On the uniqueness of the $n$-fold pseudo-hyperspace suspension for locally connected continua

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Let $X$ be a metric continuum. Let $n$ be a positive integer, we consider the hyperspace $C_n(X)$ of all nonempty closed subsets of $X$ with at most $n$ components and $F_1(X) = \{\{x\}: x \in X\}$. The $n$-fold pseudo-hyperspace suspension of $X$ is the quotient space $C_n(X)/F_1(X)$ and it is denoted by $PHS_n(X)$. In this paper we prove that: (1) if $X$ is a meshed continuum and $Y$ is a continuum such that $PHS_n(X)$ is homeomorphic to $PHS_n(Y)$, then $X$ is homeomorphic to $Y$, for each $n > 1$. (2) There are locally connected continua without unique hyperspace $PHS_n(X)$.

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1. Introduction

A continuum is a nondegenerate compact connected metric space. The set of positive integers is denoted by $\mathbb{N}$. Given a continuum $X$ and $n \in \mathbb{N}$, we consider the following hyperspaces of $X$:

\begin{align*}
2^X &= \{A \subseteq X: A \text{ is a nonempty closed subset of } X\}, \\
C_n(X) &= \{A \in 2^X: A \text{ has at most } n \text{ components}\}, \\
F_n(X) &= \{A \in 2^X: A \text{ has at most } n \text{ points}\} \text{ and} \\
C(X) &= C_1(X).
\end{align*}

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All the hyperspaces considered are metrized by the Hausdorff metric \( H \) [13, Theorem 2.2].

Related to a continuum \( X \), Sam B. Nadler, Jr. [20], introduced the hyperspace suspension of a continuum, \( HS(X) \), as the quotient space \( C(X)/F_1(X) \). Twenty five years later in [15], Sergio Macías gave a generalization of it, defining the \( n \)-fold hyperspace suspension of a continuum, \( H\text{S}_n(X) \), as the quotient space \( C_n(X)/F_n(X) \). In 2008, Juan C. Macías [16] introduced the \( n \)-fold pseudo-hyperspace suspension of a continuum, \( PHS\text{S}_n(X) \), as the quotient space \( C_n(X)/F_1(X) \). Given a continuum \( X \), let \( \mathcal{H}(X) \) be any of the hyperspaces \( 2^X, C_n(X), F_n(X), HS\text{S}_n(X), \) or \( PHS\text{S}_n(X) \). The continuum \( X \) is said to have unique hyperspace \( \mathcal{H}(X) \) provided that the following implication holds: if \( Y \) is a continuum and \( \mathcal{H}(X) \) is homeomorphic to \( \mathcal{H}(Y) \), then \( X \) is homeomorphic to \( Y \).

One of the problems that has been widely studied lately on the theory of continua and their hyperspaces is to search for continua with unique hyperspace \( \mathcal{H}(X) \). The problem of finding conditions for \( X \) in order that \( X \) has unique \( \mathcal{H}(X) \) has been widely studied for several families of continua, especially for finite graphs, meshed continua and almost meshed locally connected continua. In [12], Alejandro Illanes proved that finite graphs have unique \( C_n(X) \) and later, in [6] Rodrigo Hernández-Gutiérrez, A. Illanes and Verónica Martínez-de-la-Vega studied the uniqueness of the hyperspace \( C_n(X) \) for locally connected continua and proved that meshed continua have unique \( C_n(X) \). Later, adopting some of the techniques presented in [12] it was proved that finite graphs have unique \( HS\text{S}_n(X) \), see [7]. Later, in [8] María de J. López jointly with the second and third authors proved that framed continua have unique \( HS\text{S}_n(X) \). In relation to this topic, Germán Montero-Rodriguez, M. de J. López jointly with the second and third authors proved that finite graphs have unique hyperspace \( F_n(X)/F_1(X) \), for each \( n \geq 4 \), see [19, Theorem 3.8]. Recently, in [18] it was proved that finite graphs have unique \( PHS\text{S}_n(X) \). Following the study of this property in the hyperspace \( PHS\text{S}_n(X) \), in the present work we prove that

1. Meshed continua have unique \( n \)-fold pseudo-hyperspace suspension, for \( n > 1 \), see Theorem 4.8.
2. There are almost meshed locally connected continua without unique \( n \)-fold pseudo-hyperspace suspension, see Theorem 5.3.
3. There exists an almost meshed locally connected continuum that is not meshed with unique 2-fold pseudo-hyperspace suspension, see Example 5.4.
4. There exist locally connected continua that are not almost meshed without unique \( n \)-fold pseudo-hyperspace suspension, see Theorem 5.5.

2. Definitions

Let \( X \) be a continuum. Given a subset \( A \) of \( X \), \( \text{int}_X(A) \), \( \text{cl}_X(A) \), and \( \text{bd}_X(A) \), denote the interior, the closure, and the boundary of \( A \) in \( X \), respectively, and when there is no possible confusion with the underlying continuum in which \( A \) lies, we simply will use \( A^o \) instead of \( \text{int}_X(A) \). Through this paper, we write \( d \) for the metric associated to the continuum \( X \). Let \( \varepsilon > 0 \) and \( p \in X \); the set \( \{ x \in X : d(p, x) < \varepsilon \} \) is denoted by \( B_X(p, \varepsilon) \), when there is no possible confusion with the underlying continuum in which \( d \) lies, we use \( B(p, \varepsilon) \) instead of \( B_X(p, \varepsilon) \). The Hausdorff metric \( H \) is defined as follows: for each \( A, B \in 2^X \),

\[
H(A, B) = \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\},
\]

where \( N(\varepsilon, A) = \{ x \in X : d(x, A) < \varepsilon \} \). The hyperspaces \( F_n(X) \) and \( C_n(X) \) are called the \( n \)-fold symmetric product of \( X \) and the \( n \)-fold hyperspace of \( X \), respectively. The cardinality of \( A \) is denoted by \( |A| \). Let \( p \in X \) and \( \beta \) be a cardinal number. We say that \( p \) has order less than or equal to \( \beta \) in \( X \), written \( \text{ord}(p, X) \leq \beta \), whenever \( p \) has a basis of neighborhoods \( \mathcal{B} \) in \( X \) such that the cardinality of \( \text{bd}_X(U) \) is less than or equal to \( \beta \), for each \( U \in \mathcal{B} \). We say that \( p \) has order equal to \( \beta \) in \( X \) (\( \text{ord}(p, X) = \beta \)) provided that \( \text{ord}(p, X) \leq \beta \) and \( \text{ord}(p, X) \neq \alpha \) for any cardinal number \( \alpha < \beta \). Let \( E(X) = \{ x \in X : \text{ord}(x, X) = 1 \} \), \( O(X) = \{ x \in \)}
X: \( \text{ord}(x, X) = 2 \), and \( R(X) = \{ x \in X : \text{ord}(x, X) \geq 3 \} \). The elements of \( E(X) \) (respectively, \( O(X) \) and \( R(X) \)) are called end points (respectively, ordinary points and ramification points) of \( X \). A map is a continuous function.

A finite graph is a continuum which is a finite union of arcs such that every two of them meet at a subset of their end points.

Given a continuum \( X \), a free arc in \( X \) is an arc \( J \) with end points \( p \) and \( q \) such that \( J - \{ p, q \} \) is an open subset of \( X \). A maximal free arc in \( X \) is a free arc in \( X \) that is maximal with respect to the inclusion. A cycle in \( X \) is a simple closed curve \( J \) in \( X \) such that \( J - \{ a \} \) is an open subset of \( X \), for some \( a \in J \). Notice that if \( X \) is not a simple closed curve and \( J \) is a cycle in \( X \), then \( J \cap R(X) = \{ a \} \). Let

\[
\mathcal{A}_R(X) = \{ J \subset X : J \text{ is a cycle in } X \},
\]

\[
\mathcal{A}_E(X) = \{ J \subset X : J \text{ is a maximal free arc in } X \text{ and } |J \cap R(X)| = 1 \},
\]

\[
\mathcal{A}_S(X) = \{ J \subset X : J \text{ is a maximal free arc in } X \} \cup \mathcal{A}_R(X),
\]

\[
\mathcal{G}(X) = \{ x \in X : x \text{ has a neighborhood in } X \text{ which is a finite graph} \} \text{ and }
\]

\[
\mathcal{P}(X) = X - \mathcal{G}(X).
\]

According to [6, p. 1584] a continuum \( X \) is said to be almost meshed whenever the set \( \mathcal{G}(X) \) is dense in \( X \). An almost meshed continuum \( X \) is meshed provided that \( X \) has a basis of neighborhoods \( \mathcal{B} \) such that \( U - \mathcal{P}(X) \) is connected, for each \( U \in \mathcal{B} \).

Given a continuum \( X \) and \( n \in \mathbb{N} \), the function \( q^n_X : C_n(X) \to \text{PHS}_n(X) \) is the natural projection, and \( F^n_X \) denotes the element \( q^n_X(F_1(X)) \). Notice that

\[
q^n_X|_{C_n(X)-F_1(X)} : C_n(X) - F_1(X) \to \text{PHS}_n(X) - \{ F^n_X \} \text{ is a homeomorphism.} \tag{2.1}
\]

Given \( m \in \mathbb{N} \) and \( U_1, \ldots, U_m \) subsets of \( X \), let

\[
\langle U_1, \ldots, U_m \rangle_n = \{ A \in C_n(X) : A \subset U_1 \cup \cdots \cup U_m \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \in \{1, \ldots, m\} \}.
\]

By [13, Theorem 1.2], it is known that the family of all sets \( \langle U_1, \ldots, U_m \rangle_n \), where each \( U_i \) is an open subset of \( X \), forms a basis for the topology in \( C_n(X) \).

A topological manifold \( M \) (possibly with boundary) of dimension \( n < \infty \) is a metrizable topological space \( M \) such that each point \( x \) in \( M \) admits an open neighborhood \( U \) and a homeomorphism \( \kappa : U \to \kappa(U) \) onto an open subset of the Euclidean half-space \( \mathbb{R}^+_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0 \} \). The points \( x \) in \( M \) that correspond to points \( \kappa(x) \) in the hyperplane \( \{(x_1, \ldots, x_n) \in \mathbb{R}^+_n : x_1 = 0 \} \) form the manifold boundary of \( M \). The manifold interior of \( M \) is defined as the complement of the manifold boundary on \( M \), as in [14, p. 7].

We use the following notations: \( \dim[X] \) stands for the dimension of \( X \), and \( \text{dim}_p[X] \) stands for the dimension of \( X \) at the point \( p \in X \), as in [22, p. 5].

Given a continuum \( X \) and \( n \in \mathbb{N} \), let

\[
\mathcal{L}_n(X) = \{ A \in C_n(X) : A \text{ has a neighborhood in } C_n(X) \text{ which is a } 2n\text{-cell} \},
\]

\[
\partial\mathcal{L}_n(X) = \{ A \in C_n(X) : A \text{ has a neighborhood } \mathcal{N} \text{ in } C_n(X) \text{ such that } \mathcal{N} \text{ is a } 2n\text{-cell and } A \text{ belongs to the manifold boundary of } \mathcal{N} \},
\]

\[
\mathcal{D}_n(X) = \{ A \in C_n(X) : A \notin \mathcal{L}_n(X) \text{ and } A \text{ has a basis of neighborhoods } \mathcal{A} \text{ in } C_n(X) \text{ such that for each } \mathcal{U} \in \mathcal{A}, \dim[\mathcal{U}] = 2n \text{ and } \mathcal{U} \cap \mathcal{L}_n(X) \text{ is arcwise connected} \}.
\]
\[ \mathcal{PHL}_n(X) = \{ B \in PHS_n(X) : B \text{ has a neighborhood in } PHS_n(X) \text{ which is a } 2n\text{-cell} \}, \]
\[ \partial \mathcal{PHL}_n(X) = \{ B \in PHS_n(X) : B \text{ has a neighborhood } N \text{ in } PHS_n(X) \text{ such that } N \text{ is a } 2n\text{-cell and } B \text{ belongs to the manifold boundary of } N \}, \]
\[ \mathcal{PHD}_n(X) = \{ B \in PHS_n(X) : B \notin \mathcal{PHL}_n(X) \text{ and } B \text{ has a basis of neighborhoods } B \text{ in } PHS_n(X) \text{ such that for each } V \in B, \dim[V] = 2n \]
\[ \text{and } V \cap \mathcal{PHL}_n(X) \text{ is arcwise connected} \}, \]
\[ \mathcal{PHE}_n(X) = \{ B \in PHS_n(X) : \dim_B[PHS_n(X)] = 2n \}. \]

By (2.1), we have the following remark.

**Remark 2.1.** Let \( X \) be a continuum and \( n \in \mathbb{N} \). Then

(a) \( q^n_X(L_n(X) - F_1(X)) = \mathcal{PHL}_n(X) - \{ F^n_X \} \),
(b) \( q^n_X(\partial L_n(X) - F_1(X)) = \partial \mathcal{PHL}_n(X) - \{ F^n_X \} \) and
(c) \( q^n_X(D_n(X) - F_1(X)) = \mathcal{PHD}_n(X) - \{ F^n_X \} \).

3. Preliminary results

**Lemma 3.1.** Let \( X \) be a locally connected continuum and \( J, K \in \mathfrak{A}_S(X) \). Then

(a) \( J^o \cap R(X) = \emptyset \),
(b) \( \text{bd}_X(K) \subset R(X) \) and
(c) if \( J^o \cap K \neq \emptyset \), then \( J = K \).

**Proof.** (a) Take \( p \in J^o \). Let \( U \) be an open subset of \( X \) such that \( p \in U \). Then, there exists an arc \( L \) in \( J \) such that \( p \in \text{int}_J(L) \subset L \subset U \cap J^o \). Then \( \text{int}_J(L) \) is an open connected subset of \( X \). Moreover, \( \text{bd}_X(\text{int}_J(L)) \subset L - \text{int}_J(L) \) and \( L - \text{int}_J(L) \) has at most 2 elements. Thus, \( p \notin R(X) \). Consequently, \( J^o \cap R(X) = \emptyset \).

