S(X)$, let $\mathcal{E}(J) = \mathrm{cl}_{C(X)}(\langle J^\circ\rangle_1)$. Notice that

$$\mathcal{E}(J) = \begin{cases} C(J) - \{A \in C(J) : A \text{ is an arc and } \mathrm{int}_J(A) \cap R(X) \neq \emptyset\}, & \text{if } J \text{ is a cycle,} \\ C(J), & \text{if } J \text{ is an arc.} \end{cases}$$

Let $D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $D_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + \frac{1}{2})^2 \leq \frac{1}{4}\}$. Let $L_0 = D_1 - \mathrm{int}_{\mathbb{R}^2}(D_2)$. Notice that if J is a cycle, then $\mathcal{E}(J)$ is homeomorphic to the continuum L_0.

The proof of following result is a modification of [18, Lemma 3.4].

Lemma 3.8. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$, $p \in X$ and let $J \in \mathfrak{A}_S(X)$.*

(1) *If J is an arc, then $\{q^2_X(\{p\} \cup A) : A \in \mathcal{E}(J)\}$ is a 2-cell in $PHS_2(X)$.*
(2) *If J is a cycle, then $\{q^2_X(\{p\} \cup A) : A \in \mathcal{E}(J)\}$ is homeomorphic to the continuum L_0.*

Proof. Let g be the embedding of $C(X)$ into $C_2(X)$ given by $g(A) = \{p\} \cup A$. Since the set $g(\mathcal{E}(J)) \cap F_1(X)$ is either the set $\emptyset$ or the set $\{p\}$, we have that $g(\mathcal{E}(J))/F_1(X)$ is homeomorphic to $\mathcal{E}(J)$. Notice that in (1), the set $\mathcal{E}(J)$ is a 2-cell, and in (2), it is homeomorphic to continuum L_0. Now, we finish the proof by mentioning that $g(\mathcal{E}(J))/F_1(X)$ is clearly homeomorphic to $\{q^2_X(\{p\} \cup A) : A \in \mathcal{E}(J)\}$. $\square$

Lemma 3.9. *Let X be a locally connected continuum. If Y and Z are either arcs or simple closed curves of X such that $Y \cap Z = \emptyset$, then $\langle Y, Z\rangle_2$ is a 4-cell and $\{y, z\}$ belongs to its manifold boundary, for each $y \in Y, z \in Z$.*

Proof. Let $f\colon \langle Y, Z\rangle_2 \longrightarrow C(Y) \times C(Z)$ be defined as $f(A) = (A \cap Y, A \cap Z)$. Notice that f is a bijection. Moreover, given a sequence $\{A_n\}_{n=1}^{\infty}$ contained in $\langle Y, Z\rangle_2$ which converges to A, for some $A \in \langle Y, Z\rangle_2$, we have that $\{A_n \cap Y\}_{n=1}^{\infty}$ converges to $A \cap Y$ and $\{A_n \cap Z\}_{n=1}^{\infty}$ converges to $A \cap Z$. Thus, $\{(A_n \cap Y, A_n \cap Z)\}_{n=1}^{\infty}$ converges to $(A \cap Y, A \cap Z)$. Hence, f is a homeomorphism.

By [13, 5.1.1 and 5.2], we have that $C(Y)$ and $C(Z)$ are 2-cells such that $F_1(Y)$ is contained in the manifold boundary of $C(Y)$ and $F_1(Z)$ is contained in the manifold boundary of $C(Z)$. Hence, $\langle Y, Z\rangle_2$ is a 4-cell. Let $y \in Y$ and $z \in Z$. Since $\{y\}$ belongs to the manifold boundary of $C(Y)$, there exist an open neighborhood $\mathcal{U}$ of $\{y\}$ in $C(Y)$ and a homeomorphism $\kappa_1 : \mathcal{U} \longrightarrow \kappa_1(\mathcal{U})$ onto an open subset of $\mathbb{R}^2_+$ such that $\kappa_1(\{y\}) = (0, r)$, for some $r \in \mathbb{R}$. Similarly, there exist an open neighborhood $\mathcal{V}$ of $\{z\}$ in $C(Z)$ and a homeomorphism $\kappa_2 : \mathcal{V} \longrightarrow \kappa_2(\mathcal{V})$ onto an open subset of $\mathbb{R}^2_+$ such that $\kappa_2(\{z\}) = (0, s)$, for some $s \in \mathbb{R}$. Notice that $\mathcal{U} \times \mathcal{V}$ is an open neighborhood of $(\{y\}, \{z\})$ in $C(Y) \times C(Z)$. Let $\kappa_+ : \mathcal{U} \times \mathcal{V} \longrightarrow \kappa_+(\mathcal{U} \times \mathcal{V})$ be defined as $\kappa_+(A, B) = (\kappa_1(A), \kappa_2(B))$. Thus, κ_+ is a homeomorphism, moreover, $\kappa_+(\mathcal{U} \times \mathcal{V}) = \kappa_1(\mathcal{U}) \times \kappa_2(\mathcal{V})$ is an open subset of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$.

Now, let $g : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \longrightarrow \mathbb{R}^4_+$ be defined as $g((a, b), (c, d)) = (2ac, b, a^2 - c^2, d)$ and let $h : \mathbb{R}^4_+ \longrightarrow \mathbb{R}^2_+ \times \mathbb{R}^2_+$ be defined as

$$h(a, b, c, d) = \left(\left(\sqrt{\tfrac{1}{2}(\sqrt{a^2 + c^2} + c)}, b\right), \left(\sqrt{\tfrac{1}{2}(\sqrt{a^2 + c^2} - c)}, d\right)\right).$$

Notice that g and h are maps. Moreover, $h \circ g = \mathrm{id}_{\mathbb{R}^2_+ \times \mathbb{R}^2_+}$ and $g \circ h = \mathrm{id}_{\mathbb{R}^4_+}$. Hence, g is a homeomorphism. By definition of f, $f^{-1}(\mathcal{U} \times \mathcal{V})$ is an open neighborhood of $\{y, z\}$ in $\langle Y, Z\rangle_2$. Let $\kappa : f^{-1}(\mathcal{U} \times \mathcal{V}) \longrightarrow \kappa(f^{-1}(\mathcal{U} \times \mathcal{V}))$ be defined as $\kappa(A) = g \circ \kappa_+ \circ f(A)$. Thus, κ is a homeomorphism, $\kappa(f^{-1}(\mathcal{U} \times \mathcal{V})) = g(\kappa_1(\mathcal{U}) \times \kappa_2(\mathcal{V}))$ is an open subset of $\mathbb{R}^4_+$ and $\kappa(\{y, z\}) = (0, r, 0, s)$. Therefore, $\{y, z\}$ belongs to the manifold boundary of $\langle Y, Z\rangle_2$. $\square$

Given $J, K \in \mathfrak{A}_S(X)$, let

$$\mathcal{D}(J, K) = \mathrm{cl}_{C_2(X)}(\partial\mathcal{L}_2(X) \cap \langle J^\circ, K^\circ\rangle_2) \cap \mathrm{cl}_{C_2(X)}(\partial\mathcal{L}_2(X) - \langle J^\circ, K^\circ\rangle_2) \text{ and}$$

$$\mathcal{PHD}(J, K) = \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) \cap q^2_X(\langle J^\circ, K^\circ\rangle_2)) \cap \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) - q^2_X(\langle J^\circ, K^\circ\rangle_2)).$$

Lemma 3.10. *Let X be a locally connected continuum such that $R(X) \neq \emptyset$ and let $J, K \in \mathfrak{A}_S(X)$. Then $F^2_X \in \mathcal{PHD}(J, K)$ if and only if $J \cap K \neq \emptyset$.*

Proof. Suppose that $F^2_X \in \mathcal{PHD}(J, K)$. Then, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ contained in $\langle J^\circ, K^\circ\rangle_2$ such that $\lim q^2_X(A_n) = F^2_X$. Since q^2_X is a map, $\lim A_n = \{a\}$, for some $a \in X$. Thus, $\{a\} \in \langle J, K\rangle_2$. Therefore, $J \cap K \neq \emptyset$.

Now suppose that $J \cap K \neq \emptyset$. We consider the following cases.

Case 1. $J \neq K$.

Let $p \in J \cap K \cap R(X)$. Then, there are two sequences $\{j_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ contained in J° and K°, respectively, such that $\lim j_n = p$ and $\lim k_n = p$. Thus, $\lim q^2_X(\{j_n, k_n\}) = F^2_X$. Let J_n and K_n be subarcs of J° and K°, respectively, such that $j_n \in J^\circ_n$ and $k_n \in K^\circ_n$, for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Notice that $\langle J_n, K_n\rangle_2$ is a neighborhood of $\{j_n, k_n\}$ in $C_2(X)$. Since J_n and K_n are disjoint arcs, by Lemma 3.9, we have that $\langle J_n, K_n\rangle_2$ is a 4-cell such that $\{j_n, k_n\}$ belongs to its manifold boundary. This implies that $\{j_n, k_n\} \in \partial\mathcal{L}_2(X)$. By Remark 2.1 (b), $q^2_X(\{j_n, k_n\}) \in \partial\mathcal{PHL}_2(X)$. Therefore, $F^2_X \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) \cap q^2_X(\langle J^\circ, K^\circ\rangle_2))$.

Now, let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be two sequences contained in K° such that $\lim p_n = p$, $\lim q_n = p$ and $p_n \neq q_n$, for each $n \in \mathbb{N}$. Let P_n and Q_n be disjoint subarcs of K such that $p_n \in P_n^\circ$ and $q_n \in Q_n^\circ$, for each $n \in \mathbb{N}$. By Lemma 3.9, we have that $\langle P_n, Q_n\rangle_2$ is a 4-cell and $\{p_n, q_n\}$ belongs to its manifold boundary. By Remark 2.1 (*b*), $\{q_X^2(\{p_n, q_n\})\}_{n=1}^{\infty}$ is a sequence contained in $\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ, K^\circ\rangle_2)$. Therefore, $F_X^2 \in \mathcal{PHD}(J, K)$.

Case 2. $J = K$.

Let $p \in J \cap R(X)$. Then, there exist two sequences $\{j_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ contained in J° such that $\lim j_n = p$, $\lim k_n = p$, and $j_n \neq k_n$, for each $n \in \mathbb{N}$. Let J_n and K_n be disjoint subarcs of J° such that $j_n \in J_n^\circ$ and $k_n \in K_n^\circ$, for each $n \in \mathbb{N}$. By Lemma 3.9, we have that $\langle J_n, K_n\rangle_2$ is a 4-cell such that $\{j_n, k_n\}$ belongs to its manifold boundary. This implies that $\{j_n, k_n\} \in \partial\mathcal{L}_2(X)$. By Remark 2.1 (*b*), $q_X^2(\{j_n, k_n\}) \in \partial\mathcal{PHL}_2(X)$. Therefore, $F_X^2 \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) \cap q_X^2(\langle J^\circ\rangle_2))$.

Since $p \in R(X)$, there exists $L \in \mathfrak{A}_S(X) - \{J\}$ such that $p \in L$. Thus, $p \in J \cap L \cap R(X)$. In a similar way as Case 1, we can prove that $F_X^2 \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ\rangle_2))$. Therefore, $F_X^2 \in \mathcal{PHD}(J, K)$. $\square$

The proof of following result is a modification of [7, Lemma 2.15].