(b) If \( R(X) = \emptyset \), by [21, 8.40], we have that \( X \) is an arc or a simple closed curve and the result follows. Suppose that \( R(X) \neq \emptyset \). Let \( p \in \text{bd}_X(K) \) and \( \mathfrak{B} \) be a basis of neighborhoods of \( p \) in \( X \).

**Case 1.** \( K \) is a cycle.

Let \( q \in X - K \) and \( L \) be an arc in \( X \) with end points \( p \) and \( q \). Since \( K - \{ p \} \) is an open subset of \( X \), we have that \( K \cap L = \{ p \} \). Let \( r = d(p, q) \) and \( U \in \mathfrak{B} \) be such that \( U \subset B(p, r) \) and \( K \not\subset U \). Notice that \( \text{bd}_X(U) \) has at least 3 elements. This implies that \( p \notin E(X) \cup O(X) \). Therefore, \( p \in R(X) \).

**Case 2.** \( K \) is an arc.

Notice that \( p \) is an end point of \( K \). Let \( a \) be the other end point of \( K \). Let \( s = \min\{ \frac{\text{diam}(K)}{2}, \frac{d(a, p)}{2} \} \) and let \( W \) be an open connected subset of \( X \) such that \( p \in W \subset B(p, s) \). By [21, 8.26], \( W \) is arcwise connected. Let \( q \in W - K \) and \( L \) be an arc in \( W \) with end points \( p \) and \( q \). Notice that \( K \not\subset L \) and \( a \notin L \). Since \( K - \{ a, p \} \) is an open subset of \( X \), we have that \( K \cap L \subset \{ a, p \} \). Hence, \( K \cap L = \{ p \} \). Suppose that there exists \( \delta > 0 \) such that \( B(p, \delta) \subset K \cup L \). Let \( C_p \) be the component of \( B(p, \delta) \) such that \( p \in C_p \) and \( L_p = \text{cl}_X(C_p) \). Hence, \( L_p \) is an arc. Since \( X \) is locally connected, \( C_p \) is an open subset of \( X \). Let \( l, k \) be the end points of \( L_p \), where \( l \in L \) and \( k \in K \). Notice that \( K \cup L_p - \{ a, l \} = C_p \cup (K - \{ a, p \}) \). Thus, \( K \cup L_p \) is a free arc. This contradicts the maximality of \( K \). Therefore, for any \( \varepsilon > 0 \), \( B(p, \varepsilon) \not\subset K \cup L \). This implies that there exists an arc \( M \) such that \( (K \cup L) \cap M = \{ p \} \). Let \( z \) be the other end point of \( M \) and \( r = \min\{ d(p, a), d(p, q), d(p, z) \} \). Thus, there exists \( V \in \mathfrak{B} \) such that \( V \subset B(p, r) \). Notice that \( \text{bd}_X(V) \) has at least 3 elements. This implies that \( p \notin E(X) \cup O(X) \). Therefore, \( p \in R(X) \).
(c) Given \( p \in J^0 \cap K \), by (a), we know that \( p \notin R(X) \). Using (b), we have that \( p \in K^0 \). Hence, \( J^0 \cap K^0 = J^0 \cap K \). Consequently, \( J^0 \cap K \) is a nonempty open and closed subset of the connected set \( J^0 \).
Thus, \( J^0 = J^0 \cap K \) and \( J \subset K \). By the maximality of \( J \), we have that \( J = K \). \( \square \)

In [17], Verónica Martínez-de-la-Vega computed the dimension of the \( n \)-fold hyperspace for a finite graph \( G \) with the following formula

\[
\dim_A[C_n(G)] = 2n + \sum_{p \in A \cap R(G)} (\text{ord}(p, G) - 2), \text{ where } A \in C_n(G). \tag{3.1}
\]

**Lemma 3.2.** [6, Theorem 4] Let \( X \) be a locally connected continuum, \( n \in \mathbb{N} \) and \( A \in C_n(X) \). Then the following conditions are equivalent.

(a) \( \dim_A[C_n(X)] \) is finite,
(b) there exists a finite graph \( G \) contained in \( X \) such that \( A \subset \text{int}_X(G) \),
(c) \( A \cap \mathcal{P}(X) = \emptyset \).

**Lemma 3.3.** [6, Lemma 28] Let \( X \) be a locally connected continuum and \( n \geq 3 \). Then \( \mathcal{D}_n(X) = \{ A \in C_n(X) : A \text{ is connected and there exists } J \in \mathcal{A}_S(X) \text{ such that } A \subset \text{int}_X(J) \} \).

The proof of following result is a modification of [7, Lemma 2.3].

**Lemma 3.4.** Let \( X \) be a locally connected continuum and \( n \in \mathbb{N} \). If \( A \in C_n(X) - F_1(X) \) and \( A \cap R(X) \neq \emptyset \), then \( \dim_{q_X(A)}[PHS_n(X)] \geq 2n + 1 \).

**Proof.** From (2.1), we have that \( \dim_{q_X(A)}[PHS_n(X)] = \dim_A[C_n(X)] \). If \( \dim_A[C_n(X)] \) is not finite, the result follows. Suppose that \( \dim_A[C_n(X)] \) is finite. By Lemma 3.2, there exists a finite graph \( G \) such that \( A \subset \text{int}_X(G) \). Notice that \( \dim_A[C_n(X)] = \dim_A[C_n(G)] \). Since \( A \cap R(X) \neq \emptyset \) and \( A \subset \text{int}_X(G) \), we have that \( A \cap R(G) \neq \emptyset \). Thus, by (3.1), \( \dim_A[C_n(G)] \geq 2n + 1 \). Therefore, the result follows. \( \square \)

The proof of following result is a modification of [7, Lemma 2.4].

**Lemma 3.5.** Let \( X \) be a locally connected continuum such that \( R(X) \neq \emptyset \) and \( n \in \mathbb{N} \). Then for each neighborhood \( U \) of \( F^n_X \) in \( PHS_n(X) \), \( \dim[U] \geq 2n + 1 \).

**Proof.** Let \( U \) be an open neighborhood of \( F^n_X \) in \( PHS_n(X) \) and \( \mathcal{V} = (q_X)^{-1}(U) \). Then \( \mathcal{V} \) is an open subset of \( C_n(X) \). Fix a point \( p \in R(X) \). Since \( \{p\} \in \mathcal{V} \), there exists \( r > 0 \) such that \( B_{C_n(X)}(\{p\}, r) \subset \mathcal{V} \). Let \( C \) be the component of \( B(p, r) \) containing \( p \). Since \( C \) is an open connected subset of \( X \), by [21, 8.26], \( C \) is arcwise connected. Hence, there exists an arc \( A \) such that \( p \in A \subset B(p, r) \). Notice that \( A \in \mathcal{V} \). Thus, \( q_X(A) \in U \). Therefore, by Lemma 3.4, \( \dim_{q_X(A)}[U] \geq 2n + 1 \). \( \square \)

The proof of following result is a modification of [7, Lemma 2.9 (b)].

**Lemma 3.6.** Let \( X \) be a locally connected continuum such that \( R(X) \neq \emptyset \), \( n \in \mathbb{N} \) with \( n \geq 3 \). Then \( \mathcal{PD}_n(X) = \{ q^n_X(A) \in PHS_n(X) : A \in C(X) - F_1(X) \text{ and } A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset \} \).

**Proof.** Given \( B \in \mathcal{PD}_n(X) \), there exists \( A \in C_n(X) \) such that \( B = q^n_X(A) \). Since \( R(X) \neq \emptyset \), by Lemma 3.5, \( B \neq F^n_X \), thus, \( A \notin F_1(X) \). Moreover, by Remark 2.1 (c), \( A \in \mathcal{D}_n(X) \). By Lemma 3.3, \( A \in C(X) - F_1(X) \) and \( A \subset \text{int}_X(J) \), for some \( J \in \mathcal{A}_S(X) \). This implies that \( A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset \).
On the other hand, to prove the opposite inclusion, let \( A \in C(X) - F_1(X) \) be such that \( A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset \). In order to prove that \( q_X^R(A) \in \mathcal{PHD}_n(X) \), by Remark 2.1 (c), it will be enough to prove that \( A \in D_n(X) \). By Lemma 3.2, there exists a finite graph \( G \) contained in \( X \) such that \( A \subset \text{int}_X(G) \). Since \( A \cap R(X) = \emptyset \), we have that \( A \cap R(G) = \emptyset \). Thus, there exists a free arc \( L \) in \( G \) such that \( A \subset \text{int}_G(L) \).

Since \( A \subset \text{int}_X(G) \), \( A \subset \text{int}_X(L) \) so we may assume that \( L \subset \text{int}_X(G) \). This implies that \( L \) is a free arc in \( X \). By [6, Lemma 10], there exists \( J \in \mathfrak{A}_S(X) \) such that \( L \subset J \). Therefore, by Lemma 3.3, \( A \in D_n(X) \). \( \square \)

The proof of following result is a modification of [7, Lemma 2.10 (a) and (d)].

**Lemma 3.7.** Let \( X \) be a locally connected continuum such that \( R(X) \neq \emptyset \) and \( n \in \mathbb{N} \).

(a) For \( n \geq 3 \), the components of \( \mathcal{PHD}_n(X) \) are the sets \( q_X^R((J^o)_1) - \{F_X^n\} \), where \( J \in \mathfrak{A}_S(X) \).

(b) The components of \( \mathcal{PH}E_n(X) \) are the sets \( q_X^n((J^o)_1) - \{F_X^n\} \), where \( J_1, \ldots, J_m \in \mathfrak{A}_S(X) \) and \( m \leq n \).

**Proof.** (a) By Lemma 3.6, \( \mathcal{PHD}_n(X) = \bigcup \{q_X^R((J^o)_1) - \{F_X^n\} : J \in \mathfrak{A}_S(X) \} \). It is easy to see that the sets \( q_X^R((J^o)_1) - \{F_X^n\} \) are arcwise connected and, therefore, connected. Moreover, the sets \( q_X^R((J^o)_1) - \{F_X^n\} \) are open in \( \mathcal{PHD}_n(X) \) and pairwise disjoint. We conclude that they are the components of \( \mathcal{PHD}_n(X) \).

(b) By Lemma 3.5, \( F_X^n \notin \mathcal{PH}E_n(X) \). Given \( B \in \mathcal{PH}E_n(X) \), there exists \( A \in C_n(X) \) such that \( B = q_X^R(A) \). Notice that \( \dim_A[C_n(X)] = \dim_B[PHS_n(X)] = 2n \). By [6, Lemma 11], there exist \( J_1, \ldots, J_m \in \mathfrak{A}_S(X) \), with \( m \leq n \), such that \( A \subset \langle J_1, \ldots, J_m \rangle \). This implies that \( \mathcal{PH}E_n(X) = \bigcup \{q_X^R((J^o)_1) - \{F_X^n\} : J_1, \ldots, J_m \in \mathfrak{A}_S(X) \} \). To prove the other inclusion, let \( A \in \langle J_1, \ldots, J_m \rangle - F_1(X) \). Thus, \( A \cap [R(X) \cup \mathcal{P}(X)] = \emptyset \). By Lemma 3.2, there exists a finite graph \( G \) contained in \( X \) such that \( A \subset \text{int}_X(G) \). Since \( A \cap R(X) = \emptyset \), we have that \( A \cap R(G) = \emptyset \). Hence, by (3.1), \( \dim_A[C_n(G)] = 2n \). Since \( \dim_{q_X^R(A)}[PHS_n(X)] = \dim_{q_X^R(A)}[C_n(G)] \), \( q_X^R(A) \in \mathcal{PH}E_n(X) \). Therefore, \( \mathcal{PH}E_n(X) = \bigcup \{q_X^R((J^o)_1) - \{F_X^n\} : J_1, \ldots, J_m \in \mathfrak{A}_S(X) \} \). The rest of the proof is similar to the proof of (a). \( \square \)

Let \( X \) be a locally connected continuum such that \( R(X) \neq \emptyset \). Given \( J \in \mathfrak{A}_S(X) \), let \( \mathcal{E}(J) = \text{cl}_{C(X)}((J^o)_1) \). Notice that

\[
\mathcal{E}(J) = \begin{cases} 
C(J) - \{A \in C(J) : A \text{ is an arc and } \text{int}_J(A) \cap R(X) \neq \emptyset\}, & \text{if } J \text{ is a cycle}, \\
C(J), & \text{if } J \text{ is an arc}.
\end{cases}
\]

Let \( D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \) and \( D_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + \frac{1}{2})^2 \leq \frac{1}{4}\} \). Let \( L_0 = D_1 - \text{int}_{\mathbb{R}^2}(D_2) \).

Notice that if \( J \) is a cycle, then \( \mathcal{E}(J) \) is homeomorphic to the continuum \( L_0 \).

The proof of following result is a modification of [18, Lemma 3.4].

**Lemma 3.8.** Let \( X \) be a locally connected continuum such that \( R(X) \neq \emptyset \), \( p \in X \) and \( J \in \mathfrak{A}_S(X) \).

(1) If \( J \) is an arc, then \( \{q_X^p(p) \cup A : A \in \mathcal{E}(J)\} \) is a 2-cell in \( \text{PHS}_2(X) \).

(2) If \( J \) is a cycle, then \( \{q_X^p(p) \cup A : A \in \mathcal{E}(J)\} \) is homeomorphic to the continuum \( L_0 \).