Lemma 3.11. *Let X be a locally connected continuum with $R(X) \neq \emptyset$. If $J, K \in \mathfrak{A}_S(X)$, then $\mathcal{PHD}(J, K) = \{q_X^2(\{p\} \cup G) : p \in \mathrm{bd}_X(J)$ and $G \in \mathcal{E}(K)$ or $p \in \mathrm{bd}_X(K)$ and $G \in \mathcal{E}(J)\}$.*

Proof. Let $B \in \mathcal{PHD}(J, K)$. By Lemma 3.10, we may assume that $B \neq F_X^2$. Let $A \in C_2(X) - F_1(X)$ be such that $q_X^2(A) = B$. Since $B \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) \cap q_X^2(\langle J^\circ, K^\circ\rangle_2))$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ contained in $\langle J^\circ, K^\circ\rangle_2 - F_1(X)$ such that $\lim q_X^2(A_n) = B$ and $q_X^2(A_n) \in \partial\mathcal{PHL}_2(X)$, for each $n \in \mathbb{N}$. By the continuity of q_X^2, $\lim A_n = A$. By Remark 2.1 (*b*), $A_n \in \partial\mathcal{L}_2(X)$, for each $n \in \mathbb{N}$. Hence, $A \in \mathrm{cl}_{C_2(X)}(\partial\mathcal{L}_2(X) \cap \langle J^\circ, K^\circ\rangle_2)$. Moreover, since $B \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ, K^\circ\rangle_2))$, there exists a sequence $\{B_n\}_{n=1}^{\infty}$ contained in $\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ, K^\circ\rangle_2)$ such that $\lim B_n = B$ and $B_n \neq F_X^2$, for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, let D_n be the unique element of $C_2(X) - F_1(X)$ such that $q_X^2(D_n) = B_n$. Then $\lim D_n = A$. By Remark 2.1 (*b*), $D_n \in \partial\mathcal{L}_2(X) - \langle J^\circ, K^\circ\rangle_2$, for each $n \in \mathbb{N}$. Hence, $A \in \mathrm{cl}_{C_2(X)}(\partial\mathcal{L}_2(X) - \langle J^\circ, K^\circ\rangle_2)$. We have shown that $A \in \mathcal{D}(J, K)$. By [6, Lemma 33], $A = \{p\} \cup G$, where $p \in \mathrm{bd}_X(J)$ and $G \in \mathcal{E}(K)$ or $p \in \mathrm{bd}_X(K)$ and $G \in \mathcal{E}(J)$. This completes the proof of the first inclusion.

To prove the opposite inclusion, let $B = q_X^2(\{p\} \cup G)$, where $p \in \mathrm{bd}_X(J)$ and $G \in \mathcal{E}(K)$ or $p \in \mathrm{bd}_X(K)$ and $G \in \mathcal{E}(J)$. By Lemma 3.10, we may assume that $G \neq \{p\}$. Let $A = \{p\} \cup G$. By [6, Lemma 33], $A \in \mathcal{D}(J, K)$. Then, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ contained in $\partial\mathcal{L}_2(X) \cap \langle J^\circ, K^\circ\rangle_2$ such that $\lim A_n = A$ and $A_n \notin F_1(X)$, for each $n \in \mathbb{N}$. Hence, $q_X^2(A_n) \in \partial\mathcal{PHL}_2(X) \cap q_X^2(\langle J^\circ, K^\circ\rangle_2)$. Thus, $B \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) \cap q_X^2(\langle J^\circ, K^\circ\rangle_2))$. Similarly, $B \in \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ, K^\circ\rangle_2))$. Therefore, $B \in \mathcal{PHD}(J, K)$. $\square$

Now, we are ready to describe models of $\mathcal{PHD}(J, K)$ for each possible case. Let $J, K \in \mathfrak{A}_S(X)$, where X is a locally connected continuum such that $R(X) \neq \emptyset$. We consider nine cases.

Case I. $J = K$, J is an arc and $J \notin \mathfrak{A}_E(X)$.

By Lemma 3.11, $\mathcal{PHD}(J, J) = \{q_X^2(\{p\} \cup G) : G \in \mathcal{E}(J)\} \cup \{q_X^2(\{q\} \cup G) : G \in \mathcal{E}(J)\}$, where $p, q \in J \cap R(X)$. By Lemma 3.8, we have that $\mathcal{PHD}(J, J)$ is the union of two 2-cells whose intersection is the set $\{F_X^2, q_X^2(\{p, q\}), q_X^2(J)\}$. It is easy to see that this set is contained in the manifold boundary of both 2-cells.

Case II. $J = K$, J is an arc and $J \in \mathfrak{A}_E(X)$.

Then $J \cap R(X) = \{p\}$. Thus, $\mathcal{PHD}(J, J) = \{q_X^2(\{p\} \cup G) : G \in \mathcal{E}(J)\}$ which is a 2-cell.

Case III. $J = K$ and $J \in \mathfrak{A}_R(X)$.

Then $J \cap R(X) = \{q\}$. Thus, $\mathcal{PHD}(J, J) = \{q_X^2(\{q\} \cup G) : G \in \mathcal{E}(J)\}$ which is homeomorphic to L_0.

For the remaining cases we assume that $J \neq K$.

Case IV. J and K are arcs and $J, K \notin \mathfrak{A}_E(X)$.

Let $p_1, p_2 \in J \cap R(X)$ and $q_1, q_2 \in K \cap R(X)$. Then $\mathcal{PHD}(J,K) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$, where $\mathcal{P}_1 = \{q_X^2(\{p_1\} \cup G) : G \in \mathcal{E}(K)\}$, $\mathcal{P}_2 = \{q_X^2(\{p_2\} \cup G) : G \in \mathcal{E}(K)\}$, $\mathcal{Q}_1 = \{q_X^2(\{q_1\} \cup G) : G \in \mathcal{E}(J)\}$ and $\mathcal{Q}_2 = \{q_X^2(\{q_2\} \cup G) : G \in \mathcal{E}(J)\}$. By Lemma 3.8, $\mathcal{PHD}(J,K)$ is the union of four 2-cells. Now let us consider three subcases.

$IV(a)$. $J \cap K = \emptyset$.

Then $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset = \mathcal{Q}_1 \cap \mathcal{Q}_2$. Also, $\mathcal{P}_i \cap \mathcal{Q}_j = \{q_X^2(\{p_i, q_j\})\}$ with $i, j \in \{1, 2\}$.

$IV(b)$. $J \cap K$ is an one point set. Suppose that $p_1 = q_1$.

Similar to case $IV(a)$ with the exception that $\mathcal{P}_1 \cap \mathcal{Q}_1 = \{F_X^2\}$.

$IV(c)$. $J \cap K$ is a two point set. Suppose that $p_1 = q_1$ and $p_2 = q_2$.

Then $\mathcal{P}_1 \cap \mathcal{P}_2 = \{F_X^2, q_X^2(\{p_1, p_2\}), q_X^2(K)\}$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \{F_X^2, q_X^2(\{p_1, p_2\}), q_X^2(J)\}$. Moreover, $\mathcal{P}_i \cap \mathcal{Q}_j = \{F_X^2, q_X^2(\{p_1, p_2\})\}$ with $i, j \in \{1, 2\}$.

Case V. J and K are arcs, $J \notin \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_E(X)$.

Let $p_1, p_2 \in J \cap R(X)$ and $q \in K \cap R(X)$. Then $\mathcal{PHD}(J,K) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{Q}$, where $\mathcal{P}_1 = \{q_X^2(\{p_1\} \cup G) : G \in \mathcal{E}(K)\}$, $\mathcal{P}_2 = \{q_X^2(\{p_2\} \cup G) : G \in \mathcal{E}(K)\}$ and $\mathcal{Q} = \{q_X^2(\{q\} \cup G) : G \in \mathcal{E}(J)\}$. Thus, $\mathcal{PHD}(J,K)$ is the union of three 2-cells. Now let us consider two subcases.

$V(a)$. $J \cap K = \emptyset$.

Then $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. Also, $\mathcal{P}_i \cap \mathcal{Q} = \{q_X^2(\{p_i, q\})\}$ with $i \in \{1, 2\}$.

$V(b)$. $J \cap K$ is an one point set. Suppose that $p_1 = q$.

Similar to case $V(a)$ with the slightly difference that $\mathcal{P}_1 \cap \mathcal{Q} = \{F_X^2\}$.

Case VI. $J, K \in \mathfrak{A}_E(X)$.

Then $\mathcal{PHD}(J,K) = \{q_X^2(\{p\} \cup G) : G \in \mathcal{E}(K)\} \cup \{q_X^2(\{q\} \cup G) : G \in \mathcal{E}(J)\}$, where $p \in J \cap R(X)$ and $q \in K \cap R(X)$. Thus, $\mathcal{PHD}(J,K)$ is the union of two 2-cells whose intersection is the set $\{q_X^2(\{p, q\})\}$, or $\{F_X^2\}$ in the case that $p = q$.

Case VII. J is an arc, $J \notin \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_R(X)$.

Similar to case V with the slightly difference that $\mathcal{PHD}(J,K)$ is the union of a 2-cell and two continua L_0.

Case VIII. $J \in \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_R(X)$.

Similar to case VI with the slightly difference that $\mathcal{PHD}(J,K)$ is the union of a 2-cell and a continuum L_0.

Case IX. $J, K \in \mathfrak{A}_R(X)$.

Similar to case VI with the difference that $\mathcal{PHD}(J,K)$ is the union of two continua L_0.

Remark 3.12. Let X and Y be locally connected continua such that $R(X) \neq \emptyset$ and $R(Y) \neq \emptyset$, and let $J, K \in \mathfrak{A}_S(X)$ and $J_h, K_h \in \mathfrak{A}_S(Y)$. If $\mathcal{PHD}(J,K)$ is homeomorphic to $\mathcal{PHD}(J_h, K_h)$, then

(a) J and K are as in Case I if and only if J_h and K_h are as in Case I,
(b) J and K are as in Case II if and only if J_h and K_h are as in Case II and
(c) J and K are as in Case III if and only if J_h and K_h are as in Case III.

4. Main results

In this section we present the proof of our first main result. The first step is to mention that Ulises Morales-Fuentes has proven that the finite graphs have unique n-fold pseudo-hyperspace suspension, see [18, Theorem 5.7]. We prove that if X is a meshed continuum such that $|\bigcap \mathfrak{A}_S(X)| = 2$, then X is a finite graph, and therefore it has unique n-fold pseudo-hyperspace suspension. Finally, we prove that for a meshed continuum X such that $R(X) \neq \emptyset$ and $|\bigcap \mathfrak{A}_S(X)| \neq 2$ the uniqueness of the n-fold pseudo-hyperspace suspension holds, see Theorem 4.8.

Using [6, Lemma 2] and [5, Theorem 3.1] we have the following properties for meshed continua, which will be used without quoting them in the proof of Theorem 4.7.

Lemma 4.1. *If X is a meshed continuum, then*

(*a*) *X is locally connected,*
(*b*) *$J \cap \mathcal{P}(X) = \emptyset$, for each $J \in \mathfrak{A}_S(X)$, and*
(*c*) *$\mathcal{G}(X) = \bigcup \mathfrak{A}_S(X)$.*

The following result is proved in [4, Theorem 5.1] for case $n = 1$ and [16, Theorem 4.1 (a)] for case $n \geq 2$.