**Proof.** Let \( g \) be the embedding of \( C(X) \) into \( C_2(X) \) given by \( g(A) = \{p\} \cup A \). Since the set \( g(\mathcal{E}(J)) \cap F_1(X) \) is either the set \( \emptyset \) or the set \( \{p\} \), we have that \( g(\mathcal{E}(J))/F_1(X) \) is homeomorphic to \( \mathcal{E}(J) \). Notice that in (1), the set \( \mathcal{E}(J) \) is a 2-cell, and in (2), it is homeomorphic to continuum \( L_0 \). Now, we finish the proof by mentioning that \( g(\mathcal{E}(J))/F_1(X) \) is clearly homeomorphic to \( \{q_X^p(p) \cup A : A \in \mathcal{E}(J)\} \). \( \square \)
Lemma 3.9. Let $X$ be a locally connected continuum. If $Y$ and $Z$ are either arcs or simple closed curves of $X$ such that $Y \cap Z = \emptyset$, then $\langle Y, Z \rangle_2$ is a 4-cell and $\{y, z\}$ belongs to its manifold boundary, for each $y \in Y, z \in Z$.

Proof. Let $f : \langle Y, Z \rangle_2 \rightarrow C(Y) \times C(Z)$ be defined as $f(A) = (A \cap Y, A \cap Z)$. Notice that $f$ is a bijection. Moreover, given a sequence $\{A_n\}_{n=1}^\infty$ contained in $\langle Y, Z \rangle_2$ which converges to $A$, for some $A \in \langle Y, Z \rangle_2$, we have that $\{A_n \cap Y\}_{n=1}^\infty$ converges to $A \cap Y$ and $\{A_n \cap Z\}_{n=1}^\infty$ converges to $A \cap Z$. Thus, $\{(A_n \cap Y, A_n \cap Z)\}_{n=1}^\infty$ converges to $(A \cap Y, A \cap Z)$. Hence, $f$ is a homeomorphism.

By [13, 5.1.1 and 5.2], we have that $(Y, Z)$ and $(Z, Y)$ are 2-cells such that $F_1(Y)$ is contained in the manifold boundary of $(Y, Z)$ and $F_1(Z)$ is contained in the manifold boundary of $(Z, Y)$. Hence, $(Y, Z)_2$ is a 4-cell. Let $y \in Y$ and $z \in Z$. Since $\{y\}$ belongs to the manifold boundary of $(Y, Z)$, there exist an open neighborhood $\mathcal{U}$ of $\{y\}$ in $(Y, Z)$ and a homeomorphism $\kappa_1 : \mathcal{U} \rightarrow \kappa_1(\mathcal{U})$ onto an open subset of $\mathbb{R}^2_+$ such that $\kappa_1(\{y\}) = (0, r)$, for some $r \in \mathbb{R}$. Similarly, there exist an open neighborhood $\mathcal{V}$ of $\{z\}$ in $(Z, Y)$ and a homeomorphism $\kappa_2 : \mathcal{V} \rightarrow \kappa_2(\mathcal{V})$ onto an open subset of $\mathbb{R}^2_+$ such that $\kappa_2(\{z\}) = (0, s)$, for some $s \in \mathbb{R}$. Notice that $\mathcal{U} \times \mathcal{V}$ is an open neighborhood of $(\{y\}, \{z\})$ in $(Y, Z)_2$. Let $\kappa_+ : \mathcal{U} \times \mathcal{V} \rightarrow \kappa_+(\mathcal{U} \times \mathcal{V})$ be defined as $\kappa_+(A, B) = (\kappa_1(A), \kappa_2(B))$. Thus, $\kappa_+$ is a homeomorphism, moreover, $\kappa_+(\mathcal{U} \times \mathcal{V}) = \kappa_1(\mathcal{U}) \times \kappa_2(\mathcal{V})$ is an open subset of $\mathbb{R}^2_+$. Now, let $g : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^4_+$ be defined as $g((a, b), (c, d)) = (2ac, b, a^2 - c^2, d)$ and let $h : \mathbb{R}^4_+ \rightarrow \mathbb{R}^2_+ \times \mathbb{R}^2_+$ be defined as

$$h(a, b, c, d) = \left(\left(\sqrt{\frac{1}{2}(\sqrt{a^2 + c^2} + c)}, b\right), \left(\sqrt{\frac{1}{2}(\sqrt{a^2 + c^2} - c)}, d\right)\right).$$

Notice that $g$ and $h$ are maps. Moreover, $h \circ g = \text{id}_{\mathbb{R}^2_+ \times \mathbb{R}^2_+}$ and $g \circ h = \text{id}_{\mathbb{R}^4_+}$. Hence, $g$ is a homeomorphism. By definition of $f$, $f^{-1}(\mathcal{U} \times \mathcal{V})$ is an open neighborhood of $(\{y, z\})$ in $(Y, Z)_2$. Let $\kappa : f^{-1}(\mathcal{U} \times \mathcal{V}) \rightarrow \kappa(f^{-1}(\mathcal{U} \times \mathcal{V}))$ be defined as $\kappa(A) = g \circ \kappa_+ \circ f(A)$. Thus, $\kappa$ is a homeomorphism, $\kappa(f^{-1}(\mathcal{U} \times \mathcal{V})) = g(\kappa_1(\mathcal{U}) \times \kappa_2(\mathcal{V}))$ is an open subset of $\mathbb{R}^4_+$ and $\kappa((\{y, z\}) = (0, r, 0, s)$. Therefore, $(\{y, z\}$ belongs to the manifold boundary of $(Y, Z)_2$. \hfill $\square$

Given $J, K \in \mathcal{A}_S(X)$, let

$$D(J, K) = \text{cl}_{C_2(X)}(\partial \mathcal{L}_2(X) \cap (J^o, K^o)_2) \cap \text{cl}_{C_2(X)}(\partial \mathcal{L}_2(X) - (J^o, K^o)_2)$$

and

$$\mathcal{PHD}(J, K) = \text{cl}_{\mathcal{PHL}_2(X)}(\partial \mathcal{PHL}_2(X) \cap q_X^2((J^o, K^o)_2)) \cap \text{cl}_{\mathcal{PHL}_2(X)}(\partial \mathcal{PHL}_2(X) - q_X^2((J^o, K^o)_2)).$$

Lemma 3.10. Let $X$ be a locally connected continuum such that $R(X) \neq \emptyset$ and let $J, K \in \mathcal{A}_S(X)$. Then $F_X^2 \in \mathcal{PHD}(J, K)$ if and only if $J \cap K \neq \emptyset$.

Proof. Suppose that $F_X^2 \in \mathcal{PHD}(J, K)$. Then, there exists a sequence $\{A_n\}_{n=1}^\infty$ contained in $(J^o, K^o)_2$ such that $\lim q_X^2(A_n) = F_X^2$. Since $q_X^2$ is a map, $\lim A_n = \{a\}$, for some $a \in X$. Thus, $\{a\} \in (J, K)_2$. Therefore, $J \cap K \neq \emptyset$. Now suppose that $J \cap K \neq \emptyset$. We consider the following cases.

Case 1. $J \neq K$.

Let $p \in J \cap K \cap R(X)$. Then, there are two sequences $\{j_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ contained in $J^o$ and $K^o$, respectively, such that $\lim j_n = p$ and $\lim k_n = p$. Thus, $\lim q_X^2(\{j_n, k_n\}) = F_X^2$. Let $J_n$ and $K_n$ be subarcs of $J^o$ and $K^o$, respectively, such that $j_n \in J_n^o$ and $k_n \in K_n^o$, for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Notice that $(J_n, K_n)_2$ is a neighborhood of $(\{j_n, k_n\})$ in $C_2(X)$. Since $J_n$ and $K_n$ are disjoint arcs, by Lemma 3.9, we have that $(J_n, K_n)_2$ is a 4-cell such that $\{j_n, k_n\}$ belongs to its manifold boundary. This implies that $\{j_n, k_n\} \in \partial C_2(X)$. By Remark 2.1 (b), $q_X^2(\{j_n, k_n\}) \in \partial \mathcal{PHL}_2(X)$. Therefore, $F_X^2 \in \text{cl}_{\mathcal{PHL}_2(X)}(\partial \mathcal{PHL}_2(X) \cap q_X^2((J^o, K^o)_2))$. 


Now, let \( \{p_n\}_{n=1}^\infty \) and \( \{q_n\}_{n=1}^\infty \) be two sequences contained in \( K^o \) such that \( \lim p_n = p \), \( \lim q_n = q \) and \( p_n \neq q_n \), for each \( n \in \mathbb{N} \). Let \( P_n \) and \( Q_n \) be disjoint subarcs of \( K \) such that \( p_n \in P_n^o \) and \( q_n \in Q_n^o \), for each \( n \in \mathbb{N} \). By Lemma 3.9, we have that \( (P_n, Q_n) \) is a 4-cell and \( \{p_n, q_n\} \) belongs to its manifold boundary. By Remark 2.1 (b), \( \{q^2_X((p_n, q_n))\}_{n=1}^\infty \) is a sequence contained in \( \partial \mathcal{PL}_2(X) - q^2_X((J^o, K^o))^2 \). Therefore, \( F^2_X \in \mathcal{PD}(J, K) \).

Case 2. \( J = K \).

Let \( p \in J \cap R(X) \). Then, there exist two sequences \( \{j_n\}_{n=1}^\infty \) and \( \{k_n\}_{n=1}^\infty \) contained in \( J^o \) such that \( \lim j_n = p \), \( \lim k_n = p \) and \( j_n \neq k_n \), for each \( n \in \mathbb{N} \). Let \( J_n \) and \( K_n \) be disjoint subarcs of \( J^o \) such that \( j_n \in J_n^o \) and \( k_n \in K_n^o \), for each \( n \in \mathbb{N} \). By Lemma 3.9, we have that \( (J_n, K_n) \) is a 4-cell such that \( \{j_n, k_n\} \) belongs to its manifold boundary. This implies that \( \{j_n, k_n\} \in \partial \mathcal{L}_2(X) \). By Remark 2.1 (b), \( q^2_X((j_n, k_n)) \in \partial \mathcal{PL}_2(X) \). Therefore, \( F^2_X \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{PL}_2(X) - q^2_X((J^o, K^o))^2) \).

Since \( p \in R(X) \), there exists \( L \in \mathfrak{A}_S(X) - \{J\} \) such that \( p \in L \). Thus, \( p \in J \cap L \cap R(X) \). In a similar way as Case 1, we can prove that \( F^2_X \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{PL}_2(X) - q^2_X((J^o, K^o))^2) \). Therefore, \( F^2_X \in \mathcal{PD}(J, K) \). \( \square \)

The proof of following result is a modification of [7, Lemma 2.15].

**Lemma 3.11.** Let \( X \) be a locally connected continuum with \( R(X) \neq \emptyset \). If \( J, K \in \mathfrak{A}_S(X) \), then \( \mathcal{PD}(J, K) = \{q^2_X(\{p\} \cup G) : p \in bd_X(J) \text{ and } G \in \mathcal{E}(K) \text{ or } p \in bd_X(K) \text{ and } G \in \mathcal{E}(J) \} \).

**Proof.** Let \( B \in \mathcal{PD}(J, K) \). By Lemma 3.10, we may assume that \( B \neq F^2_X \). Let \( A \in C_2(X) - F_1(X) \) be such that \( q^2_X(A) = B \). Since \( B \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{PL}_2(X) \cap q^2_X((J^o, K^o))^2) \), there exists a sequence \( \{A_n\}_{n=1}^\infty \) contained in \( (J^o, K^o)^2 - F_1(X) \) such that \( \lim q^2_X(A_n) = B \) and \( q^2_X(A_n) \in \partial \mathcal{PL}_2(X) \), for each \( n \in \mathbb{N} \). By the continuity of \( q^2_X \), \( \lim A_n = A \). By Remark 2.1 (b), \( A_n \in \partial \mathcal{L}_2(X) \), for each \( n \in \mathbb{N} \). Hence, \( A \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{L}_2(X) \cap (J^o, K^o)^2) \). Moreover, since \( B \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{PL}_2(X) - q^2_X((J^o, K^o))^2) \), there exists a sequence \( \{B_n\}_{n=1}^\infty \) contained in \( \partial \mathcal{PL}_2(X) - q^2_X((J^o, K^o))^2 \) such that \( \lim B_n = B \) and \( B_n \neq F^2_X \), for each \( n \in \mathbb{N} \). Given \( n \in \mathbb{N} \), let \( D_n \) be the unique element of \( C_2(X) - F_1(X) \) such that \( q^2_X(D_n) = B_n \). Then \( \lim D_n = A \). By Remark 2.1 (b), \( D_n \in \partial \mathcal{L}_2(X) - (J^o, K^o)^2 \), for each \( n \in \mathbb{N} \). Hence, \( A \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{L}_2(X) - (J^o, K^o)^2) \). We have shown that \( A \in \mathcal{PD}(J, K) \). By [6, Lemma 33], \( A = \{p\} \cup G \), where \( p \in bd_X(J) \) and \( G \in \mathcal{E}(K) \) or \( p \in bd_X(K) \) and \( G \in \mathcal{E}(J) \). This completes the proof of the first inclusion.

To prove the opposite inclusion, let \( B = q^2_X(\{p\} \cup G) \), where \( p \in bd_X(J) \) and \( G \in \mathcal{E}(K) \) or \( p \in bd_X(K) \) and \( G \in \mathcal{E}(J) \). By Lemma 3.10, we may assume that \( G \neq \{p\} \). Let \( A = \{p\} \cup G \). By [6, Lemma 33], \( A \in \mathcal{PD}(J, K) \). Then, there exists a sequence \( \{A_n\}_{n=1}^\infty \) contained in \( \partial \mathcal{L}_2(X) \cap (J^o, K^o)^2 \) such that \( \lim A_n = A \) and \( A_n \notin F_1(X) \), for each \( n \in \mathbb{N} \). Hence, \( q^2_X(A_n) \in \partial \mathcal{PL}_2(X) \cap q^2_X((J^o, K^o)^2) \). Thus, \( B \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{PL}_2(X) \cap q^2_X((J^o, K^o)^2)) \). Similarly, \( B \in \mathcal{CL}_{S_2}(X)(\partial \mathcal{PL}_2(X) - q^2_X((J^o, K^o)^2)) \). Therefore, \( B \in \mathcal{PD}(J, K) \). \( \square \)

Now, we are ready to describe models of \( \mathcal{PD}(J, K) \) for each possible case. Let \( J, K \in \mathfrak{A}_S(X) \), where \( X \) is a locally connected continuum such that \( R(X) \neq \emptyset \). We consider nine cases.