Lemma 4.2. *Let X be a continuum and $n \in \mathbb{N}$. Then X is locally connected if and only if $PHS_n(X)$ is locally connected.*

Given a continuum X and $n \in \mathbb{N}$, let

$$\mathfrak{F}_n(X) = \{A \in C_n(X) : \dim_A[C_n(X)] \text{ is finite}\}.$$

Theorem 4.3. *Let X be a meshed continuum and $n \in \mathbb{N}$. If Y is a continuum such that $PHS_n(X)$ is homeomorphic to $PHS_n(Y)$, then Y is a meshed continuum.*

Proof. Let $h : PHS_n(X) \longrightarrow PHS_n(Y)$ be a homeomorphism. Since X is a locally connected continuum, using Lemma 4.2, we have that Y is a locally connected continuum. Let $A \in C_n(X)$ and $B \in C_n(Y)$ be such that $h(q^n_X(A)) = F^n_Y$ and $h^{-1}(q^n_Y(B)) = F^n_X$. Let $\mathcal{K} = C_n(X) - (F_1(X) \cup \{A\})$ and $\mathcal{L} = C_n(Y) - (F_1(Y) \cup \{B\})$. Then $g\colon \mathcal{K} \longrightarrow \mathcal{L}$ defined by $g = (q^n_Y|_{\mathcal{L}})^{-1} \circ h \circ q^n_X|_{\mathcal{K}}$ is a homeomorphism. Moreover, $g(\mathfrak{F}_n(X) \cap \mathcal{K}) = \mathfrak{F}_n(Y) \cap \mathcal{L}$. Since X is meshed, by [6, Theorem 5], we know that $\mathfrak{F}_n(X)$ is a dense subset of $C_n(X)$. This implies that $\mathfrak{F}_n(Y) \cap \mathcal{L}$ is dense in $\mathcal{L}$. Finally, by the density of $\mathcal{L}$ in $C_n(Y)$, we conclude that $\mathfrak{F}_n(Y)$ is a dense subset of $C_n(Y)$. Therefore, by [6, Theorem 5], Y is a meshed continuum. $\square$

The following result extends [18, Lemma 5.2].

Lemma 4.4. *Let $n \geq 2$. If X is a locally connected continuum with $R(X) \neq \emptyset$ and $|\mathfrak{A}_S(X)| \geq 2$, then*

$$\bigcap \{\mathrm{cl}_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\} = \begin{cases} \{F^n_X\} & \textit{if } |\bigcap \mathfrak{A}_S(X)| \neq 2, \\ \{F^n_X, q^n_X(\{p,q\})\} & \textit{if } \bigcap \mathfrak{A}_S(X) = \{p,q\}. \end{cases}$$

Proof. Let $J \in \mathfrak{A}_S(X)$ and $a \in J^\circ$. Since $\{a\}$ can be approximated by elements in $\langle J^\circ \rangle_1 - F_1(X)$, we have that $\{a\} \in \mathrm{cl}_{C_n(X)}(\langle J^\circ \rangle_n - F_1(X))$. Hence, $F^n_X \in \mathrm{cl}_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\})$. Moreover, if $\bigcap \mathfrak{A}_S(X) = \{p,q\}$, then $p, q \in J$ and since $n \geq 2$, $\{p,q\}$ can be approximated by elements in $\langle J^\circ \rangle_n - F_1(X)$. Hence, $q^n_X(\{p,q\}) \in \mathrm{cl}_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\})$. This implies the second inclusion.

Now, let $B \in \bigcap \{\mathrm{cl}_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\}$.
Suppose that $B \neq F^n_X$. Let $A \in C_n(X) - F_1(X)$ be such that $q^n_X(A) = B$. Let $J \in \mathfrak{A}_S(X)$. Since $B \in \mathrm{cl}_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\})$, there exists a sequence $\{B_m\}_{m=1}^\infty$ contained in $q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}$ which converges to B. Let $A_m \in \langle J^\circ \rangle_n - F_1(X)$ be such that $q^n_X(A_m) = B_m$, for each $m \in \mathbb{N}$. Notice that $\{A_m\}_{m=1}^\infty$ converges to A. Hence, $A \subset J$, for each $J \in \mathfrak{A}_S(X)$. Therefore, $A \subset \bigcap \mathfrak{A}_S(X)$. Since $|\mathfrak{A}_S(X)| \geq 2$, we have that $|\bigcap \mathfrak{A}_S(X)| \leq 2$.

Consider the following cases.

Case 1. $|\bigcap \mathfrak{A}_S(X)| \neq 2$.
Then $|\bigcap \mathfrak{A}_S(X)| \leq 1$. Hence, $|A| \leq 1$. This is a contradiction since $A \in C_n(X) - F_1(X)$. Therefore, $B = F^n_X$.

Case 2. $\bigcap \mathfrak{A}_S(X) = \{p,q\}$.
Since $A \in C_n(X) - F_1(X)$, we have that $A = \{p,q\}$. Hence, $B \in \{F^n_X, q^n_X(\{p,q\})\}$, as desired.

From these cases, the result follows. $\square$

Theorem 4.5. *Let X be a meshed continuum such that $R(X) \neq \emptyset$. If $|\bigcap \mathfrak{A}_S(X)| = 2$, then X is a finite graph.*

Proof. Let $p, q \in \bigcap \mathfrak{A}_S(X)$. Thus, p and q are the end points of each maximal free arc. Suppose that there exists $a \in \mathcal{P}(X)$. By [5, Theorem 3.3], there is a sequence of pairwise distinct elements contained in $R(X) \cap \mathcal{G}(X)$ which converges to a. However, this is not possible since $R(X) \cap \mathcal{G}(X) \subset \{p, q\}$. Hence, $\mathcal{P}(X) = \emptyset$. Therefore, X is a finite graph. $\square$

Using Theorem 4.5 and [18, Theorem 5.7] we obtain the following result.

Theorem 4.6. *Let X be a meshed continuum such that $R(X) \neq \emptyset$. If $|\bigcap \mathfrak{A}_S(X)| = 2$, then X has unique n-fold pseudo-hyperspace suspension.*

The following result extends [18, Lemma 5.1 and Lemma 5.5].

Theorem 4.7. *Let X and Y be meshed continua such that $R(X) \neq \emptyset$, $R(Y) \neq \emptyset$ and $|\bigcap \mathfrak{A}_S(X)| \neq 2$, $|\bigcap \mathfrak{A}_S(Y)| \neq 2$, $n \geq 2$ and let $h : PHS_n(X) \longrightarrow PHS_n(Y)$ be a homeomorphism. Suppose that for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^n(\langle J^\circ \rangle_1) - \{F_X^n\}) \subset q_Y^n(\langle J_h^\circ \rangle_n)$ and $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(X)\}$. Then*

(*a*) *for each $J \in \mathfrak{A}_S(X)$, $h(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\}) = q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\}$,*
(*b*) *for each $J \in \mathfrak{A}_S(X)$, $h^{-1}(q_Y^n(\langle J_h^\circ \rangle_n \cap C(Y)) - \{F_Y^n\}) \subset q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\}$,*
(*c*) *the association $J \to J_h$ is a bijection between $\mathfrak{A}_S(X)$ and $\mathfrak{A}_S(Y)$.*
(*d*) *$h(F_X^n) = F_Y^n$.*

If we also suppose that

(1) *if $J \in \mathfrak{A}_R(X)$, then $J_h \in \mathfrak{A}_R(Y)$ and*
(2) *if $J \in \mathfrak{A}_E(X)$, then $J_h \in \mathfrak{A}_E(Y)$,*

then X is homeomorphic to Y.

Proof. (*a*) Let $J \in \mathfrak{A}_S(X)$ and A be a subarc of J° such that $h(q_X^n(A)) \neq F_Y^n$. By Lemma 3.7 (*b*), we have that $h(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\})$ and $q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\}$ are components of $\mathcal{PHE}_n(X)$. Notice that $h(q_X^n(A)) \in h(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\}) \cap (q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\})$. Therefore, $h(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\}) = q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\}$.

Clearly, (*b*) follows from (*a*).

To prove (*c*), it is enough to prove that the correspondence is one to one. Let $J, L \in \mathfrak{A}_S(X)$ and suppose that $J_h = L_h$. Using (*a*) we conclude that $q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\} = q_X^n(\langle L^\circ \rangle_n) - \{F_X^n\}$. Let A be a subarc of J°. Then $q_X^n(A) \in q_X^n(\langle L^\circ \rangle_n)$ and $A \subset L^\circ$. Therefore, by Lemma 3.1 (*c*), $J = L$.

(*d*) By Lemma 4.4 and using (*a*) we have that

$$\begin{aligned} h(\{F_X^n\}) &= \bigcap \{\mathrm{cl}_{PHS_n(Y)}(h(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\})) : J \in \mathfrak{A}_S(X)\} \\ &= \bigcap \{\mathrm{cl}_{PHS_n(Y)}(q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\}) : J \in \mathfrak{A}_S(X)\} \\ &= \bigcap \{\mathrm{cl}_{PHS_n(Y)}(q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\}) : J_h \in \mathfrak{A}_S(Y)\} = \{F_Y^n\}. \end{aligned}$$

Therefore, $h(F_X^n) = F_Y^n$.

Let $g : C_n(X) - F_1(X) \longrightarrow C_n(Y) - F_1(Y)$ be defined as $g = (q_Y^n)^{-1} \circ h \circ q_X^n$. Notice that g is a homeomorphism. Given $J \in \mathfrak{A}_S(X)$, let $\mathcal{K}_n(J, X) = \mathrm{cl}_{C_n(X)}(\langle J^\circ \rangle_n) - F_1(X)$.

The proofs of Claim 1 and Claim 2 are similar to the proofs of Claim 1 and Claim 2 from [7, Theorem 3.1], respectively. The proof of Claim 3 is similar to arguments given in [7, Theorem 3.1, p. 88–89].

Claim 1. *If $J \in \mathfrak{A}_S(X)$, then*

(*e*) $\mathcal{K}_n(J_h, Y) = g(\mathcal{K}_n(J, X))$,
(*f*) $\{\dim_A[C_n(X)] : A \in \mathcal{K}_n(J, X)\} = \{\dim_B[C_n(Y)] : B \in \mathcal{K}_n(J_h, Y)\}$,
(*g*) $|J \cap R(X)| = |J_h \cap R(Y)|$,
(*h*) *if $A \in \mathcal{K}_n(J, X)$, then $|A \cap R(X)| = |g(A) \cap R(Y)|$.*

Proof of Claim 1. Let $J \in \mathfrak{A}_S(X)$. Notice that $\mathrm{cl}_{C_n(X)}(\langle J^\circ\rangle_n) - F_1(X) = \mathrm{cl}_{C_n(X)-F_1(X)}(\langle J^\circ\rangle_n)$. From this, clearly (e) is true and (f) follows from (e). Now, since X is a meshed continuum, $J \cap \mathcal{P}(X) = \emptyset$. Thus, by Lemma 3.2, there exists a finite graph G contained in X such that $J \subset \mathrm{int}_X(G)$. Using (3.1), we have that $|\{\dim_A[C_n(X)] : A \in \mathcal{K}_n(J, X)\}| \geq 3$ if and only if $|J \cap R(X)| = 2$ and $|\{\dim_A[C_n(X)] : A \in \mathcal{K}_n(J, X)\}| = 2$ if and only if $|J \cap R(X)| = 1$. Notice that J_h also satisfies the same conditions as J, such as $J_h \cap \mathcal{P}(Y) = \emptyset$. This proves (g). Moreover, given $A \in \mathcal{K}_n(J, X)$. If $|A \cap R(X)| = 2$, then $|J \cap R(X)| = 2$. Thus, $|J_h \cap R(Y)| = 2$ and $\dim_A[C_n(X)] = \max\{\dim_E[C_n(X)] : E \in \mathcal{K}_n(J, X)\}$. Hence, $\dim_{g(A)}[C_n(Y)] = \max\{\dim_B[C_n(Y)] : B \in \mathcal{K}_n(J_h, Y)\}$. This implies that $|g(A) \cap R(Y)| = 2$. Similarly, if $|g(A) \cap R(Y)| = 2$, then $|A \cap R(X)| = 2$. If $|A \cap R(X)| = 0$, then $2n = \dim_A[C_n(G)] = \dim_A[C_n(X)] = \dim_{g(A)}[C_n(Y)]$. Hence, $|g(A) \cap R(Y)| = 0$. Similarly, if $|g(A) \cap R(Y)| = 0$, then $|A \cap R(X)| = 0$. Finally, if $|A \cap R(X)| = 1$, then $|g(A) \cap R(Y)| \notin \{0, 2\}$. Thus, $|g(A) \cap R(Y)| = 1$. This completes the proof of Claim 1. □

Claim 2. *If $J \in \mathfrak{A}_S(X)$ and $v \in J \cap R(X)$, then $\mathcal{K}(v, J) = \{A \in \mathcal{K}_n(J, X) : A \cap R(X) = \{v\}\}$ is arcwise connected.*

Now, given $v \in R(X) \cap \mathcal{G}(X)$, there is $J \in \mathfrak{A}_S(X)$ such that $v \in J$. Let $A \in \mathcal{K}(v, J)$. By Claim 1, $g(A) \in \mathcal{K}_n(J_h, Y)$ and there exists a unique point $v_h(A) \in R(Y) \cap g(A)$. Notice that $v_h(A) \in J_h$ and $v_h(A) \in R(Y) \cap \mathcal{G}(Y)$.

Claim 3. *Let $v \in R(X) \cap \mathcal{G}(X)$ and $J, L \in \mathfrak{A}_S(X)$ with $v \in J \cap L$. If $A \in \mathcal{K}(v, J)$ and $E \in \mathcal{K}(v, L)$, then $v_h(A) = v_h(E)$ (in other words, $v_h(A)$ depends neither on the choice of J nor on the choice of A).*

Proof of Claim 3. In order to prove this, take A_1 and E_1 arcs in J and L, respectively, such that v is an end point of A_1 and E_1, $A_1 \neq J$ and $E_1 \neq L$. Notice that $A_1 \in \mathcal{K}(v, J)$ and $E_1 \in \mathcal{K}(v, L)$. By Claim 2, there exist maps $\alpha_A\colon [0,1] \longrightarrow \mathcal{K}(v, J)$ and $\alpha_E\colon [0,1] \longrightarrow \mathcal{K}(v, L)$ such that $\alpha_A(0) = A$, $\alpha_A(1) = A_1$, $\alpha_E(0) = E_1$ and $\alpha_E(1) = E$. Moreover, since $A_1 \cup E_1$ is an arc, we may define a map $\alpha_0\colon [0,1] \longrightarrow C(A_1 \cup E_1)$ with the following properties: $\alpha_0(0) = A_1$, $\alpha_0(1) = E_1$ and for each $t \in [0,1]$, $\alpha_0(t) \cap R(X) = \{v\}$ and $\alpha_0(t) \notin F_1(X)$. Let $\alpha\colon [0,1] \longrightarrow \mathcal{K}(v, J) \cup C(A_1 \cup E_1) \cup \mathcal{K}(v, L)$ be defined as

$$\alpha(t) = \begin{cases} \alpha_A(3t) & \text{if } t \in [0, \frac{1}{3}], \\ \alpha_0(3t-1) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ \alpha_E(3t-2) & \text{if } t \in [\frac{2}{3}, 1]. \end{cases}$$

Notice that $\alpha(t) \subset J \cup L$. Thus, $g(\alpha(t)) \subset J_h \cup L_h$, for each $t \in [0,1]$. Let $i_0 = \mathrm{ord}(v, X)$. Since $(J \cup L) \cap \mathcal{P}(X) = \emptyset$, by Lemma 3.2 and (3.1), we have that for each $t \in [0,1]$,

$$2n + (i_0 - 2) = \dim_{\alpha(t)}[C_n(X)] = \dim_{g(\alpha(t))}[C_n(Y)].$$

Since $v_h(A)$ is the only ramification point of Y in the set $g(A) = g(\alpha(0))$, this implies that $\mathrm{ord}(v_h(A), Y) = i_0$. Let $T = \{t \in [0,1] : v_h(A) \in g(\alpha(t))\}$. Notice that T is a closed subset of $[0,1]$ and $0 \in T$. Suppose that $T \neq [0,1]$ and let R be a component of $[0,1] - T$. Then $t_0 = \inf R \in T$ and there exists a sequence $\{r_m\}_{m=1}^{\infty}$ of elements of R which converges to t_0. Since $(J_h \cup L_h) \cap R(Y)$ is finite, we may assume that there exists $w \in (J_h \cup L_h) \cap R(Y)$ such that $w \in g(\alpha(r_m))$. Hence, $w, v_h(A) \in g(\alpha(t_0))$. Notice that $w \neq v_h(A)$. Hence, $\dim_{g(\alpha(t_0))}[C_n(Y)] > 2n + (i_0 - 2)$, a contradiction. Therefore, $T = [0,1]$. On the other hand, we know that $v_h(E)$ is the only ramification point of Y in the set $g(E) = g(\alpha(1))$. Consequently, $v_h(A) = v_h(E)$. This proves Claim 3. □

From now on, we simply write v_h instead of $v_h(A)$. Thus, we have a function

$$\varphi : R(X) \cap \mathcal{G}(X) \longrightarrow R(Y) \cap \mathcal{G}(Y)$$
$$v \longmapsto v_h$$

Since Y satisfies similar conditions to those of X, we have that φ is a bijection.

Claim 4. *There exists a homeomorphism $\phi : \mathcal{G}(X) \longrightarrow \mathcal{G}(Y)$ such that $\phi|_{R(X)\cap\mathcal{G}(X)} = \varphi$.*

Proof of Claim 4. Let $J \in \mathfrak{A}_S(X)$.

Case 1. $|J \cap R(X)| = 2$.
Suppose that $J \cap R(X) = \{p, q\}$. Thus, $p_h, q_h \in J_h$. Since J and J_h are arcs, we may consider a homeomorphism $\varphi_J : J \longrightarrow J_h$ such that $\varphi_J(p) = p_h$ and $\varphi_J(q) = q_h$.

Case 2. $|J \cap R(X)| = 1$, assuming that $J \cap R(X) = \{a\}$.
Notice that $J_h \cap R(Y) = \{a_h\}$. By (1) and (2), we may take a homeomorphism $\varphi_J : J \longrightarrow J_h$ such that $\varphi_J(a) = a_h$. Hence, we define $\phi : \mathcal{G}(X) \longrightarrow \mathcal{G}(Y)$ given by $\phi(x) = \varphi_J(x)$, where $x \in J$. Therefore, ϕ is a homeomorphism. □

If X is a finite graph, then $\mathcal{G}(X) = X$. Thus, $\phi(X) = \mathcal{G}(Y)$ is a nonempty open and closed subset of Y. Therefore, $\mathcal{G}(Y) = Y$ and X is homeomorphic to Y. Now, suppose that X and Y are not finite graphs.

Claim 5. *If $a \in \mathcal{P}(X)$ and $\{a_m\}_{m=1}^{\infty}$ is a sequence contained in $\mathcal{G}(X) \cap R(X)$ which converges to a, then $\{\phi(a_m)\}_{m=1}^{\infty}$ converges.*

Proof of Claim 5. Let $\{\phi(b_l)\}_{l=1}^{\infty}$ be a convergent subsequence which converges to some $z \in Y$. By [5, Theorem 3.3], $z \in \mathcal{P}(Y)$. We are going to prove that $\lim \phi(a_m) = z$. Suppose to the contrary that

$$\text{there is } \varepsilon_1 > 0 \text{ such that for each } N \in \mathbb{N}, \text{ there exists } k > N \text{ such that } \phi(a_k) \notin B(z, \varepsilon_1). \tag{4.1}$$

Since $\lim \phi(b_l) = z$, there exists $N_1 \in \mathbb{N}$ such that if $l > N_1$, then $\phi(b_l) \in B(z, \frac{\varepsilon_1}{2})$. By [6, Lemma 3], there exists a basis $\mathcal{B}$ of open connected subsets of X such that, for each $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected. Let $V_1 \in \mathcal{B}$ be such that $a \in V_1$ and $\mathrm{diam}(V_1) < 1$. Thus, there is $N_2 > N_1$ such that if $m > N_2$, then $a_m \in V_1 - \mathcal{P}(X)$. Let $l_1 > N_2$. Hence, $b_{l_1} \in \phi^{-1}(B(z, \frac{\varepsilon_1}{2})) \cap (V_1 - \mathcal{P}(X))$. By (4.1), there exists $k_1 > N_2$ such that $\phi(a_{k_1}) \notin B(z, \varepsilon_1)$. Notice that $a_{k_1}, b_{l_1} \in V_1 - \mathcal{P}(X)$. Since $V_1 - \mathcal{P}(X)$ is an open connected subset of X, by [21, 8.26], $V_1 - \mathcal{P}(X)$ is arcwise connected. Then, there exists an arc α_1 in $V_1 - \mathcal{P}(X)$ with end points a_{k_1} and b_{l_1}. Hence, $\gamma_1 = \phi(\alpha_1)$ is an arc with end points $\phi(a_{k_1})$ and $\phi(b_{l_1})$. Notice that $\mathrm{diam}(\gamma_1) \geq \frac{\varepsilon_1}{2}$. Now, let $V_2 \in \mathcal{B}$ be such that $a \in V_2$, $\mathrm{diam}(V_2) < \frac{1}{2}$ and $\alpha_1 \cap V_2 = \emptyset$. Thus, there is $N_3 > N_2$ such that if $m > N_3$, then $a_m \in V_2 - \mathcal{P}(X)$. Let $l_2 > N_3$. Hence, $b_{l_2} \in \phi^{-1}(B(z, \frac{\varepsilon_1}{2})) \cap (V_2 - \mathcal{P}(X))$. By (4.1), there exists $k_2 > N_3$ such that $\phi(a_{k_2}) \notin B(z, \varepsilon_1)$. Notice that $a_{k_2}, b_{l_2} \in V_2 - \mathcal{P}(X)$. Then, there exists an arc α_2

in $V_2-\mathcal{P}(X)$ with end points a_{k_2} and b_{l_2}. Therefore, $\gamma_2=\phi(\alpha_2)$ is an arc with end points $\phi(a_{k_2})$ and $\phi(b_{l_2})$ and $\operatorname{diam}(\gamma_2)\geq\frac{\varepsilon_1}{2}$. Proceeding in a recursive way, we obtain

- a sequence $\{V_i-\mathcal{P}(X)\}_{i=1}^{\infty}$ such that each $V_i-\mathcal{P}(X)$ is an open connected subset of X, $a\in V_i$ and $\operatorname{diam}(V_i)<\frac{1}{i}$,
- a sequence $\{\phi(a_{k_i})\}_{i=1}^{\infty}$ such that $\phi(a_{k_i})\notin B(z,\varepsilon_1)$ and $a_{k_i}\in V_i-\mathcal{P}(X)$,
- a subsequence $\{\phi(b_{l_i})\}_{i=1}^{\infty}$ of the sequence $\{\phi(b_l)\}_{l=1}^{\infty}$ such that $\lim\phi(b_{l_i})=z$ and $b_{l_i}\in\phi^{-1}(B(z,\frac{\varepsilon_1}{2}))\cap(V_i-\mathcal{P}(X))$,
- a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of pairwise disjoint arcs such that $\alpha_i\subset V_i-\mathcal{P}(X)$ whose end points are a_{k_i} and b_{l_i}, and $\alpha_i\cap V_{i+1}=\emptyset$,
- a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of pairwise disjoint arcs such that $\gamma_i\subset\mathcal{G}(Y)$, where $\gamma_i=\phi(\alpha_i)$, $\operatorname{diam}(\gamma_i)\geq\frac{\varepsilon_1}{2}$, and $\phi(a_{k_i})$, $\phi(b_{l_i})$ are the end points of γ_i.