Case I. \( J = K \), \( J \) is an arc and \( J \notin \mathfrak{A}_E(X) \).

By Lemma 3.11, \( \mathcal{PD}(J, J) = \{q^2_X(\{p\} \cup G) : G \in \mathcal{E}(J)\} \cup \{q^2_X(\{q\} \cup G) : G \in \mathcal{E}(J)\} \), where \( p, q \in J \cap R(X) \). By Lemma 3.8, we have that \( \mathcal{PD}(J, J) \) is the union of two 2-cells whose intersection is the set \( \{F^2_X, q^2_X(\{p, q\}), q^2_X(J)\} \). It is easy to see that this set is contained in the manifold boundary of both 2-cells.

Case II. \( J = K \), \( J \) is an arc and \( J \in \mathfrak{A}_E(X) \).

Then \( J \cap R(X) = \{p\} \). Thus, \( \mathcal{PD}(J, J) = \{q^2_X(\{p\} \cup G) : G \in \mathcal{E}(J)\} \) which is a 2-cell.

Case III. \( J = K \) and \( J \in \mathfrak{A}_R(X) \).

Then \( J \cap R(X) = \{q\} \). Thus, \( \mathcal{PD}(J, J) = \{q^2_X(\{q\} \cup G) : G \in \mathcal{E}(J)\} \) which is homeomorphic to \( L_0 \). For the remaining cases we assume that \( J \neq K \).

Case IV. \( J \) and \( K \) are arcs and \( J, K \notin \mathfrak{A}_E(X) \).
Let $p_1, p_2 \in J \cap R(X)$ and $q_1, q_2 \in K \cap R(X)$. Then $\mathcal{PD}(J,K) = P_1 \cup P_2 \cup Q_1 \cup Q_2$, where $P_1 = \{q_X^2 \{p_1\} : G \in \mathcal{E}(K)\}$, $P_2 = \{q_X^2 \{p_2\} : G \in \mathcal{E}(K)\}$, $Q_1 = \{q_X^2 \{q_1\} : G \in \mathcal{E}(J)\}$ and $Q_2 = \{q_X^2 \{q_2\} : G \in \mathcal{E}(J)\}$. By Lemma 3.8, $\mathcal{PD}(J,K)$ is the union of four 2-cells. Now let us consider three subcases.

$\text{IV}(a)$. Let $J \cap K = \emptyset$.

Then $P_1 \cap P_2 = \emptyset = Q_1 \cap Q_2$. Also, $P_i \cap Q_j = \{q_X^2 \{p_i, q_j\}\}$ with $i, j \in \{1, 2\}$.

$\text{IV}(b)$. Let $J \cap K$ is an one point set. Suppose that $p_1 = q_1$.

Similar to case $\text{IV}(a)$ with the exception that $P_1 \cap Q_1 = \{F_X^2\}$.

$\text{IV}(c)$. Let $J \cap K$ is a two point set. Suppose that $p_1 = q_1$ and $p_2 = q_2$.

Then $P_1 \cap P_2 = \{F_X^2, q_X^2 \{p_1, p_2\}\}$ and $Q_1 \cap Q_2 = \{F_X^2, q_X^2 \{p_1, p_2\}\}$. Moreover, $P_i \cap Q_j = \{q_X^2 \{p_i, p_j\}\}$ with $i, j \in \{1, 2\}$.

$\text{Case V}$. J and K are arcs, $J \notin \mathcal{E}(X)$ and $K \in \mathcal{A}(X)$.

Let $p_1, p_2 \in J \cap R(X)$ and $q \in K \cap R(X)$. Then $\mathcal{PD}(J,K) = P_1 \cup P_2 \cup Q$, where $P_1 = \{q_X^2 \{p_1\} : G \in \mathcal{E}(K)\}$, $P_2 = \{q_X^2 \{p_2\} : G \in \mathcal{E}(K)\}$ and $Q = \{q_X^2 \{q\} : G \in \mathcal{E}(J)\}$. Thus, $\mathcal{PD}(J,K)$ is the union of three 2-cells. Now let us consider two subcases.

$\text{V}(a)$. Let $J \cap K = \emptyset$.

Then $P_1 \cap P_2 = \emptyset$. Also, $P_i \cap Q = \{q_X^2 \{p_i, q\}\}$ with $i \in \{1, 2\}$.

$\text{V}(b)$. Let $J \cap K$ is an one point set. Suppose that $p_1 = q$.

Similar to case $\text{V}(a)$ with the slightly difference that $P_1 \cap Q = \{F_X^2\}$.

$\text{Case VI}$. J, K $\in \mathcal{A}(X)$.

Then $\mathcal{PD}(J,K) = \{q_X^2 \{p\} : G \in \mathcal{E}(K)\} \cup \{q_X^2 \{q\} : G \in \mathcal{E}(J)\}$, where $p \in J \cap R(X)$ and $q \in K \cap R(X)$. Thus, $\mathcal{PD}(J,K)$ is the union of two 2-cells whose intersection is the set $\{q_X^2 \{p, q\}\}$, or $\{F_X^2\}$ in the case that $p = q$.

$\text{Case VII}$. J is an arc, $J \notin \mathcal{A}(X)$ and $K \in \mathcal{A}(X)$.

Similar to case V with the slightly difference that $\mathcal{PD}(J,K)$ is the union of a 2-cell and two continua $L_0$.

$\text{Case VIII}$. J $\in \mathcal{A}(X)$ and $K \in \mathcal{A}(X)$.

Similar to case VI with the slightly difference that $\mathcal{PD}(J,K)$ is the union of a 2-cell and a continuum $L_0$.

$\text{Case IX}$. J, K $\in \mathcal{A}(X)$.

Similar to case VI with the difference that $\mathcal{PD}(J,K)$ is the union of two continua $L_0$.

$\text{Remark 3.12}$. Let X and Y be locally connected continua such that $R(X) \neq \emptyset$ and $R(Y) \neq \emptyset$, and let $J, K \in \mathcal{A}(X)$ and $J_h, K_h \in \mathcal{A}(Y)$. If $\mathcal{PD}(J,K)$ is homeomorphic to $\mathcal{PD}(J_h,K_h)$, then

(a) $J$ and $K$ are as in Case I if and only if $J_h$ and $K_h$ are as in Case I,

(b) $J$ and $K$ are as in Case II if and only if $J_h$ and $K_h$ are as in Case II and

(c) $J$ and $K$ are as in Case III if and only if $J_h$ and $K_h$ are as in Case III.

$\text{4. Main results}$

In this section we present the proof of our first main result. The first step is to mention that Ulises Morales-Fuentes has proven that the finite graphs have unique n-fold pseudo-hyperspace suspension, see [18, Theorem 5.7]. We prove that if X is a meshed continuum such that $|\bigcap \mathcal{A}(X)| = 2$, then X is a finite graph, and therefore it has unique n-fold pseudo-hyperspace suspension. Finally, we prove that for a meshed continuum X such that $R(X) \neq \emptyset$ and $|\bigcap \mathcal{A}(X)| \neq 2$ the uniqueness of the n-fold pseudo-hyperspace suspension holds, see Theorem 4.8.

Using [6, Lemma 2] and [5, Theorem 3.1] we have the following properties for meshed continua, which will be used without quoting them in the proof of Theorem 4.7.
Lemma 4.1. If $X$ is a meshed continuum, then

(a) $X$ is locally connected,
(b) $J \cap \mathcal{P}(X) = \emptyset$, for each $J \in \mathfrak{A}_S(X)$, and
(c) $\mathcal{G}(X) = \bigcup \mathfrak{A}_S(X)$.

The following result is proved in [4, Theorem 5.1] for case $n = 1$ and [16, Theorem 4.1 (a)] for case $n \geq 2$.

Lemma 4.2. Let $X$ be a continuum and $n \in \mathbb{N}$. Then $X$ is locally connected if and only if $\text{PHS}_n(X)$ is locally connected.

Given a continuum $X$ and $n \in \mathbb{N}$, let

$$\mathfrak{F}_n(X) = \{A \in C_n(X) : \text{dim}_A(C_n(X)) \text{ is finite}\}.$$

Theorem 4.3. Let $X$ be a meshed continuum and $n \in \mathbb{N}$. If $Y$ is a continuum such that $\text{PHS}_n(X)$ is homeomorphic to $\text{PHS}_n(Y)$, then $Y$ is a meshed continuum.

Proof. Let $h : \text{PHS}_n(X) \rightarrow \text{PHS}_n(Y)$ be a homeomorphism. Since $X$ is a locally connected continuum, using Lemma 4.2, we have that $Y$ is a locally connected continuum. Let $A \in C_n(X)$ and $B \in C_n(Y)$ be such that $h(q_X^n(A)) = F_Y^n$ and $h^{-1}(q_Y^n(B)) = F_Y^n$. Let $K = C_n(X) - (F_1(X) \cup \{A\})$ and $L = C_n(Y) - (F_1(Y) \cup \{B\})$. Then $g : K \rightarrow L$ defined by $g = (q_Y^n|_L)^{-1} \circ h \circ q_X^n|_K$ is a homeomorphism. Moreover, $g(\mathfrak{F}_n(X) \cap K) = \mathfrak{F}_n(Y) \cap L$. Since $X$ is meshed, by [6, Theorem 5], we know that $\mathfrak{F}_n(X)$ is a dense subset of $C_n(X)$. This implies that $\mathfrak{F}_n(Y) \cap L$ is dense in $L$. Finally, by the density of $L$ in $C_n(Y)$, we conclude that $\mathfrak{F}_n(Y)$ is a dense subset of $C_n(Y)$. Therefore, by [6, Theorem 5], $Y$ is a meshed continuum.

The following result extends [18, Lemma 5.2].

Lemma 4.4. Let $n \geq 2$. If $X$ is a locally connected continuum with $R(X) \neq \emptyset$ and $|\mathfrak{A}_S(X)| \geq 2$, then

$$\bigcap \{\text{cl}_{\text{PHS}_n(X)}(q_X^n(\langle J^o \rangle)_n) - F_X^n : J \in \mathfrak{A}_S(X)\} = \begin{cases} \{F_X^n\} & \text{if } |\bigcap \mathfrak{A}_S(X)| \neq 2, \\
\{F_X^n, q_X^n(p, q)\} & \text{if } |\bigcap \mathfrak{A}_S(X)\} = \{p, q\}. \end{cases}$$

Proof. Let $J \in \mathfrak{A}_S(X)$ and $a \in J^o$. Since $\{a\}$ can be approximated by elements in $\langle J^o \rangle_1 - F_1(X)$, we have that $\{a\} \in \text{cl}_{C_n(X)}(\langle J^o \rangle_n - F_1(X))$. Hence, $F_X^n \in \text{cl}_{\text{PHS}_n(X)}(q_X^n(\langle J^o \rangle)_n) - F_X^n)$. Moreover, if $|\bigcap \mathfrak{A}_S(X) = \{p, q\}$, then $p, q \in J$ and since $n \geq 2$, $\{p, q\}$ can be approximated by elements in $\langle J^o \rangle_n - F_1(X)$. Hence, $q_X^n(\{p, q\}) \in \text{cl}_{\text{PHS}_n(X)}(q_X^n(\langle J^o \rangle)_n) - F_X^n)$. This implies the second inclusion.

Now, let $B \in \{\text{cl}_{\text{PHS}_n(X)}(q_X^n(\langle J^o \rangle)_n) - F_X^n) : J \in \mathfrak{A}_S(X)\}$. Suppose that $B \neq F_X^n$. Let $A \in C_n(X) - F_1(X)$ be such that $q_X^n(A) = B$. Let $J \in \mathfrak{A}_S(X)$. Since $B \in \text{cl}_{\text{PHS}_n(X)}(q_X^n(\langle J^o \rangle)_n) - F_X^n)$, there exists a sequence $\{B_m\}_{m=1}^\infty$ contained in $q_X^n(\langle J^o \rangle)_n) - F_X^n)$ which converges to $B$. Let $A_m \in \langle J^o \rangle_n - F_1(X)$ be such that $q_X^n(A_m) = B_m$, for each $m \in \mathbb{N}$. Notice that $\{A_m\}_{m=1}^\infty$ converges to $A$. Hence, $A \subset J$, for each $J \in \mathfrak{A}_S(X)$. Therefore, $A \subset \bigcap \mathfrak{A}_S(X)$. Since $|\mathfrak{A}_S(X)| \geq 2$, we have that $|\bigcap \mathfrak{A}_S(X)| \leq 2$.

Consider the following cases.

Case 1. $|\bigcap \mathfrak{A}_S(X)| \neq 2$.

Then $|\bigcap \mathfrak{A}_S(X)| \leq 1$. Hence, $|A| \leq 1$. This is a contradiction since $A \subset C_n(X) - F_1(X)$. Therefore, $B = F_X^n$.

Case 2. $\bigcap \mathfrak{A}_S(X) = \{p, q\}$.

Since $A \subset C_n(X) - F_1(X)$, we have that $A = \{p, q\}$. Hence, $B \in \{F_X^n, q_X^n(\{p, q\})\}$, as desired.

From these cases, the result follows.
Theorem 4.5. Let $X$ be a meshed continuum such that $R(X) \neq \emptyset$. If $|\bigcap \mathcal{A}_S(X)| = 2$, then $X$ is a finite graph.