We may assume that the sequence $\{\phi(a_{k_i})\}_{i=1}^{\infty}$ converges to some point $w\in Y$. Notice that the sequence $\{\gamma_i\}_{i=1}^{\infty}$ is contained in $C(Y)$. By [21, 4.17], we may suppose that $\{\gamma_i\}_{i=1}^{\infty}$ converges to some $\gamma\in C(Y)$. Since $\phi(a_{k_i})\notin B(z,\frac{\varepsilon_1}{2})$, for each $i\in\mathbb{N}$, we have that $w\neq z$. Notice that $w,z\in\gamma$. Thus, $\gamma\in C(Y)-F_1(Y)$. Since g^{-1} is a homeomorphism, we have that $\lim g^{-1}(\gamma_i)=g^{-1}(\gamma)$, where $g^{-1}(\gamma)\in C_n(X)-F_1(X)$. On the other hand, since $\lim a_{k_i}=a$, $\lim b_{l_i}=a$ and $\lim\operatorname{diam}(\alpha_i)=0$, we have that $\lim\alpha_i=\{a\}$.

Fix $i\in\mathbb{N}$. Since $a_{k_i},b_{l_i}\in\mathcal{G}(X)\cap R(X)$ and $\alpha_i\cap\mathcal{P}(X)=\emptyset$, we have that $\alpha_i=J_1\cup\cdots\cup J_{s_i}$, where $J_1,\ldots,J_{s_i}\in\mathfrak{A}_S(X)$. Thus, $\gamma_i=\phi(J_1)\cup\cdots\cup\phi(J_{s_i})$. By definition of ϕ, $\gamma_i=(J_1)_h\cup\cdots\cup(J_{s_i})_h$. Notice that $\langle(J_1)_h^\circ\cup\cdots\cup(J_{s_i})_h^\circ\rangle_1=\langle(J_1)_h^\circ\rangle_1\cup\cdots\cup\langle(J_{s_i})_h^\circ\rangle_1$. Hence,

$$q_Y^n(\langle(J_1)_h^\circ\cup\cdots\cup(J_{s_i})_h^\circ\rangle_1)-\{F_Y^n\}=q_Y^n(\langle(J_1)_h^\circ\rangle_1)\cup\cdots\cup q_Y^n(\langle(J_{s_i})_h^\circ\rangle_1)-\{F_Y^n\}.$$

By (b), we have that

$$h^{-1}(q_Y^n(\langle(J_1)_h^\circ\cup\cdots\cup(J_{s_i})_h^\circ\rangle_1)-\{F_Y^n\})\subset q_X^n(\langle J_1^\circ\rangle_n)\cup\cdots\cup q_X^n(\langle J_{s_i}^\circ\rangle_n)-\{F_X^n\}.$$

Consequently, $g^{-1}(\langle(J_1)_h^\circ\cup\cdots\cup(J_{s_i})_h^\circ\rangle_1-F_1(Y))\subset\langle J_1^\circ\cup\cdots\cup J_{s_i}^\circ\rangle_n-F_1(X)$. This implies that $g^{-1}(\langle\gamma_i\rangle_1-F_1(Y))\subset\langle\alpha_i\rangle_n-F_1(X)$ and $g^{-1}(\gamma_i)\subset\alpha_i$. Therefore, $g^{-1}(\gamma)\subset\{a\}$, a contradiction. This proves Claim 5. $\square$

Claim 6. *If $a\in\mathcal{P}(X)$ and $\{a_m\}_{m=1}^{\infty}$ is a sequence contained in $\mathcal{G}(X)$ such that $\lim a_m=a$, then $\{\phi(a_m)\}_{m=1}^{\infty}$ converges.*

We may assume that there exists a sequence $\{J_m\}_{m=1}^{\infty}$ of pairwise distinct elements of $\mathfrak{A}_S(X)$ such that $a_m\in J_m$, for each $m\in\mathbb{N}$. By [6, Lemma 8], we obtain that $\{J_m\}_{m=1}^{\infty}$ converges to $\{a\}$. Let $r_m\in J_m\cap R(X)$, for each $m\in\mathbb{N}$. Thus, $\{r_m\}_{m=1}^{\infty}$ is a sequence contained in $\mathcal{G}(X)\cap R(X)$ which converges to a. By Claim 5, there exists $z\in Y$ such that $\lim\phi(r_m)=z$. Notice that $\phi(r_m)\in(J_m)_h$, for each $m\in\mathbb{N}$. By [6, Lemma 8], we obtain that $\{(J_m)_h\}_{m=1}^{\infty}$ converges to $\{z\}$. Since $\phi(a_m)\in(J_m)_h$, $\lim\phi(a_m)=z$, for each $m\in\mathbb{N}$. This proves Claim 6.

Moreover, let $a\in\mathcal{P}(X)$, $\{a_m\}_{m=1}^{\infty}$ and $\{a'_m\}_{m=1}^{\infty}$ be sequences in $\mathcal{G}(X)$ which converge to a. By Claim 6, $\{\phi(a_m)\}_{m=1}^{\infty}$ and $\{\phi(a'_m)\}_{m=1}^{\infty}$ are convergent sequences. Now, let $b_{2k-1}=a_k$ and $b_{2k}=a'_k$, for $k\in\mathbb{N}$. Hence, $\{b_m\}_{m=1}^{\infty}$ is a sequence in $\mathcal{G}(X)$ which converges to a. By Claim 6, there exists $z\in Y$ such that $\lim\phi(b_m)=z$. Since $\{\phi(a_m)\}_{m=1}^{\infty}$ and $\{\phi(a'_m)\}_{m=1}^{\infty}$ are convergent subsequences of $\phi(\{b_m\})_{m=1}^{\infty}$, we have that $\lim\phi(a_m)=z$ and $\lim\phi(a'_m)=z$. From this, we may associate to each $a\in\mathcal{P}(X)$ a unique element of $\mathcal{P}(Y)$ which will denote by a_ϕ. Consequently, we define a map $\Phi\colon X\longrightarrow Y$ given by

$$\Phi(x) = \begin{cases} \phi(x) & \text{if } x \in \mathcal{G}(X), \\ x_\phi & \text{if } x \in \mathcal{P}(X). \end{cases}$$

Since Y satisfies similar conditions as X, the following claim is true.

Claim 7. *If $b \in \mathcal{P}(Y)$ and $\{b_m\}_{m=1}^\infty$ is a sequence contained in $\mathcal{G}(Y)$ which converges to b, then $\{\phi^{-1}(b_m)\}_{m=1}^\infty$ converges to an unique element $b_{\phi^{-1}} \in \mathcal{P}(X)$, which does not depend on the sequence $\{b_m\}_{m=1}^\infty$.*

From Claim 7, we have that Φ is one to one. Now, let $b \in \mathcal{P}(Y)$. By [5, Theorem 3.3], there exists a sequence $\{b_m\}_{m=1}^\infty$ contained in $\mathcal{G}(Y) \cap R(Y)$ which converges to b. Thus, by Claim 7, the sequence $\{\phi^{-1}(b_m)\}_{m=1}^\infty$ converges to an unique element $b_{\phi^{-1}} \in \mathcal{P}(X)$. Notice that $\Phi(b_{\phi^{-1}}) = b$. Hence, Φ is surjective. Therefore, Φ is a homeomorphism and X is homeomorphic to Y. $\square$

The proof of following result, except Case 2, is a modification of [7, Theorem 3.2].

Theorem 4.8. *Let X be a meshed continuum such that $R(X) \neq \emptyset$ and $n \geq 2$. If $|\bigcap \mathfrak{A}_S(X)| \neq 2$, then X has unique n-fold pseudo-hyperspace suspension.*

Proof. Let Y be a continuum and let $h\colon PHS_n(X) \longrightarrow PHS_n(Y)$ be a homeomorphism. By Theorem 4.3, we know that Y is a meshed continuum. Moreover, if Y is an arc or a simple closed curve, by [18, Theorem 5.7] it follows that X is homeomorphic to Y. This is a contradiction since $R(X) \neq \emptyset$. Hence, $R(Y) \neq \emptyset$. Moreover, by Theorem 4.6, we have that $|\bigcap \mathfrak{A}_S(Y)| \neq 2$. We consider two cases:

Case 1. $n \geq 3$.

Since the definition of $\mathcal{PHL}_n(X)$ is given in terms of topological properties, we have that $h(\mathcal{PHL}_n(X)) = \mathcal{PHL}_n(Y)$. This implies that $h(\mathcal{PHD}_n(X)) = \mathcal{PHD}_n(Y)$. Given $J \in \mathfrak{A}_S(X)$, by Lemma 3.7 (a), we know that $h(q_X^n(\langle J^\circ\rangle_1) - \{F_X^n\})$ is a component of $\mathcal{PHD}_n(X)$. Hence, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^n(\langle J^\circ\rangle_1) - \{F_X^n\}) = q_Y^n(\langle J_h^\circ\rangle_1) - \{F_Y^n\} \subset q_Y^n(\langle J_h^\circ\rangle_n)$. Moreover, with similar arguments for Y, we have that $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(X)\}$. Thus, (a), (b), (c) and (d) from Theorem 4.7 are satisfied.

Now we verify conditions (1) and (2) from Theorem 4.7. Let $J \in \mathfrak{A}_S(X)$ be such that $|J \cap R(X)| = 1$. We will show that if J is an arc, then J_h is an arc (and, by symmetry, the converse implication also holds). Suppose that J is an arc with end points p and q, where $q \in R(X)$. Suppose that J_h is a cycle. Let A be a subarc of J such that $p \in A$ and $q \notin A$. We know that $h(q_X^n(\langle J^\circ\rangle_1) - \{F_X^n\}) = q_Y^n(\langle J_h^\circ\rangle_1) - \{F_Y^n\}$. Let $D = q_X^n(A)$ and $E = h(D)$. Thus, $E \in q_Y^n(\langle J_h^\circ\rangle_1) - \{F_Y^n\}$. Then there exists $B \in \langle J_h^\circ\rangle_1 - F_1(Y)$ such that $q_Y^n(B) = E$. Notice that B is a subarc of J_h. Since X and Y are meshed continua, we have that $J \cap P(X) = \emptyset = J_h \cap P(Y)$. By Lemma 3.2, there exist finite graphs M in X and M_h in Y such that $J \subset M^\circ$ and $J_h \subset M_h^\circ$. By (3.1), $2n = \dim_A[C_n(M)] = \dim_A[C_n(X)] = \dim_D[PHS_n(X)] = \dim_E[PHS_n(Y)] = \dim_B[C_n(Y)]$. Thus, $B \cap R(Y) = \emptyset$. Since $C(J_h)$ is a 2-cell such that its manifold boundary is $F_1(J_h)$, we have that B has a neighborhood $\mathcal{M}$ in $\langle J_h^\circ\rangle_1 - F_1(Y)$ which is a 2-cell and B belongs to its manifold interior. Hence, $q_Y^n(\mathcal{M})$ is a neighborhood of E in $q_Y^n(\langle J_h^\circ\rangle_1) - \{F_Y^n\}$ such that $q_Y^n(\mathcal{M})$ is a 2-cell and E belongs to its manifold interior. Since $h(F_X^n) = F_Y^n$, it implies that $(q_X^n)^{-1} \circ h \circ q_Y^n(\mathcal{M})$ is a neighborhood of A in $\langle J_h^\circ\rangle_1 - F_1(Y)$ which is a 2-cell and A belongs to its manifold interior. This is a contradiction since A belongs to the manifold boundary of $C(J)$. Therefore, J_h is an arc. Moreover, by Claim 1 (g) of Theorem 4.7, we have that $|J_h \cap R(Y)| = 1$ and $J_h \in \mathfrak{A}_E(Y)$. Consequently, $J \in \mathfrak{A}_E(X)$ if and only if $J_h \in \mathfrak{A}_E(Y)$. Thus, conditions (1) and (2) from Theorem 4.7 are satisfied. Therefore, X is homeomorphic to Y.