Proof. Let $p, q \in \bigcap \mathcal{A}_S(X)$. Thus, $p$ and $q$ are the end points of each maximal free arc. Suppose that there exists $a \in \mathcal{P}(X)$. By [5, Theorem 3.3], there is a sequence of pairwise distinct elements contained in $R(X) \cap \mathcal{G}(X)$ which converges to $a$. However, this is not possible since $R(X) \cap \mathcal{G}(X) \subset \{p, q\}$. Hence, $\mathcal{P}(X) = \emptyset$. Therefore, $X$ is a finite graph. □

Using Theorem 4.5 and [18, Theorem 5.7] we obtain the following result.

Theorem 4.6. Let $X$ be a meshed continuum such that $R(X) \neq \emptyset$. If $|\bigcap \mathcal{A}_S(X)| = 2$, then $X$ has unique $n$-fold pseudo-hyperspace suspension.

The following result extends [18, Lemma 5.1 and Lemma 5.5].

Theorem 4.7. Let $X$ and $Y$ be meshed continua such that $R(X) \neq \emptyset, R(Y) \neq \emptyset$ and $|\bigcap \mathcal{A}_S(X)| \neq 2, |\bigcap \mathcal{A}_S(Y)| \neq 2$, $n \geq 2$ and let $h : \text{PHS}_n(X) \to \text{PHS}_n(Y)$ be a homeomorphism. Suppose that for each $J \in \mathcal{A}_S(X)$, there exists $J_h \in \mathcal{A}_S(Y)$ such that $h(q_X^n((J^n)_1) - \{F^n_Y\}) \subset q_Y^n((J_h^n)_n)$ and $\mathcal{A}_S(Y) = \{J_h : J \in \mathcal{A}_S(X)\}$. Then

(a) for each $J \in \mathcal{A}_S(X)$, $h(q_X^n((J^n)_1) - \{F^n_Y\}) = q_Y^n((J_h^n)_n) - \{F^n_Y\}$,

(b) for each $J \in \mathcal{A}_S(X)$, $h^{-1}(q_Y^n((J_h^n)_n \cap C(Y)) - \{F^n_Y\}) \subset q_X^n((J^n)_1) - \{F^n_Y\}$,

(c) the association $J \to J_h$ is a bijection between $\mathcal{A}_S(X)$ and $\mathcal{A}_S(Y)$.

(d) $h(F^n_X) = F^n_Y$.

If we also suppose that

(1) if $J \in \mathcal{A}_R(X)$, then $J_h \in \mathcal{A}_R(Y)$ and

(2) if $J \in \mathcal{A}_E(X)$, then $J_h \in \mathcal{A}_E(Y)$,

then $X$ is homeomorphic to $Y$.

Proof. (a) Let $J \in \mathcal{A}_S(X)$ and $A$ be a subarc of $J^o$ such that $h(q_X^n(A)) \neq F^n_Y$. By Lemma 3.7 (b), we have that $h(q_X^n((J^o)_1) - \{F^n_Y\})$ and $q_Y^n((J_h^n)_n) - \{F^n_Y\}$ are components of $\mathcal{PH}(n)(X)$. Notice that $h(q_X^n(A)) \in h(q_X^n((J^o)_1) - \{F^n_Y\}) \cap (q_Y^n((J_h^n)_n) - \{F^n_Y\})$. Therefore, $h(q_X^n((J^o)_1) - \{F^n_Y\}) = q_Y^n((J_h^n)_n) - \{F^n_Y\}$.

Clearly, (b) follows from (a).

To prove (c), it is enough to prove that the correspondence is one to one. Let $J, L \in \mathcal{A}_S(X)$ and suppose that $J_h = L_h$. Using (a) we conclude that $q_X^n((J^o)_1) - \{F^n_Y\} = q_X^n((L^o)_1) - \{F^n_Y\}$. Let $A$ be a subarc of $J^o$. Then $q_X^n(A) \in q_X^n((L^o)_1)$ and $A \subset L^o$. Therefore, by Lemma 3.1 (c), $J = L$.

(d) By Lemma 4.4 and using (a) we have that

$$h(\{F^n_Y\}) = \bigcap\{\text{cl}_{\text{PHS}_n(Y)}(h(q_X^n((J^o)_1) - \{F^n_Y\})) : J \in \mathcal{A}_S(X)\}$$

$$= \bigcap\{\text{cl}_{\text{PHS}_n(Y)}(q_Y^n((J_h^n)_n) - \{F^n_Y\}) : J \in \mathcal{A}_S(X)\}$$

$$= \bigcap\{\text{cl}_{\text{PHS}_n(Y)}(q_Y^n((J_h^n)_n) - \{F^n_Y\}) : J_h \in \mathcal{A}_S(Y)\} = \{F^n_Y\}.$$

Therefore, $h(F^n_X) = F^n_Y$.

Let $g : C_n(X) - F_1(X) \to C_n(Y) - F_1(Y)$ be defined as $g = (q_Y^n)^{-1} \circ h \circ q_X^n$. Notice that $g$ is a homeomorphism. Given $J \in \mathcal{A}_S(X)$, let $K_n(J, X) = \text{cl}_{\text{C}_{\text{ph}}(X)}((J^n)_1) - F_1(X)$. 


The proofs of Claim 1 and Claim 2 are similar to the proofs of Claim 1 and Claim 2 from [7, Theorem 3.1], respectively. The proof of Claim 3 is similar to arguments given in [7, Theorem 3.1, p. 88–89].

Claim 1. If $J \in \mathfrak{A}_S(X)$, then

(e) $K_n(J_h, Y) = g(K_n(J, X))$,
(f) $\{ \dim_A[C_n(X)] : A \in K_n(J, X) \} = \{ \dim_B[C_n(Y)] : B \in K_n(J_h, Y) \}$,
(g) $|J \cap R(X)| = |J_h \cap R(Y)|$,
(h) if $A \in K_n(J, X)$, then $|A \cap R(X)| = |g(A) \cap R(Y)|$.

Proof of Claim 1. Let $J \in \mathfrak{A}_S(X)$. Notice that $\text{cl}_{C_n(X)}((J^0)_A) - F_1(X) = \text{cl}_{C_n(X)} - F_1(X)((J^0)_A)$. From this, clearly (e) is true and (f) follows from (e). Now, since $X$ is a meshed continuum, $J \cap \mathcal{P}(X) = \emptyset$. Thus, by Lemma 3.2, there exists a finite graph $G$ contained in $X$ such that $J \subset \text{int}_X(G)$. Using (3.1), we have that $\{ \dim_A[C_n(X)] : A \in K_n(J, X) \} \geq 3$ and only if $|J \cap R(X)| = 2$ and $\{ \dim_A[C_n(X)] : A \in K_n(J, X) \} = 2$ if and only if $|J \cap R(X)| = 3$. Notice that $J_h$ also satisfies the same conditions as $J$, such as $J_h \cap \mathcal{P}(Y) = \emptyset$. This proves (g). Moreover, given $A \in K_n(J, X)$. If $|A \cap R(X)| = 2$, then $|J \cap R(Y)| = 2$. Thus, $|J_h \cap R(Y)| = 2$ and $\dim_A[C_n(X)] = \max\{ \dim_E[C_n(X)] : E \in K_n(J, X) \}$. Hence, $\dim_{g(A)}[C_n(Y)] = \max\{ \dim_B[C_n(Y)] : B \in K_n(J_h, Y) \}$. This implies that $|g(A) \cap R(Y)| = 2$. Similarly, if $|g(A) \cap R(Y)| = 2$, then $|A \cap R(X)| = 2$. Similarly, if $|g(A) \cap R(Y)| = 0$, then $|A \cap R(X)| = 0$. Finally, if $|A \cap R(X)| = 1$, then $|g(A) \cap R(Y)| \notin \{ 0, 2 \}$. Thus, $|g(A) \cap R(Y)| = 1$. This completes the proof of Claim 1. $\square$

Claim 2. If $J \in \mathfrak{A}_S(X)$ and $v \in J \cap R(X)$, then $K(v, J) = \{ A \in K_n(J, X) : A \cap R(X) = \{ v \} \}$ is arcwise connected.

Now, given $v \in R(X) \cap \mathcal{G}(X)$, there is $J \in \mathfrak{A}_S(X)$ such that $v \in J$. Let $A \in K(v, J)$. By Claim 1, $g(A) \in K_n(J_h, Y)$ and there exists a unique point $v_h(A) \in R(Y) \cap g(A)$. Notice that $v_h(A) \in J_h$ and $v_h(A) \in g(A) \cap \mathcal{G}(Y)$.

Claim 3. Let $v \in R(X) \cap \mathcal{G}(X)$ and $J, L \in \mathfrak{A}_S(X)$ with $v \in J \cap L$. If $A \in K(v, J)$ and $E \in K(v, L)$, then $v_h(A) = v_h(E)$ (in other words, $v_h(A)$ depends neither on the choice of $J$ nor on the choice of $A$).

Proof of Claim 3. In order to prove this, take $A_1$ and $E_1$ arcs in $J$ and $L$, respectively, such that $v$ is an end point of $A_1$ and $E_1$, $A_1 \neq J$ and $E_1 \neq L$. Notice that $A_1 \in K(v, J)$ and $E_1 \in K(v, L)$. By Claim 2, there exist maps $\alpha_1 : [0, 1] \to K(v, J)$ and $\alpha_E : [0, 1] \to K(v, L)$ such that $\alpha_1(0) = A_1$, $\alpha_1(1) = A_1$, $\alpha_E(0) = E_1$ and $\alpha_E(1) = E$. Moreover, since $A_1 \cup E_1$ is an arc, we may define a map $\alpha_0 : [0, 1] \to C(A_1 \cup E_1)$ with the following properties: $\alpha_0(0) = A_1$, $\alpha_0(1) = E_1$ and for each $t \in [0, 1]$, $\alpha_0(t) \cap R(X) = \{ v \}$ and $\alpha_0(t) \notin F_1(X)$. Let $\alpha : [0, 1] \to K(v, J) \cup C(A_1 \cup E_1) \cup K(v, L)$ be defined as

$$\alpha(t) = \begin{cases} 
\alpha_1(3t) & \text{if } t \in [0, \frac{1}{3}], \\
\alpha_0(3t - 1) & \text{if } t \in \left[ \frac{1}{3}, \frac{2}{3} \right], \\
\alpha_E(3t - 2) & \text{if } t \in \left[ \frac{2}{3}, 1 \right].
\end{cases}$$

Notice that $\alpha(t) \subset J \cup L$. Thus, $g(\alpha(t)) \subset J_h \cup L_h$, for each $t \in [0, 1]$. Let $i_0 = \text{ord}(v, X)$. Since $\alpha(t) \cap \mathcal{P}(X) = \emptyset$, by Lemma 3.2 and (3.1), we have that for each $t \in [0, 1],$

$$2n + (i_0 - 2) = \dim_{\alpha(t)}[C_n(X)] = \dim_{g(\alpha(t))}[C_n(Y)].$$
Since $v_h(A)$ is the only ramification point of $Y$ in the set $g(A) = g(\alpha(0))$, this implies that $\text{ord}(v_h(A), Y) = i_0$. Let $T = \{t \in [0, 1] : v_h(A) \in g(\alpha(t))\}$. Notice that $T$ is a closed subset of $[0, 1]$ and $0 \in T$. Suppose that $T \neq [0, 1]$ and let $R$ be a component of $[0, 1] - T$. Then $t_0 = \inf R \in T$ and there exists a sequence $\{r_m\}_{m=1}^{\infty}$ of elements of $R$ which converges to $t_0$. Since $(J_h \cup L_h) \cap R(Y)$ is finite, we may assume that there exists $w \in (J_h \cup L_h) \cap R(Y)$ such that $w \in g(\alpha(r_m))$. Hence, $w, v_h(A) \in g(\alpha(t_0))$. Notice that $w \neq v_h(A)$. Hence, $\dim g(\alpha(t_0))[C_n(Y)] > 2n + (i_0 - 2)$, a contradiction. Therefore, $T = [0, 1]$. On the other hand, we know that $v_h(E)$ is the only ramification point of $Y$ in the set $g(E) = g(\alpha(1))$. Consequently, $v_h(A) = v_h(E)$. This proves Claim 3. □

From now on, we simply write $v_h$ instead of $v_h(A)$. Thus, we have a function

$$\varphi : R(X) \cap G(X) \longrightarrow R(Y) \cap G(Y)$$

$$v \longmapsto v_h$$

Since $Y$ satisfies similar conditions to those of $X$, we have that $\varphi$ is a bijection.

Claim 4. There exists a homeomorphism $\phi : G(X) \longrightarrow G(Y)$ such that $\phi|_{R(X) \cap G(X)} = \varphi$.

Proof of Claim 4. Let $J \in \mathfrak{A}_S(X)$.

Case 1. $|J \cap R(X)| = 2$.

Suppose that $J \cap R(X) = \{p, q\}$. Thus, $p_h, q_h \in J_h$. Since $J$ and $J_h$ are arcs, we may consider a homeomorphism $\varphi_J : J \longrightarrow J_h$ such that $\varphi_J(p) = p_h$ and $\varphi_J(q) = q_h$.

Case 2. $|J \cap R(X)| = 1$, assuming that $J \cap R(X) = \{a\}$.

Notice that $J_h \cap R(Y) = \{a_h\}$. By (1) and (2), we may take a homeomorphism $\varphi_J : J \longrightarrow J_h$ such that $\varphi_J(a) = a_h$. Hence, we define $\phi : G(X) \longrightarrow G(Y)$ given by $\phi(x) = \varphi_J(x)$, where $x \in J$. Therefore, $\phi$ is a homeomorphism. □

If $X$ is a finite graph, then $G(X) = X$. Thus, $\phi(X) = G(Y)$ is a nonempty open and closed subset of $Y$. Therefore, $G(Y) = Y$ and $X$ is homeomorphic to $Y$. Now, suppose that $X$ and $Y$ are not finite graphs.