Case 2. $n = 2$.

Notice that $h(\mathcal{PHE}_2(X)) = \mathcal{PHE}_2(Y)$. Given $J \in \mathfrak{A}_S(X)$, by Lemma 3.7 (b), there exist $J_h, K_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^2(\langle J^\circ\rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^\circ, K_h^\circ\rangle_2) - \{F_Y^2\}$. By Lemma 3.5, we have that $F_X^2 \notin \partial\mathcal{PHL}_2(X)$, $F_Y^2 \notin \partial\mathcal{PHL}_2(Y)$ and $h(\partial\mathcal{PHL}_2(X)) = \partial\mathcal{PHL}_2(Y)$. Thus,

$$h(\partial\mathcal{PHL}_2(X) \cap q_X^2(\langle J^\circ\rangle_2)) = \partial\mathcal{PHL}_2(Y) \cap q_Y^2(\langle J_h^\circ, K_h^\circ\rangle_2), \text{ and}$$
$$h(\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ\rangle_2)) = \partial\mathcal{PHL}_2(Y) - q_Y^2(\langle J_h^\circ, K_h^\circ\rangle_2).$$

Hence, $h(\mathcal{PHD}(J,J)) = \mathcal{PHD}(J_h, K_h)$. By Remark 3.12, we have that $J_h = K_h$. Consequently, $h(q_X^2(\langle J^\circ\rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^\circ\rangle_2) - \{F_Y^2\}$ and $h(q_X^2(\langle J^\circ\rangle_1) - \{F_X^2\}) \subset q_Y^2(\langle J_h^\circ\rangle_2)$. Moreover, under similar arguments for Y, we have that $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(X)\}$. Finally, by Remark 3.12 (b) and (c), conditions (1) and (2) from Theorem 4.7 are satisfied. Therefore, X is homeomorphic to Y. $\square$

The notions of framed and almost framed continua appear in [11, p. 48]. Given a continuum X, notice that $\bigcup\{J : J \text{ is a free arc in } X\}$ is dense in X if and only if $\bigcup\{J^\circ : J \text{ is a free arc in } X\}$ is dense in X. By [6, Lemma 1], we have that $\bigcup\{J : J \text{ is a free arc in } X\}$ is dense in X if and only if $\mathcal{G}(X)$ is dense in X. From this the following remark holds.

Remark 4.9. Let X be a locally connected continuum. Then X is almost framed if and only if X is almost meshed. Moreover, X is framed if and only if X is meshed distinct to a simple closed curve.

Theorem 4.10. *If X is a meshed continuum and $n \in \mathbb{N}$, then X has unique n-fold pseudo-hyperspace suspension.*

Proof. Suppose that X is a meshed continuum and let $n \in \mathbb{N}$. By [18, Theorem 5.7], we may assume that X is not a finite graph. So that we consider the following two cases:

Case 1. $R(X) \neq \emptyset$ and $n = 1$.

Since $PHS_1(X) = HS_1(X)$, by [8, Theorem 3.4] the result follows.

Case 2. $R(X) \neq \emptyset$ and $n \geq 2$.

As a consequence of Theorem 4.6 and Theorem 4.8, we have that X has unique n-fold pseudo-hyperspace suspension. $\square$

5. Locally connected continua without unique hyperspace

Given a continuum X, a nonempty closed subset K of X, and $n \in \mathbb{N}$, let

$$F_n(X,K) = \{A \in F_n(X) : A \cap K \neq \emptyset\} \text{ and}$$
$$C_n(X,K) = \{A \in C_n(X) : A \cap K \neq \emptyset\}.$$

For two disjoint continua X and Y, and given points $p \in X$ and $q \in Y$, let $X \cup_p Y$ be the continuum obtained by attaching X to Y, identifying p to q.

Given a continuum X with metric d, a closed subset A of X is said to be a Z-set in X provided that, for each $\varepsilon > 0$, there is a map $f_\varepsilon : X \longrightarrow X - A$ such that $d(f_\varepsilon(x), x) < \varepsilon$ for all $x \in X$. A map between compacta $f : X \longrightarrow Y$ is called a Z-map provided that $f(X)$ is a Z-set in Y. Let $\varepsilon > 0$ and $A \in 2^X$, the *generalized closed d-ball* in X of radius ε about A, denoted by $C_d(\varepsilon, A)$, is defined as follows: $C_d(\varepsilon, A) = \{x \in X : d(x, A) \leq \varepsilon\}$. Whenever $A = \{p\}$, we write $C(\varepsilon, p)$ instead of $C(\varepsilon, \{p\})$. A metric d for X is said to be *convex* provided that, for any $p, q \in X$, there exists $m \in X$ such that $d(p,m) = \frac{1}{2}d(p,q) = d(m,q)$. By [2, 22], if X is a locally connected continuum, then X admits a metric convex.

Given a locally connected continuum X with convex metric d and $\varepsilon > 0$, define $\Phi_\varepsilon : 2^X \longrightarrow 2^X$ by $\Phi_\varepsilon(A) = C_d(A, \varepsilon)$. By [13, Proposition 10.5], Φ_ε is a map.

Lemma 5.1. *Let $n \in \mathbb{N}$ and K, L be closed subsets of a locally connected continuum X. Then $F_m(X, L)$ is a Z-set in $C_n(X, K)$, for each $m \in \{1, \ldots, n\}$.*

Proof. Let $\varepsilon > 0$ and $m \in \{1, \ldots, n\}$. We assume that the metric for X is convex. Given $A \in C_n(X, K)$, by [13, Proposition 10.6], we have that $C_d(\frac{\varepsilon}{2}, A) \in C_n(X, K)$. Moreover, $C_d(\varepsilon, A) \notin F_m(X)$. Let $f_\varepsilon = \Phi_{\frac{\varepsilon}{2}}|_{C_n(X,K)}$. Hence, f_ε is a map from $C_n(X, K)$ to $C_n(X, K) - F_m(X, L)$. Notice that $C_d(\frac{\varepsilon}{2}, A) \subset N(\varepsilon, A)$ and, clearly, $A \subset N(\varepsilon, C_d(\frac{\varepsilon}{2}, A))$. Thus, $H(C_d(\frac{\varepsilon}{2}, A), A) < \varepsilon$, which is equivalent to $H(f_\varepsilon(A), A) < \varepsilon$. Therefore, $F_m(X, L)$ is a Z-set in $C_n(X, K)$. □

Theorem 5.2. *[1, Corollary 10.3] (Anderson's homogeneity theorem). If $h : A \longrightarrow B$ is a homeomorphism between Z-sets in a Hilbert cube $\mathcal{Q}$, then h extends to a homeomorphism of $\mathcal{Q}$ onto $\mathcal{Q}$.*

Theorem 5.3. *Let X be an almost meshed locally connected continuum and $n \in \mathbb{N}$. Suppose that there exist a contractible closed subset R of $\mathcal{P}(X)$ and pairwise disjoint nonempty open subsets $U_1, \ldots, U_{n+1}$ of X such that*

(a) $X - R = U_1 \cup \cdots \cup U_{n+1}$ and
(b) $R \subset \mathrm{cl}_X(U_i)$, for each $i \in \{1, \ldots, n+1\}$.

Then X does not have unique hyperspace $PHS_m(X)$, for each $m \leq n$.

Proof. Let $m \leq n$ and fix $p \in R$. By [6, Theorem 18], there exists a dendrite D without free arcs and disjoint to X such that $Y = X \cup_p D$ is a locally connected continuum not homeomorphic to X.

By the proof of [6, Theorem 22], we have that $C_m(Y)$ is homeomorphic to $C_m(X)$. In fact, the homeomorphism $h : C_m(X) \longrightarrow C_m(Y)$ constructed in such proof satisfies $h(A) = A$, for each $A \in C_m(X) - C_m(X, R)$. In particular, $h(F_1(\mathcal{G}(X))) = F_1(\mathcal{G}(X))$ and since X is almost meshed, we obtain that

$$h(F_1(X)) = h(\mathrm{cl}_{C_m(X)} F_1(\mathcal{G}(X))) = \mathrm{cl}_{C_m(Y)} F_1(\mathcal{G}(X)) = F_1(X).$$

Let $q^m_{X,Y} : C_m(Y) \longrightarrow C_m(Y)/F_1(X)$ be the quotient function and $q^m_{X,Y}(F_1(X)) = \{F^m_{X,Y}\}$. Since $q^m_X|_{C_m(X)-F_1(X)}$, $h|_{C_m(X)-F_1(X)}$ and $q^m_{X,Y}|_{C_m(Y)-F_1(X)}$ are homeomorphisms, $PHS_m(X) - \{F^m_X\}$ is homeomorphic to $C_m(Y)/F_1(X) - \{F^m_{X,Y}\}$. Thus, $PHS_m(X)$ is homeomorphic to $C_m(Y)/F_1(X)$.

In order to conclude, we only need to show $C_m(Y)/F_1(X)$ is homeomorphic to $PHS_m(Y)$. First, we are going to prove that $q^m_Y(C_m(Y, R \cup D))$ and $q^m_{X,Y}(C_m(Y, R \cup D))$ are Hilbert cubes. By [6, Theorem 16], we know that $C_m(Y, R \cup D)$ is a Hilbert cube. Notice that $q^m_Y(C_m(Y, R \cup D))$ is homeomorphic to $C_m(Y, R \cup D)/F_1(Y, R \cup D)$ and $q^m_{X,Y}(C_m(Y, R \cup D))$ is homeomorphic to $C_m(Y, R \cup D)/F_1(Y, R)$. By [3, Theorem 1.2 (21)], we know that D is contractible. Thus, $R \cup_p D$ is contractible. Hence, $F_1(Y, R \cup D)$ and $F_1(Y, R)$ are contractible. Since Y is locally connected, by Lemma 5.1, we have that $F_1(Y, R \cup D)$ and $F_1(Y, R)$ are Z-sets of $C_m(Y, R \cup D)$. By [10, Corollary 2.7], we have that $C_m(Y, R \cup D)/F_1(Y, R \cup D)$ and $C_m(Y, R \cup D)/F_1(Y, R)$ are Hilbert cubes. Therefore, $q^m_Y(C_m(Y, R \cup D))$ and $q^m_{X,Y}(C_m(Y, R \cup D))$ are Hilbert cubes.

Claim. *The space $\mathrm{bd}_{PHS_m(Y)}(q^m_Y(C_m(Y, R \cup D)))$ is a Z-set of $q^m_Y(C_m(Y, R \cup D))$.*

Proof of Claim. We denote the metric of $PHS_m(Y)$ by $\overline{H}$. Let $\varepsilon > 0$. Since $C_m(Y)$ is compact, we have that q^m_Y is uniformly continuous. Thus, there exists $\delta > 0$ such that if $A, B \in C_m(Y)$ with $H(A, B) < \delta$, then $\overline{H}(q^m_Y(A), q^m_Y(B)) < \frac{\varepsilon}{2}$. By [6, Theorem 22, Claim 2], there exists a map

$$g_\delta : C_m(Y, R \cup D) \longrightarrow C_m(Y, R \cup D) - \mathrm{bd}_{C_m(Y)}(C_m(Y, R \cup D))$$

such that $H(g_\delta(A), A) < \delta$, for each $A \in C_m(Y, R \cup D)$.

On the other hand, by [10, Remark 2.6], the one point sets of the Hilbert cube are Z-sets. Thus, there is a map

$$\gamma : q_Y^m(C_m(Y, R\cup D)) \longrightarrow q_Y^m(C_m(Y, R\cup D)) - \{F_Y^m\}$$

such that $\overline{H}(\gamma(B), B) < \frac{\varepsilon}{2}$, for each $B \in q_Y^m(C_m(Y, R\cup D))$. Let $f = q_Y^m|_{C_m(Y)-F_1(Y)}$. By [10, Lemma 2.8], we know that $\mathrm{bd}_{PHS_m(Y)}(q_Y^m(C_m(Y, R\cup D))) = q_Y^m(\mathrm{bd}_{C_m(Y)}(C_m(Y, R\cup D)))$. Hence, we define the map

$$f_\varepsilon : q_Y^m(C_m(Y, R\cup D)) \longrightarrow q_Y^m(C_m(Y, R\cup D)) - \mathrm{bd}_{PHS_m(Y)}(q_Y^m(C_m(Y, R\cup D)))$$

by $f_\varepsilon(B) = q_Y^m \circ g_\delta \circ f^{-1} \circ \gamma(B)$, for each $B \in q_Y^m(C_m(Y, R\cup D))$. Given $B \in q_Y^m(C_m(Y, R\cup D))$, we have that $H(g_\delta(f^{-1}(\gamma(B))), f^{-1}(\gamma(B))) < \delta$. Thus, $\overline{H}(q_X^m(g_\delta(f^{-1}(\gamma(B)))), q_X^m(f^{-1}(\gamma(B)))) < \frac{\varepsilon}{2}$. Therefore, $\overline{H}(f_\varepsilon(B), \gamma(B)) < \frac{\varepsilon}{2}$. Since $\overline{H}(\gamma(B), B) < \frac{\varepsilon}{2}$, we have that $\overline{H}(f_\varepsilon(B), B) < \varepsilon$. This proves the claim. □

Using arguments that are analogous to those of the previous claim, we obtain that $\mathrm{bd}_{C_m(Y)/F_1(X)}(q_{X,Y}^m(C_m(Y, R\cup D)))$ is a Z-set of $q_{X,Y}^m(C_m(Y, R\cup D))$.

By [10, Lemma 2.9 (b)], there exists a homeomorphism $h_1 : q_{X,Y}^m(C_m(X)) \longrightarrow q_Y^m(C_m(X))$ such that $h_1(q_{X,Y}^m(A)) = q_Y^m(A)$, for each $A \in C_m(X)$. Thus,

$$h_1(q_{X,Y}^m(\mathrm{bd}_{C_m(Y)}(C_m(Y, R\cup D)))) = q_Y^m(\mathrm{bd}_{C_m(Y)}(C_m(Y, R\cup D)))$$

and therefore,

$$h_1(\mathrm{bd}_{C_m(Y)/F_1(X)}(q_{X,Y}^m(C_m(Y, R\cup D)))) = \mathrm{bd}_{PHS_m(Y)}(q_Y^m(C_m(Y, R\cup D))).$$

Hence, $h_1|_{\mathrm{bd}_{C_m(Y)/F_1(X)}(q_{X,Y}^m(C_m(Y,R\cup D)))}$ is a homeomorphism between the Z-sets $\mathrm{bd}_{C_m(Y)/F_1(X)}(q_{X,Y}^m(C_m(Y, R\cup D)))$ and $\mathrm{bd}_{PHS_m(Y)}(q_Y^m(C_m(Y, R\cup D)))$, by Anderson's homogeneity theorem (Theorem 5.2) there exists a homeomorphism

$$h_2 : q_{X,Y}^m(C_m(Y, R\cup D)) \longrightarrow q_Y^m(C_m(Y, R\cup D))$$

such that $h_2(A) = h_1(A)$, for each $A \in \mathrm{bd}_{C_m(Y)/F_1(X)}(q_{X,Y}^m(C_m(Y, R\cup D)))$.

Let $h : C_m(Y)/F_1(X) \longrightarrow PHS_m(Y)$ be given by

$$h(A) = \begin{cases} h_1(A) & \text{if } A \in C_m(Y)/F_1(X) - q_{X,Y}^m(C_m(Y, R\cup D)), \\ h_2(A) & \text{if } A \in q_{X,Y}^m(C_m(Y, R\cup D)). \end{cases}$$

Then, h is a homeomorphism, and the theorem is proved. □

Let $m \in \mathbb{N}$ and

$$Z_3 = ([-1, 1] \times \{0\}) \cup (\bigcup\{\{-\tfrac{1}{m}\} \times [0, \tfrac{1}{m}] : m \geq 2\}) \cup (\bigcup\{\{\tfrac{1}{m}\} \times [0, \tfrac{1}{m}] : m \geq 2\}).$$

The continuum Z_3 has unique hyperspace $C_2(Z_3)$ [6, Example 39].

Example 5.4. The continuum Z_3 has unique hyperspace $PHS_2(Z_3)$ but it does not have unique hyperspace $PHS_1(Z_3) = HS_1(Z_3)$.

Notice that Z_3 is an almost meshed locally connected continuum such that $\mathcal{P}(Z_3) = \{(0,0)\}$ and Z_3 is not meshed continuum. Using Theorem 5.3, we have that Z_3 does not have unique hyperspace $PHS_1(Z_3)$.

Let $\theta = (0,0)$. Suppose that Y is a continuum such that $PHS_2(Z_3)$ and $PHS_2(Y)$ are homeomorphic. Let $h : PHS_2(Z_3) \longrightarrow PHS_2(Y)$ be a homeomorphism. By Lemma 4.2, we have that Y is locally connected. Moreover, by [18, Theorem 5.7], Y is not a finite graph. Hence, $R(Y) \neq \emptyset$. Since $|\mathfrak{A}_S(Z_3)| \geq 2$, using Lemma 3.7 (b), we have that $|\mathfrak{A}_S(Y)| \geq 2$. Also, given $J \in \mathfrak{A}_S(Z_3)$, by Lemma 3.7 (b), there exist $J_h, K_h \in \mathfrak{A}_S(Y)$ such that $h(q^2_{Z_3}(\langle J^\circ\rangle_2) - \{F^2_{Z_3}\}) = q^2_Y(\langle J^\circ_h, K^\circ_h\rangle_2) - \{F^2_Y\}$. Notice that $h(\partial\mathcal{PHL}_2(Z_3)) = \partial\mathcal{PHL}_2(Y)$ and, by Lemma 3.5, we have that $F^2_{Z_3} \notin \partial\mathcal{PHL}_2(Z_3)$ and $F^2_Y \notin \partial\mathcal{PHL}_2(Y)$. Thus,

$$h(\partial\mathcal{PHL}_2(Z_3) \cap q^2_{Z_3}(\langle J^\circ\rangle_2)) = \partial\mathcal{PHL}_2(Y) \cap q^2_Y(\langle J^\circ_h, K^\circ_h\rangle_2), \text{ and}$$
$$h(\partial\mathcal{PHL}_2(Z_3) - q^2_{Z_3}(\langle J^\circ\rangle_2)) = \partial\mathcal{PHL}_2(Y) - q^2_Y(\langle J^\circ_h, K^\circ_h\rangle_2).$$

Hence, $h(\mathcal{PHD}(J,J)) = \mathcal{PHD}(J_h, K_h)$. By Remark 3.12, we have that $J_h = K_h$. Consequently, $h(q^2_{Z_3}(\langle J^\circ\rangle_2) - \{F^2_{Z_3}\}) = q^2_Y(\langle J^\circ_h\rangle_2) - \{F^2_Y\}$ and $h(q^2_{Z_3}(\langle J^\circ\rangle_1) - \{F^2_{Z_3}\}) \subset q^2_Y(\langle J^\circ_h\rangle_2)$. Moreover, under similar arguments for Y, we have that $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(Z_3)\}$. In the same way as in the proof of Theorem 4.7, we conclude the association $J \to J_h$ is a bijection between $\mathfrak{A}_S(Z_3)$ and $\mathfrak{A}_S(Y)$, and $h(F^2_{Z_3}) = F^2_Y$. Thus, $g : C_2(Z_3) - F_1(Z_3) \longrightarrow C_2(Y) - F_1(Y)$ defined as $g = (q^2_Y)^{-1} \circ h \circ q^2_{Z_3}$ is a homeomorphism. Hence, (e) and (f) of Claim 1 from Theorem 4.7 hold. Notice that $J \cap \mathcal{P}(Z_3) = \emptyset$, for each $J \in \mathfrak{A}_S(Z_3)$. Using (f) and Lemma 3.2, we conclude $J_h \cap \mathcal{P}(Y) = \emptyset$, for each $J_h \in \mathfrak{A}_S(Y)$.

By Remark 3.12 (b) and (c), we have that

(1) Y does not have cycles and
(2) $J \in \mathfrak{A}_E(Z_3)$ if and only if $J_h \in \mathfrak{A}_E(Y)$.

Since, $J \cap \mathcal{P}(Z_3) = \emptyset$ and $J_h \cap \mathcal{P}(Y) = \emptyset$, for each $J \in \mathfrak{A}_S(Z_3)$, proceeding as in Claims 1 to 4 from Theorem 4.7, we define a homeomorphism $\phi : \mathcal{G}(Z_3) \longrightarrow \mathcal{G}(Y)$. Let

$$\mathcal{G}_{\mathcal{I}}(Z_3) = ([-1,0) \times \{0\}) \cup (\bigcup \{\{-\tfrac{1}{m}\} \times [0, \tfrac{1}{m}] : m \geq 2\})$$

and

$$\mathcal{G}_{\mathcal{D}}(Z_3) = ((0,1] \times \{0\}) \cup (\bigcup \{\{\tfrac{1}{m}\} \times [0, \tfrac{1}{m}] : m \geq 2\}).$$

Notice that $\mathcal{G}(Z_3) = \mathcal{G}_I(Z_3) \cup \mathcal{G}_D(Z_3)$. Let $\mathcal{G}_I(Y) = \phi(\mathcal{G}_I(Z_3))$ and $\mathcal{G}_D(Y) = \phi(\mathcal{G}_D(Z_3))$. Thus, $\mathcal{G}(Y) = \mathcal{G}_I(Y) \cup \mathcal{G}_D(Y)$. Let $\theta_I \in \mathrm{cl}_Y(\mathcal{G}_I(Y)) - \mathcal{G}_I(Y)$ and $\theta_D \in \mathrm{cl}_Y(\mathcal{G}_D(Y)) - \mathcal{G}_D(Y)$.

Let $\varepsilon_1 = 1$. Since $\theta_I \in \mathrm{cl}_Y(\mathcal{G}_I(Y))$, there exists $l_1 \in \mathcal{G}_I(Y)$ such that $d_Y(\theta_I, l_1) < \varepsilon_1$. Let $(I_1)_h \in \mathfrak{A}_S(Y)$ be such that $l_1 \in (I_1)_h$. Let $\varepsilon_2 = \min\{d_Y(\theta_I, (I_1)_h), \frac{1}{2}\}$ and $l_2 \in \mathcal{G}_I(Y)$ be such that $d_Y(\theta_I, l_2) < \varepsilon_2$. Let $(I_2)_h \in \mathfrak{A}_S(Y)$ be such that $l_2 \in (I_2)_h$. Notice that $(I_2)_h \neq (I_1)_h$. Let $\varepsilon_3 = \min\{d_Y(\theta_I, (I_2)_h), \frac{1}{3}\}$ and $l_3 \in \mathcal{G}_I(Y)$ be such that $d_Y(\theta_I, l_3) < \varepsilon_3$. Let $(I_3)_h \in \mathfrak{A}_S(Y)$ be such that $l_3 \in (I_3)_h$. Notice that $(I_3)_h \notin \{(I_1)_h, (I_2)_h\}$. Proceeding in a recursive way, we construct the sequence $\{l_m\}_{m=1}^\infty$ contained in $\mathcal{G}(Y)$ which converges to θ_I and a sequence of pairwise different elements $\{(I_m)_h\}_{m=1}^\infty$ contained in $\mathfrak{A}_S(Y)$ such that $l_m \in (I_m)_h \subset \mathcal{G}_I(Y)$, for each $m \in \mathbb{N}$. Using [6, Lemma 8], we have that $\{(I_m)_h\}_{m=1}^\infty$ converges to $\{\theta_I\}$. Analogously, there exists a sequence of pairwise different elements $\{(D_m)_h\}_{m=1}^\infty$ contained in $\mathfrak{A}_S(Y)$ which converges to $\{\theta_D\}$ and $(D_m)_h \subset \mathcal{G}_D(Y)$, for each $m \in \mathbb{N}$. Thus, $\{(I_m)_h \cup (D_m)_h\}_{m=1}^\infty$ converges to $\{\theta_I, \theta_D\}$.