Claim 5. If $a \in \mathcal{P}(X)$ and $\{a_m\}_{m=1}^{\infty}$ is a sequence contained in $G(X) \cap R(X)$ which converges to $a$, then $\{\phi(a_m)\}_{m=1}^{\infty}$ converges.

Proof of Claim 5. Let $\{\phi(b_i)\}_{i=1}^{\infty}$ be a convergent subsequence which converges to some $z \in Y$. By [5, Theorem 3.3], $z \in \mathcal{P}(Y)$. We are going to prove that $\lim \phi(a_m) = z$. Suppose to the contrary that

$$\text{there is } \varepsilon_1 > 0 \text{ such that for each } N \in \mathbb{N}, \text{ there exists } k > N \text{ such that } \phi(a_k) \notin B(z, \varepsilon_1). \tag{4.1}$$

Since $\lim \phi(b_i) = z$, there exists $N_1 \in \mathbb{N}$ such that if $l > N_1$, then $\phi(b_l) \in B(z, \frac{\varepsilon_1}{2})$. By [6, Lemma 3], there exists a basis $\mathcal{B}$ of open connected subsets of $X$ such that, for each $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected. Let $V_1 \in \mathcal{B}$ be such that $a \in V_1$ and $\text{diam}(V_1) < 1$. Thus, there is $N_2 > N_1$ such that if $m > N_2$, then $a_m \in V_1 - \mathcal{P}(X)$. Let $l_1 > N_2$. Hence, $b_{l_1} \in \phi^{-1}(B(z, \frac{\varepsilon_1}{2})) \cap (V_1 - \mathcal{P}(X))$. By (4.1), there exists $k_1 > N_2$ such that $\phi(a_{k_1}) \notin B(z, \varepsilon_1)$. Notice that $a_{k_1}, b_{l_1} \in V_1 - \mathcal{P}(X)$. Since $V_1 - \mathcal{P}(X)$ is an open connected subset of $X$, by [21, 8.26], $V_1 - \mathcal{P}(X)$ is arcwise connected. Then, there exists an arc $\alpha_1$ in $V_1 - \mathcal{P}(X)$ with end points $a_{k_1}$ and $b_{l_1}$. Hence, $\gamma_1 = \phi(\alpha_1)$ is an arc with end points $\phi(a_{k_1})$ and $\phi(b_{l_1})$. Notice that $\text{diam}(\gamma_1) \geq \frac{\varepsilon_1}{2}$. Now, let $V_2 \in \mathcal{B}$ be such that $a \in V_2$, $\text{diam}(V_2) < \frac{1}{2}$ and $\alpha_1 \cap V_2 = \emptyset$. Thus, there is $N_3 > N_2$ such that if $m > N_3$, then $a_m \in V_2 - \mathcal{P}(X)$. Let $l_2 > N_3$. Hence, $b_{l_2} \in \phi^{-1}(B(z, \frac{\varepsilon_1}{2})) \cap (V_2 - \mathcal{P}(X))$. By (4.1), there exists $k_2 > N_3$ such that $\phi(a_{k_2}) \notin B(z, \varepsilon_1)$. Notice that $a_{k_2}, b_{l_2} \in V_2 - \mathcal{P}(X)$. Then, there exists an arc $\alpha_2$
in $V_2 - \mathcal{P}(X)$ with end points $a_{k_2}$ and $b_{i_2}$. Therefore, $\gamma_2 = \phi(\alpha_2)$ is an arc with end points $\phi(a_{k_2})$ and $\phi(b_{i_2})$ and $\text{diam}(\gamma_2) \geq \frac{\varepsilon}{2}$. Proceeding in a recursive way, we obtain

- a sequence $\{V_i - \mathcal{P}(X)\}_{i=1}^{\infty}$ such that each $V_i - \mathcal{P}(X)$ is an open connected subset of $X$, $a \in V_i$ and $\text{diam}(V_i) < \frac{1}{i}$,
- a sequence $\{\phi(a_{k_i})\}_{i=1}^{\infty}$ such that $\phi(a_{k_i}) \notin B(z, \varepsilon_1)$ and $a_{k_i} \in V_i - \mathcal{P}(X)$,
- a subsequence $\{\phi(b_{i_i})\}_{i=1}^{\infty}$ of the sequence $\{\phi(b_{i_i})\}_{i=1}^{\infty}$ such that $\lim \phi(b_{i_i}) = z$ and $b_{i_i} \in \phi^{-1}(B(z, \frac{\varepsilon}{2})) \cap (V_i - \mathcal{P}(X))$,
- a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of pairwise disjoint arcs such that $\alpha_i \subset V_i - \mathcal{P}(X)$ whose end points are $a_{k_i}$ and $b_{i_i}$, and $\alpha_i \cap V_{i+1} = \emptyset$,
- a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of pairwise disjoint arcs such that $\gamma_i \subset \mathcal{G}(Y)$, where $\gamma_i = \phi(\alpha_i)$, $\text{diam}(\gamma_i) \geq \frac{\varepsilon}{4}$, and $\phi(a_{k_i}), \phi(b_{i_i})$ are the end points of $\gamma_i$.

We may assume that the sequence $\{\phi(a_{k_i})\}_{i=1}^{\infty}$ converges to some point $w \in Y$. Notice that the sequence $\{\gamma_i\}_{i=1}^{\infty}$ is contained in $C(Y)$. By [21, 4.17], we may suppose that $\{\gamma_i\}_{i=1}^{\infty}$ converges to some $\gamma \in C(Y)$. Since $\phi(a_{k_i}) \notin B(z, \frac{\varepsilon}{2})$, for each $i \in \mathbb{N}$, we have that $w \neq z$. Notice that $w, z \in \gamma$. Thus, $\gamma \in C(Y) - F_1(Y)$. Since $g^{-1}$ is a homeomorphism, we have that $\lim g^{-1}(\gamma_i) = g^{-1}(\gamma)$, where $g^{-1}(\gamma) \in C_n(X) - F_1(X)$.

On the other hand, since $\lim a_{k_i} = a$, $\lim b_{i_i} = a$ and $\lim \text{diam}(\alpha_i) = 0$, we have that $\lim \alpha_i = \{a\}$.

Fix $i \in \mathbb{N}$. Since $a_{k_i}, b_{i_i} \in \mathcal{G}(X) \cap R(X)$ and $\alpha_i \cap \mathcal{P}(X) = \emptyset$, we have that $\alpha_i = J_1 \cup \cdots \cup J_{s_i}$, where $J_1, \ldots, J_{s_i} \in \mathcal{A}_S(X)$. Thus, $\gamma_i = \phi(J_1) \cup \cdots \cup \phi(J_{s_i})$. By definition of $\phi$, $\gamma_i = (J_1)^h \cup \cdots \cup (J_{s_i})^h$.

Hence, $q_X^n((J_1)^n_1 \cup \cdots \cup (J_{s_i})^n_1) \setminus \{F_1^n\} = q_Y^n((J_1)^h_1 \cup \cdots \cup (J_{s_i})^h_1) \setminus \{F_Y^n\}$.

By (b), we have that

$$h^{-1}(q_X^n((J_1)^n_1 \cup \cdots \cup (J_{s_i})^n_1) \setminus \{F_1^n\}) \subset q_X^n((J_1)^h_1 \cup \cdots \cup (J_{s_i})^h_1) \setminus \{F_X^n\}.$$ 

Consequently, $g^{-1}((J_1)^n_1 \cup \cdots \cup (J_{s_i})^n_1) \setminus F_1(Y) \subset (J_1)^h_1 \cup \cdots \cup (J_{s_i})^h_1 - F_1(X)$. This implies that $g^{-1}(\gamma_i) \subset (\alpha_i) - F_1(X)$ and $g^{-1}(\gamma_i) \subset \alpha_i$. Therefore, $g^{-1}(\gamma) \subset \{a\}$, a contradiction. This proves Claim 5.

**Claim 6.** If $a \in \mathcal{P}(X)$ and $\{a_m\}_{m=1}^{\infty}$ is a sequence contained in $\mathcal{G}(X)$ such that $\lim a_m = a$, then $\{\phi(a_m)\}_{m=1}^{\infty}$ converges.

We may assume that there exists a sequence $\{J_m\}_{m=1}^{\infty}$ of pairwise distinct elements of $\mathcal{A}_S(X)$ such that $a_m \in J_m$, for each $m \in \mathbb{N}$. By [6, Lemma 8], we obtain that $\{J_m\}_{m=1}^{\infty}$ converges to $\{a\}$. Let $r_m \in J_m \cap R(X)$, for each $m \in \mathbb{N}$. Thus, $\{r_m\}_{m=1}^{\infty}$ is a sequence contained in $\mathcal{G}(X) \cap R(X)$ which converges to $a$. By Claim 5, there exists $z \in Y$ such that $\lim \phi(r_m) = z$. Notice that $\phi(r_m) \in (J_m)^h_a$, for each $m \in \mathbb{N}$. By [6, Lemma 8], we obtain that $\{(J_m)^h\}_{m=1}^{\infty}$ converges to $\{z\}$. Since $\phi(a_m) \in (J_m)^h_a$, $\lim \phi(a_m) = z$, for each $m \in \mathbb{N}$. This proves Claim 6.

Moreover, let $a \in \mathcal{P}(X)$, $\{a_m\}_{m=1}^{\infty}$ and $\{a'_m\}_{m=1}^{\infty}$ be sequences in $\mathcal{G}(X)$ which converge to $a$. By Claim 6, $\{\phi(a_m)\}_{m=1}^{\infty}$ and $\{\phi(a'_m)\}_{m=1}^{\infty}$ are convergent sequences. Now, let $b_{2k-1} = a_k$ and $b_{2k} = a'_k$, for $k \in \mathbb{N}$. Hence, $\{b_m\}_{m=1}^{\infty}$ is a sequence in $\mathcal{G}(X)$ which converges to $a$. By Claim 6, there exists $z \in Y$ such that $\lim \phi(b_m) = z$. Since $\{\phi(a_m)\}_{m=1}^{\infty}$ and $\{\phi(a'_m)\}_{m=1}^{\infty}$ are convergent subsequences of $\phi(\{b_m\}_{m=1}^{\infty})$, we have that $\lim \phi(a_m) = z$ and $\lim \phi(a'_m) = z$. From this, we may associate to each $a \in \mathcal{P}(X)$ a unique element of $\mathcal{P}(Y)$ which will denote by $a_\phi$. Consequently, we define a map $\Phi: X \rightarrow Y$ given by
\[ \Phi(x) = \begin{cases} \phi(x) & \text{if } x \in \mathcal{G}(X), \\ x_\phi & \text{if } x \in \mathcal{P}(X). \end{cases} \]

Since \( Y \) satisfies similar conditions as \( X \), the following claim is true.

**Claim 7.** If \( b \in \mathcal{P}(Y) \) and \( \{ b_m \}_{m=1}^\infty \) is a sequence contained in \( \mathcal{G}(Y) \) which converges to \( b \), then \( \{ \phi^{-1}(b_m) \}_{m=1}^\infty \) converges to an unique element \( b_{\phi^{-1}} \in \mathcal{P}(X) \), which does not depend on the sequence \( \{ b_m \}_{m=1}^\infty \).

From Claim 7, we have that \( \Phi \) is one to one. Now, let \( b \in \mathcal{P}(Y) \). By [5, Theorem 3.3], there exists a sequence \( \{ b_m \}_{m=1}^\infty \) contained in \( \mathcal{G}(Y) \cap R(Y) \) which converges to \( b \). Thus, by Claim 7, the sequence \( \{ \phi^{-1}(b_m) \}_{m=1}^\infty \) converges to an unique element \( b_{\phi^{-1}} \in \mathcal{P}(X) \). Notice that \( \Phi(b_{\phi^{-1}}) = b \). Hence, \( \Phi \) is surjective. Therefore, \( \Phi \) is a homeomorphism and \( X \) is homeomorphic to \( Y \).

The proof of following result, except Case 2, is a modification of [7, Theorem 3.2].

**Theorem 4.8.** Let \( X \) be a meshed continuum such that \( R(X) \neq \emptyset \) and \( n \geq 2 \). If \( | \cap \mathfrak{A}_S(X) | \neq 2 \), then \( X \) has unique \( n \)-fold pseudo-hyperspace suspension.

**Proof.** Let \( Y \) be a continuum and let \( h : \text{PHS}_n(X) \longrightarrow \text{PHS}_n(Y) \) be a homeomorphism. By Theorem 4.3, we know that \( Y \) is a meshed continuum. Moreover, if \( Y \) is an arc or a simple closed curve, by [18, Theorem 5.7] it follows that \( X \) is homeomorphic to \( Y \). This is a contradiction since \( R(X) \neq \emptyset \). Hence, \( R(Y) \neq \emptyset \). Moreover, by Theorem 4.6, we have that \( | \cap \mathfrak{A}_S(Y) | \neq 2 \). We consider two cases:

**Case 1.** \( n \geq 3 \).

Since the definition of \( \mathcal{PLC}_n(X) \) is given in terms of topological properties, we have that \( h(\mathcal{PLC}_n(X)) = \mathcal{PLC}_n(Y) \). This implies that \( h(\mathcal{PD}_n(X)) = \mathcal{PD}_n(Y) \). Given \( J \in \mathfrak{A}_S(X) \), by Lemma 3.7 (a), we know that \( h(q_X \langle \langle J^0 \rangle_1 \rangle - \{ F_X \} ) \) is a component of \( \mathcal{PD}_n(X) \). Hence, there exists \( J_h \in \mathfrak{A}_S(Y) \) such that \( h(q_X \langle \langle J^0 \rangle_1 \rangle - \{ F_X \} ) = q_Y \langle \langle J_h^0 \rangle_1 \rangle - \{ F_Y \} \). Moreover, with similar arguments for \( Y \), we have that \( \mathfrak{A}_S(Y) = \{ J_h : J \in \mathfrak{A}_S(X) \} \). Thus, (a), (b), (c), and (d) from Theorem 4.7 are satisfied.

Now we verify conditions (1) and (2) from Theorem 4.7. Let \( J \in \mathfrak{A}_S(X) \) be such that \( | J \cap R(X) | = 1 \).

We will show that if \( J \) is an arc, then \( J_h \) is an arc (and, by symmetry, the converse implication also holds). Suppose that \( J \) is an arc with end points \( p \) and \( q \), where \( q \notin R(X) \). Suppose that \( J_h \) is a cycle. Let \( A \) be a subarc of \( J \) such that \( p \in A \) and \( q \notin A \). We know that \( h(q_X \langle \langle J^0 \rangle_1 \rangle - \{ F_X \} ) = q_Y \langle \langle J_h^0 \rangle_1 \rangle - \{ F_Y \} \). Let \( D = q_X \langle A \rangle \) and \( E = h(D) \). Then \( E = q_Y \langle \langle J_h^0 \rangle_1 \rangle - \{ F_Y \} \). Then there exists \( B \in \langle J_h^0 \rangle_1 - F_1(Y) \) such that \( q_Y \langle B \rangle = E \). Notice that \( B \) is a subarc of \( J_h \). Since \( X \) and \( Y \) are meshed continua, we have that \( J \cap P(X) = \emptyset = J_h \cap P(Y) \). By Lemma 3.2, there exist finite graphs \( M \) in \( X \) and \( M_h \) in \( Y \) such that \( J \subset M \) and \( J_h \subset M_h \). By (3.1), \( 2n = \dim_A[M \cap \mathcal{C}_n(M)] = \dim_A[C_n(X)] = \dim_D[P\text{HS}_n(X)] = \dim_E[P\text{HS}_n(Y)] = \dim_B[C_n(Y)] \). Thus, \( B \cap R(Y) = \emptyset \). Since \( C(J_h) \) is a 2-cell such that its manifold boundary is \( F_1(J_h) \), we have that \( B \) has a neighborhood \( M \) in \( \langle J_h^0 \rangle_1 - F_1(Y) \) which is a 2-cell and \( B \) belongs to its manifold interior. Hence, \( q_Y \langle M \rangle \) is a neighborhood of \( E \) in \( q_Y \langle \langle J_h^0 \rangle_1 \rangle - \{ F_Y \} \) such that \( q_Y \langle M \rangle \) is a 2-cell and \( B \) belongs to its manifold interior. Since \( h(F_X) \neq F_Y \), it implies that \( q_X^{-1} \circ h \circ q_Y \langle \langle J^0 \rangle_1 \rangle - \{ F_X \} \) is a 2-cell and \( A \) belongs to its manifold interior. This is a contradiction since \( A \) belongs to the manifold boundary of \( C(J) \). Therefore, \( J_h \) is an arc. Moreover, by Claim 1 (g) of Theorem 4.7, we have that \( | J_h \cap R(Y) | = 1 \) and \( J_h \in \mathfrak{A}_E(Y) \). Consequently, \( J \in \mathfrak{A}_E(X) \) if and only if \( J_h \in \mathfrak{A}_E(Y) \). Thus, conditions (1) and (2) from Theorem 4.7 are satisfied. Therefore, \( X \) is homeomorphic to \( Y \).

**Case 2.** \( n = 2 \).

Notice that \( h(\mathcal{PLC}_2(X)) = \mathcal{PLC}_2(Y) \). Given \( J \in \mathfrak{A}_S(X) \), by Lemma 3.7 (b), there exist \( J_h, K_h \in \mathfrak{A}_S(Y) \) such that \( h(q_X \langle \langle J^0 \rangle_2 \rangle - \{ F_2^X \} ) = q_Y \langle \langle J_h^0, K_h^0 \rangle_2 \rangle - \{ F_2^Y \} \). By Lemma 3.5, we have that \( F_2^X \notin \partial \mathcal{PLC}_2(X) \), \( F_2^Y \notin \partial \mathcal{PLC}_2(Y) \) and \( h(\partial \mathcal{PLC}_2(X)) = \partial \mathcal{PLC}_2(Y) \). Thus,
Lemma 1.6.

\[
h(\partial \mathcal{PH}_2(X) \cap q_X^{-1}(J^0)_{2}) = \partial \mathcal{PH}_2(Y) \cap q_Y^{-1}(J_k^0, K_k^0)_{2},\]
\[
h(\partial \mathcal{PH}_2(X) - q_X^{-1}(J^0)_{2}) = \partial \mathcal{PH}_2(Y) - q_Y^{-1}(J_k^0, K_k^0)_{2}.
\]

Hence, \( h(\mathcal{PD}(J, J)) = \mathcal{PD}(J_h, K_h) \). By Remark 3.12, we have that \( J_h = K_h \). Consequently, \( h(q_X^{-1}(J^0)_{2}) = q_Y^{-1}(J_k^0)_{2} - \{F_X^0\} \) and \( h(q_X^{-1}(J^0)_{1}) = q_Y^{-1}(J_k^0)_{2} \). Moreover, under similar arguments for \( Y \), we have that \( \mathfrak{A}_S(Y) = \{ J_h : J \in \mathfrak{A}_S(X) \} \). Finally, by Remark 3.12 (b) and (c), conditions (1) and (2) from Theorem 4.8 are satisfied. Therefore, \( X \) is homeomorphic to \( Y \). \( \square \)

The notions of framed and almost framed continua appear in [11, p. 48]. Given a continuum \( X \), notice that \( \bigcup \{ J : J \text{ is a free arc in } X \} \) is dense in \( X \) if and only if \( \bigcup \{ J^0 : J \text{ is a free arc in } X \} \) is dense in \( X \). By [6, Lemma 1], we have that \( \bigcup \{ J : J \text{ is a free arc in } X \} \) is dense in \( X \) if and only if \( G(X) \) is dense in \( X \). From this the following remark holds.

Remark 4.9. Let \( X \) be a locally connected continuum. Then \( X \) is almost framed if and only if \( X \) is almost meshed. Moreover, \( X \) is framed if and only if \( X \) is meshed distinct to a simple closed curve.

Theorem 4.10. If \( X \) is a meshed continuum and \( n \in \mathbb{N} \), then \( X \) has unique \( n \)-fold pseudo-hyperspace suspension.

Proof. Suppose that \( X \) is a meshed continuum and let \( n \in \mathbb{N} \). By [18, Theorem 5.7], we may assume that \( X \) is not a finite graph. So that we consider the following two cases:

Case 1. \( R(X) \neq \emptyset \) and \( n = 1 \).

Since \( PHS_1(X) = HS_1(X) \), by [8, Theorem 3.4] the result follows.

Case 2. \( R(X) \neq \emptyset \) and \( n \geq 2 \).

As a consequence of Theorem 4.5 and Theorem 4.8, we have that \( X \) has unique \( n \)-fold pseudo-hyperspace suspension. \( \square \)

5. Locally connected continua without unique hyperspace

Given a continuum \( X \), a nonempty closed subset \( K \) of \( X \), and \( n \in \mathbb{N} \), let

\[
F_n(X, K) = \{ A \in F_n(X) : A \cap K \neq \emptyset \}
\]

and

\[
C_n(X, K) = \{ A \in C_n(X) : A \cap K \neq \emptyset \}.
\]

For two disjoint continua \( X \) and \( Y \), and given points \( p \in X \) and \( q \in Y \), let \( X \cup_p Y \) be the continuum obtained by attaching \( X \) to \( Y \), identifying \( p \) to \( q \).

Given a continuum \( X \) with metric \( d \), a closed subset \( A \) of \( X \) is said to be a \( Z \)-set in \( X \) provided that, for each \( \varepsilon > 0 \), there is a map \( f_\varepsilon : X \to X - A \) such that \( d(f_\varepsilon(x), x) < \varepsilon \) for all \( x \in X \). A map between compacta \( f : X \to Y \) is called a \( Z \)-map provided that \( f(X) \) is a \( Z \)-set in \( Y \). Let \( \varepsilon > 0 \) and \( A \in 2^X \), the generalized closed \( d \)-ball in \( X \) of radius \( \varepsilon \) about \( A \), denoted by \( C_d(\varepsilon, A) \), is defined as follows: \( C_d(\varepsilon, A) = \{ x \in X : d(x, A) \leq \varepsilon \} \). Whenever \( A = \{ p \} \), we write \( C(\varepsilon, p) \) instead of \( C(\varepsilon, \{ p \}) \).

A metric \( d \) for \( X \) is said to be convex provided that, for any \( p, q \in X \), there exists \( m \in X \) such that \( d(p, m) = \frac{1}{2} d(p, q) = d(m, q) \). By [2, 22], if \( X \) is a locally connected continuum, then \( X \) admits a metric convex.

Given a locally connected continuum \( X \) with convex metric \( d \) and \( \varepsilon > 0 \), define \( \Phi_\varepsilon : 2^X \to 2^X \) by \( \Phi_\varepsilon(A) = C_d(A, \varepsilon) \). By [13, Proposition 10.5], \( \Phi_\varepsilon \) is a map.

Lemma 5.1. Let \( n \in \mathbb{N} \) and \( K, L \) be closed subsets of a locally connected continuum \( X \). Then \( F_m(X, L) \) is a \( Z \)-set in \( C_n(X, K) \), for each \( m \in \{1, \ldots, n\} \).
Proof. Let $\varepsilon > 0$ and $m \in \{1, \ldots, n\}$. We assume that the metric for $X$ is convex. Given $A \in C_n(X, K)$, by [13, Proposition 10.6], we have that $C_d(\frac{1}{2}, A) \in C_n(X, K)$. Moreover, $C_d(\varepsilon, A) \notin F_m(X)$. Let $f_\varepsilon = \Phi_{\frac{1}{2}}|_{C_n(X, K)}$. Hence, $f_\varepsilon$ is a map from $C_n(X, K)$ to $C_n(X, K) - F_m(X, L)$. Notice that $C_d(\frac{1}{2}, A) \subset N(\varepsilon, A)$ and, clearly, $A \subset N(\varepsilon, C_d(\frac{1}{2}, A))$. Thus, $H(C_d(\frac{1}{2}, A), A) < \varepsilon$, which is equivalent to $H(f_\varepsilon(A), A) < \varepsilon$. Therefore, $F_m(X, L)$ is a Z-set in $C_n(X, K)$. \qed

Theorem 5.2. [1, Corollary 10.3] (Anderson’s homogeneity theorem). If $h : A \to B$ is a homeomorphism between $Z$-sets in a Hilbert cube $Q$, then $h$ extends to a homeomorphism of $Q$ onto $Q$.

Theorem 5.3. Let $X$ be an almost meshed locally connected continuum and $n \in \mathbb{N}$. Suppose that there exist a contractible closed subset $R$ of $\mathcal{P}(X)$ and pairwise disjoint nonempty open subsets $U_1, \ldots, U_{n+1}$ of $X$ such that

(a) $X - R = U_1 \cup \cdots \cup U_{n+1}$ and
(b) $R \subset \text{cl}_X(U_i)$, for each $i \in \{1, \ldots, n+1\}$.

Then $X$ does not have unique hyperspace $\text{PHS}_m(X)$, for each $m \leq n$.

Proof. Let $m \leq n$ and fix $p \in R$. By [6, Theorem 18], there exists a dendrite $D$ without free arcs and disjoint to $X$ such that $Y = X \cup_p D$ is a locally connected continuum not homeomorphic to $X$.

By the proof of [6, Theorem 22], we have that $C_m(Y)$ is homeomorphic to $C_m(X)$. In fact, the homeomorphism $h : C_m(X) \to C_m(Y)$ constructed in such proof satisfies $h(A) = A$, for each $A \in C_m(X) - C_m(X, R)$. In particular, $h(F_1(G(X))) = F_1(G(X))$ and since $X$ is almost meshed, we obtain that

$$h(F_1(X)) = h(\text{cl}_{C_m(X)} F_1(G(X))) = \text{cl}_{C_m(Y)} F_1(G(X)) = F_1(X).$$

Let $q_{X,Y}^m : C_m(Y) \to C_m(Y)/F_1(X)$ be the quotient function and $q_{X,Y}^m(F_1(X)) = \{F_{X,Y}^m\}$. Since $q_{X,Y}^m|_{C_m(X)/F_1(X)}$, $h|_{C_m(X)/F_1(X)}$ and $q_{X,Y}^m|_{C_m(Y)/F_1(X)}$ are homeomorphisms, $\text{PHS}_m(X) - \{F_{X,Y}^m\}$ is homeomorphic to $\text{PHS}_m(Y)/F_1(Y)$. Thus, $\text{PHS}_m(X)$ is homeomorphic to $\text{PHS}_m(Y)/F_1(Y)$.

In order to conclude, we only need to show $C_m(Y)/F_1(Y)$ is homeomorphic to $\text{PHS}_m(Y)$. First, we are going to prove that $q_{Y}^m(C_m(Y, R \cup D))$ and $q_{X,Y}^m(C_m(Y, R \cup D))$ are Hilbert cubes. By [6, Theorem 16], we know that $C_m(Y, R \cup D)$ is a Hilbert cube. Notice that $q_{Y}^m(C_m(Y, R \cup D))$ is homeomorphic to $C_m(Y, R \cup D)/F_1(Y, R \cup D)$ and $q_{X,Y}^m(C_m(Y, R \cup D))$ is homeomorphic to $C_m(Y, R \cup D)/F_1(Y, R \cup D)$ by [3, Theorem 1.2 (21)], we know that $D$ is contractible. Thus, $R \cup_p D$ is contractible. Hence, $F_1(Y, R \cup D)$ and $F_1(Y, R)$ are contractible. Since $Y$ is locally connected, by Lemma 5.1, we have that $F_1(Y, R \cup D)$ and $F_1(Y, R)$ are Z-sets of $C_m(Y, R \cup D)$. By [10, Corollary 2.7], we have that $C_m(Y, R \cup D)/F_1(Y, R \cup D)$ and $C_m(Y, R \cup D)/F_1(Y, R)$ are Hilbert cubes. Therefore, $q_{Y}^m(C_m(Y, R \cup D))$ and $q_{X,Y}^m(C_m(Y, R \cup D))$ are Hilbert cubes.