On the other hand, given $m \in \mathbb{N}$, by Lemma 3.7 (b), there exist $L_m, N_m \in \mathfrak{A}_S(Z_3)$ such that $g^{-1}(\langle (I_m)^\circ_h, (D_m)^\circ_h\rangle_2) = \langle L^\circ_m, N^\circ_m\rangle_2 - \{F^2_X\}$. Since $(I_m)_h \neq (D_m)_h$, by Theorem 4.7 (a), we have that $L_m \neq N_m$. Thus, $g^{-1}(\langle (I_m)^\circ_h, (D_m)^\circ_h\rangle_2) = \langle L^\circ_m, N^\circ_m\rangle_2$. Notice that we may suppose that $\{L_m\}_{m=1}^\infty$ and

$\{N_m\}_{m=1}^{\infty}$ are two sequences of pairwise different elements of $\mathfrak{A}_S(Z_3)$. Let $a_m \in L_m$, for each $m \in \mathbb{N}$. Since Z_3 is compact, we may suppose that $\{a_m\}_{m=1}^{\infty}$ converges to a, for some $a \in Z_3$. By [6, Lemma 8], we have that $\{L_m\}_{m=1}^{\infty}$ converges to $\{a\}$. Hence, by [9, Theorem 4.1], $a \in \mathcal{P}(Z_3)$. Thus, $a = \theta$. Analogously, we can prove that $\{N_m\}_{m=1}^{\infty}$ converges to $\{\theta\}$. Thus, $\{L_m \cup N_m\}_{m=1}^{\infty}$ converges to $\{\theta\}$.

Given $m \in \mathbb{N}$, notice that $g^{-1}(\mathrm{cl}_{C_2(Y)-F_1(Y)}(\langle (I_m)_h^{\circ}, (D_m)_h^{\circ}\rangle_2)) \subset \langle L_m, N_m\rangle_2$, and therefore, $g^{-1}((I_m)_h \cup (D_m)_h) \subset L_m \cup N_m$. Suppose that $\theta_I \neq \theta_D$. Thus, $\{g^{-1}((I_m)_h \cup (D_m)_h)\}_{m=1}^{\infty}$ converges to $g^{-1}(\{\theta_I, \theta_D\})$. Hence, $g^{-1}(\{\theta_I, \theta_D\}) \subset \{\theta\}$, a contradiction. Therefore, $\theta_I = \theta_D$. Since $\mathrm{cl}_Y(\mathcal{G}(Y)) = \mathrm{cl}_Y(\mathcal{G}_I(Y)) \cup \mathrm{cl}_Y(\mathcal{G}_D(Y))$, we have that $|\mathrm{cl}_Y(\mathcal{G}(Y)) - \mathcal{G}(Y)| = 1$. Let $\theta_h \in \mathrm{cl}_Y(\mathcal{G}(Y)) - \mathcal{G}(Y)$ and $\Phi : Z_3 \longrightarrow Y$ be defined as

$$\Phi(z) = \begin{cases} \phi(z) & \text{if } z \in \mathcal{G}(Z_3), \\ \theta_h & \text{if } z = \theta. \end{cases}$$

Hence, Φ is an embedding from Z_3 into Y. By definition of Φ, we know that $\Phi(Z_3) = \mathrm{cl}_Y(\mathcal{G}(Y))$. Notice that, $\Phi(Z_3) \cap \mathcal{P}(Y) = \{\theta_h\}$. This implies that $\mathcal{P}(Y)$ is a subcontinuum of Y. Let

$$\mathfrak{T}_{Z_3} = \mathrm{int}_{C_2(Z_3)-F_1(Z_3)}((C_2(Z_3) - F_1(Z_3)) - \mathfrak{F}_2(Z_3))$$

and

$$\mathfrak{T}_Y = \mathrm{int}_{C_2(Y)-F_1(Y)}((C_2(Y) - F_1(Y)) - \mathfrak{F}_2(Y)).$$

Notice that $g(\mathfrak{T}_{Z_3}) = \mathfrak{T}_Y$. Using the same arguments as in [6, Example 39], we have that $\mathfrak{T}_{Z_3}$ is disconnected and, if $Y \neq \mathrm{cl}_Y(\mathcal{G}(Y))$, then $\mathfrak{T}_Y$ is pathwise connected. Hence, $Y = \mathrm{cl}_Y(\mathcal{G}(Y))$. Therefore, Z_3 has unique hyperspace $PHS_2(Z_3)$.

Theorem 5.5. *Let X be a locally connected continuum that is not almost meshed. Suppose that there exist $p \in \mathcal{P}(X)$ and $\varepsilon > 0$ such that $B(p, 2\varepsilon) \subset \mathcal{P}(X)$ and $C_d(\varepsilon, p)$ is contractible. Then, for every $n \in \mathbb{N}$, X does not have unique hyperspace $PHS_n(X)$.*

Proof. By [6, Theorem 18], there exists a dendrite D without free arcs and disjoint to X such that $Y = X \cup_p D$ is a locally connected continuum not homeomorphic to X.

Let $E = C_d(\varepsilon, p)$. By Lemma 5.1, we have that $F_1(E)$ is a Z-set of $C_n(X, E)$ and $C_n(Y, E \cup D)$. Using [6, Theorem 22, Claim 2], we have that $\mathrm{bd}_{C_n(X)}(C_n(X, E)) \cup F_1(E)$ is a Z-set of $C_n(X, E)$ and $\mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_1(E)$ is a Z-set of $C_n(Y, E \cup D)$. Moreover, by [6, Lemma 19], we have that $\mathrm{bd}_{C_n(X)}(C_n(X, E)) \cup F_1(E) = \mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_1(E)$. Hence, the identity map

$$\mathrm{id} : \mathrm{bd}_{C_n(X)}(C_n(X, E)) \cup F_1(E) \longrightarrow \mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_1(E)$$

is a well-defined homeomorphism. By [6, Theorem 16], we know that $C_n(X, E)$ and $C_n(Y, E \cup D)$ are Hilbert cubes. Thus, by Anderson's homogeneity theorem (Theorem 5.2), the identity map can be extended to a homeomorphism $h_1 : C_n(X, E) \longrightarrow C_n(Y, E \cup D)$.

We define $h : C_n(X) \longrightarrow C_n(Y)$ by

$$h(A) = \begin{cases} h_1(A) & \text{if } A \in C_n(X, E), \\ A & \text{if } A \in C_n(X) - C_n(X, E). \end{cases}$$

Notice h is a homeomorphism such that $h(F_1(X)) = F_1(X)$.

Let $q^n_{X,Y} : C_n(Y) \longrightarrow C_n(Y)/F_1(X)$ be the quotient function and $q^n_{X,Y}(F_1(X)) = \{F^n_{X,Y}\}$. Since $q^n_X|_{C_n(X)-F_1(X)}$, $h|_{C_n(X)-F_1(X)}$ and $q^n_{X,Y}|_{C_n(Y)-F_1(X)}$ are homeomorphisms, then $PHS_n(X) - \{F^n_X\}$ is homeomorphic to $C_n(Y)/F_1(X) - \{F^n_{X,Y}\}$. Thus, $PHS_n(X)$ is homeomorphic to $C_n(Y)/F_1(X)$.

We will prove that $C_n(Y)/F_1(X)$ is homeomorphic to $PHS_n(Y)$. First, we are going to prove that $q^n_Y(C_n(Y, E \cup D))$ and $q^n_{X,Y}(C_n(Y, E \cup D))$ are Hilbert cubes. Notice that $q^n_Y(C_n(Y, E \cup D))$ is homeomorphic to $C_n(Y, D)/F_1(Y, E \cup D)$ and $q^n_{X,Y}(C_n(Y, E \cup D))$ is homeomorphic to $C_n(Y, E \cup D)/F_1(Y, E)$. By [3, Theorem 1.2 (21)], we know that D is contractible. Thus, $E \cup_p D$ is contractible. Hence, $F_1(Y, E \cup D)$ and $F_1(Y, E)$ are contractible. Since Y is locally connected, by Lemma 5.1, we have that $F_1(Y, E \cup D)$ and $F_1(E)$ are Z-sets of $C_n(Y, E \cup D)$. By [10, Corollary 2.7], we have that $C_n(Y, E \cup D)/F_1(Y, E \cup D)$ and $C_n(Y, E \cup D)/F_1(Y, E)$ are Hilbert cubes. Therefore, $q^n_Y(C_n(Y, E \cup D))$ and $q^n_{X,Y}(C_n(Y, E \cup D))$ are Hilbert cubes.

Similar to the Claim from Theorem 5.3 was proved, the following Claim can be shown.

Claim. *The space* $\mathrm{bd}_{PHS_n(Y)}(q^n_Y(C_n(Y, E \cup D)))$ *is a* Z*-set of* $q^n_Y(C_n(Y, E \cup D))$ *and the set* $\mathrm{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y, E \cup D)))$ *is a* Z*-set of* $q^n_{X,Y}(C_n(Y, E \cup D))$.

Using [10, Lemma 2.9(b)], the function $f: q^n_{X,Y}(C_n(X)) \longrightarrow q^n_Y(C_n(X))$ defined by $f(q^n_{X,Y}(A)) = q^n_Y(A)$, for each $A \in C_n(X)$, is a homeomorphism. Thus,

$$f(q^n_{X,Y}(\mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)))) = q^n_Y(\mathrm{bd}_{C_n(Y)}(C_n(Y, E \cup D)))$$

and therefore,

$$f(\mathrm{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y, E \cup D)))) = \mathrm{bd}_{PHS_n(Y)}(q^n_Y(C_n(Y, E \cup D))).$$

Hence, $f|_{\mathrm{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_f(Y,E\cup D)))}$ is a homeomorphism between Z-sets $\mathrm{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y, E \cup D)))$ and $\mathrm{bd}_{PHS_n(Y)}(q^n_Y(C_n(Y, E \cup D)))$, by Anderson's homogeneity theorem (Theorem 5.2) there exists a homeomorphism $g : q^n_{X,Y}(C_n(Y, E \cup D)) \longrightarrow q^n_Y(C_n(Y, E \cup D))$ such that $g(A) = f(A)$, for each $A \in \mathrm{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y, E \cup D)))$.

Let $\overline{h} : C_n(Y)/F_1(X) \longrightarrow PHS_n(Y)$ be given by

$$\overline{h}(A) = \begin{cases} f(A) & \text{if } A \in C_n(Y)/F_1(X) - q^n_{X,Y}(C_n(Y, E \cup D)), \\ g(A) & \text{if } A \in q^n_{X,Y}(C_n(Y, E \cup D)). \end{cases}$$

Then, $\overline{h}$ is a homeomorphism. Therefore, X does not have unique hyperspace $PHS_n(X)$. □

Question 5.6. *Is Theorem 5.3 still true if we remove the assumption that* R *is contractible?*

Regarding to Theorem 5.5, we ask:

Question 5.7. *Let* X *be a locally connected continuum such that* X *is not almost meshed and let* $n \in \mathbb{N}$*. Does* X *have unique hyperspace* $PHS_n(X)$*?*

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