Claim. The space $\text{bd}_{\text{PHS}_m(Y)}(q_{Y}^m(C_m(Y, R \cup D)))$ is a Z-set of $q_{Y}^m(C_m(Y, R \cup D))$.

Proof of Claim. We denote the metric of $\text{PHS}_m(Y)$ by $\overline{H}$. Let $\varepsilon > 0$. Since $C_m(Y)$ is compact, we have that $q_{Y}^m$ is uniformly continuous. Thus, there exists $\delta > 0$ such that if $A, B \in C_m(Y)$ with $H(A, B) < \delta$, then $\overline{H}(q_{Y}^m(A), q_{Y}^m(B)) < \frac{\varepsilon}{2}$. By [6, Theorem 22, Claim 2], there exists a map

$$g_\delta : C_m(Y, R \cup D) \to C_m(Y, R \cup D) - \text{bd}_{C_m(Y)}(C_m(Y, R \cup D))$$

such that $H(g_\delta(A), A) < \delta$, for each $A \in C_m(Y, R \cup D)$. 


On the other hand, by [10, Remark 2.6], the one point sets of the Hilbert cube are $Z$-sets. Thus, there is a map
\[
\gamma : q^m_Y(C_m(Y, R ∪ D)) \rightarrow q^m_Y(C_m(Y, R ∪ D)) - \{F^m_Y\}
\]
such that $\overline{H}(\gamma(B), B) < \frac{\delta}{2}$, for each $B ∈ q^m_Y(C_m(Y, R ∪ D))$. Let $f = q^m_Y|_{C_m(Y) - F_1(Y)}$. By [10, Lemma 2.8], we know that $\text{bd}_{PHS_m(Y)}(q^m_Y(C_m(Y, R ∪ D))) = q^m_Y(\text{bd}_{C_m(Y)}(C_m(Y, R ∪ D)))$. Hence, we define the map
\[
f_\varepsilon : q^m_Y(C_m(Y, R ∪ D)) \rightarrow q^m_Y(C_m(Y, R ∪ D)) - \text{bd}_{PHS_m(Y)}(q^m_Y(C_m(Y, R ∪ D)))
\]
by $f_\varepsilon(B) = q^m_Y \circ g_\delta \circ f^{-1} \circ \gamma(B)$, for each $B ∈ q^m_Y(C_m(Y, R ∪ D))$. Given $B ∈ q^m_Y(C_m(Y, R ∪ D))$, we have that $H(g_\delta(f^{-1}(\gamma(B))), f^{-1}(\gamma(B))) < \delta$. Thus, $\overline{H}(q^m_Y(g_\delta(f^{-1}(\gamma(B))))), q^m_Y(f^{-1}(\gamma(B))) < \frac{\delta}{2}$. Therefore, $\overline{H}(f_\varepsilon(B), \gamma(B)) < \frac{\delta}{2}$. Since $\overline{H}(\gamma(B), B) < \varepsilon$, we have that $\overline{H}(f_\varepsilon(B), B) < \varepsilon$. This proves the claim. \(\square\)

Using arguments that are analogous to those of the previous claim, we obtain that $\text{bd}_{C_m(Y)/F_1(X)}(q^m_{X,Y}(C_m(Y, R ∪ D)))$ is a $Z$-set of $q^m_{X,Y}(C_m(Y, R ∪ D))$.

By [10, Lemma 2.9 (b)], there exists a homeomorphism $h_1 : q^m_{X,Y}(C_m(X)) \rightarrow q^m_Y(C_m(X))$ such that $h_1(q^m_{X,Y}(A)) = q^m_Y(A)$, for each $A ∈ C_m(X)$. Thus,
\[
h_1(q^m_{X,Y}(\text{bd}_{C_m(Y)}(C_m(Y, R ∪ D)))) = q^m_Y(\text{bd}_{C_m(Y)}(C_m(Y, R ∪ D)))
\]
and therefore,
\[
h_1(\text{bd}_{C_m(Y)/F_1(X)}(q^m_{X,Y}(C_m(Y, R ∪ D)))) = \text{bd}_{PHS_m(Y)}(q^m_Y(C_m(Y, R ∪ D))).
\]

Hence, $h_1|_{\text{bd}_{C_m(Y)/F_1(X)}(q^m_{X,Y}(C_m(Y, R ∪ D)))}$ is a homeomorphism between the $Z$-sets $\text{bd}_{C_m(Y)/F_1(X)}(q^m_{X,Y}(C_m(Y, R ∪ D)))$ and $\text{bd}_{PHS_m(Y)}(q^m_Y(C_m(Y, R ∪ D)))$, by Anderson’s homogeneity theorem (Theorem 5.2) there exists a homeomorphism
\[
h_2 : q^m_{X,Y}(C_m(Y, R ∪ D)) \rightarrow q^m_Y(C_m(Y, R ∪ D))
\]
such that $h_2(A) = h_1(A)$, for each $A ∈ \text{bd}_{C_m(Y)/F_1(X)}(q^m_{X,Y}(C_m(Y, R ∪ D)))$.

Let $h : C_m(Y)/F_1(X) \rightarrow PHS_m(Y)$ be given by
\[
h(A) = \begin{cases} h_1(A) & \text{if } A ∈ C_m(Y)/F_1(X) - q^m_{X,Y}(C_m(Y, R ∪ D)), \\ h_2(A) & \text{if } A ∈ q^m_{X,Y}(C_m(Y, R ∪ D)). \end{cases}
\]

Then, $h$ is a homeomorphism, and the theorem is proved. \(\square\)

Let $m ∈ \mathbb{N}$ and
\[
Z_3 = ([{-1, 1}] × \{0\}) ∪ \bigcup \{\{-\frac{1}{m}\} × [0, \frac{1}{m}] : m ≥ 2\} ∪ \bigcup \{\frac{1}{m} × [0, \frac{1}{m}] : m ≥ 2\}.
\]

The continuum $Z_3$ has unique hyperspace $C_2(Z_3)$ [6, Example 39].

**Example 5.4.** The continuum $Z_3$ has unique hyperspace $PHS_2(Z_3)$ but it does not have unique hyperspace $PHS_1(Z_3) = HS_1(Z_3)$. 
Notice that $Z_3$ is an almost meshed locally connected continuum such that $\mathcal{P}(Z_3) = \{(0,0)\}$ and $Z_3$ is not meshed continuum. Using Theorem 5.3, we have that $Z_3$ does not have unique hyperspace $PHS_1(Z_3)$.

Let $\theta = (0,0)$. Suppose that $Y$ is a continuum such that $PHS_2(Z_3)$ and $PHS_2(Y)$ are homeomorphic. Let $h : PHS_2(Z_3) \to PHS_2(Y)$ be a homeomorphism. By Lemma 4.2, we have that $Y$ is locally connected. Moreover, by [18, Theorem 5.7], $Y$ is not a finite graph. Hence, $R(Y) \neq \emptyset$. Since $|\mathfrak{A}_S(Z_3)| \geq 2$, using Lemma 3.7 (b), we have that $|\mathfrak{A}_S(Y)| \geq 2$. Also, given $J \in \mathfrak{A}_S(Z_3)$, by Lemma 3.7 (b), there exist $J_h, K_h \in \mathfrak{A}_S(Y)$ such that $h(q_2^Z(J_h^0) - \{F_2^Z\}) = q_2^Y(J_h^0) - \{F_2^Y\}$ and $h(q_2^Z(J_h^0) - \{F_2^Z\}) \subset q_2^Y(J_h^0) - \{F_2^Y\}$. Moreover, under similar arguments for $Y$, we have that $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(Z_3)\}$. In the same way as in the proof of Theorem 4.7, we conclude the association $J \to J_h$ is a bijection between $\mathfrak{A}_S(Z_3)$ and $\mathfrak{A}_S(Y)$, and $h(F_2^Z) = F_2^Y$. Thus, $g : C_2(Z_3) - F_1(Z_3) \to C_2(Y) - F_1(Y)$ defined as $g = (q_2^Y)^{-1} \circ h \circ q_2^Z$ is a homeomorphism. Hence, (e) and (f) of Claim 1 from Theorem 4.7 hold. Notice that $J \cap \mathcal{P}(Z_3) = \emptyset$, for each $J \in \mathfrak{A}_S(Z_3)$. Using (f) and Lemma 3.2, we conclude $J_h \cap \mathcal{P}(Y) = \emptyset$, for each $J_h \in \mathfrak{A}_S(Y)$.

By Remark 3.12 (b) and (c), we have that

1. $Y$ does not have cycles and
2. $J \in \mathfrak{A}_E(Z_3)$ if and only if $J_h \in \mathfrak{A}_E(Y)$.

Since, $J \cap \mathcal{P}(Z_3) = \emptyset$ and $J_h \cap \mathcal{P}(Y) = \emptyset$, for each $J \in \mathfrak{A}_S(Z_3)$, proceeding as in Claims 1 to 4 from Theorem 4.7, we define a homeomorphism $\phi : \mathcal{G}(Z_3) \to \mathcal{G}(Y)$. Let

$$\mathcal{G}_I(Z_3) = \{[-1, 0) \times \{0\}) \cup \bigcup \{\{- \frac{1}{m}\} \times [0, \frac{1}{m}] : m \geq 2\}$$

and

$$\mathcal{G}_D(Z_3) = \{(0, 1] \times \{0\}) \cup \bigcup \{\{\frac{1}{m}\} \times [0, \frac{1}{m}] : m \geq 2\}.$$
\{N_m\}_{m=1}^{\infty} are two sequences of pairwise different elements of \(A(Z_3)\). Let \(a_m \in L_m\), for each \(m \in \mathbb{N}\). Since \(Z_3\) is compact, we may suppose that \(\{a_m\}_{m=1}^{\infty}\) converges to \(a\), for some \(a \in Z_3\). By [6, Lemma 8], we have that \(\{L_m\}_{m=1}^{\infty}\) converges to \(\{a\}\). Hence, by [9, Theorem 4.1], \(a \in \mathcal{P}(Z_3)\). Thus, \(a = \theta\). Analogously, we can prove that \(\{N_m\}_{m=1}^{\infty}\) converges to \(\{\theta\}\). Thus, \(\{L_m \cup N_m\}_{m=1}^{\infty}\) converges to \(\{\theta\}\).

Given \(m \in \mathbb{N}\), notice that \(g^{-1}(\text{cl}_{C_2(Y)}(\langle(C_m)_{h}, (D_m)_{h}\rangle_{2}\rangle) \subset \{L_m\}_{m=1}^{\infty}\), and therefore, \(g^{-1}(\langle(C_m)_{h}, (D_m)_{h}\rangle_{2}\rangle) \subset L_m \cup N_m\). Suppose that \(\theta_1 \neq \theta_D\). Thus, \(\{g^{-1}(\langle(C_m)_{h}, (D_m)_{h}\rangle_{2}\rangle)\}_{m=1}^{\infty}\) converges to \(g^{-1}(\{\theta_1, \theta_D\})\). Hence, \(g^{-1}(\{\theta_1, \theta_D\}) \subset \{\theta\}\), a contradiction. Therefore, \(\theta_1 = \theta_D\). Since \(\text{cl}_{Y}(G(Y)) = \text{cl}_{Y}(\Gamma(Y)) \cup \text{cl}_{Y}(G_{D}(Y))\), we have that \(|\text{cl}_{Y}(G(Y)) - G(Y)| = 1\). Let \(\theta_h \in \text{cl}_{Y}(G(Y)) - G(Y)\) and \(\Phi : Z_3 \rightarrow Y\) be defined as

\[
\Phi(z) = \begin{cases} 
\phi(z) & \text{if } z \in G(Z_3), \\
\theta_h & \text{if } z = \theta.
\end{cases}
\]

Hence, \(\Phi\) is an embedding from \(Z_3\) into \(Y\). By definition of \(\Phi\), we know that \(\Phi(Z_3) = \text{cl}_{Y}(G(Y))\). Notice that, \(\Phi(Z_3) \cap \mathcal{P}(Y) = \{\theta_h\}\). This implies that \(\mathcal{P}(Y)\) is a subcontinuum of \(Y\). Let

\[
\Xi_{Z_3} = \text{int}_{C_2(Z_3) - F_1(Z_3)}((C_2(Z_3) - F_1(Z_3)) - \bar{S}_2(Z_3))
\]

and

\[
\Xi_Y = \text{int}_{C_2(Y) - F_1(Y)}((C_2(Y) - F_1(Y)) - \bar{S}_2(Y)).
\]

Notice that \(g(\Xi_{Z_3}) = \Xi_Y\). Using the same arguments as in [6, Example 39], we have that \(\Xi_{Z_3}\) is disconnected and, if \(Y \neq \text{cl}_{Y}(G(Y))\), then \(\Xi_Y\) is pathwise connected. Hence, \(Y = \text{cl}_{Y}(G(Y))\). Therefore, \(Z_3\) has unique hyperspace \(\text{PHS}_2(Z_3)\).

**Theorem 5.5.** Let \(X\) be a locally connected continuum that is not almost meshed. Suppose that there exist \(p \in \mathcal{P}(X)\) and \(\varepsilon > 0\) such that \(B(p, 2\varepsilon) \subset \mathcal{P}(X)\) and \(C_d(\varepsilon, p)\) is contractible. Then, for every \(n \in \mathbb{N}\), \(X\) does not have unique hyperspace \(\text{PHS}_n(X)\).

**Proof.** By [6, Theorem 18], there exists a dendrite \(D\) without free arcs and disjoint to \(X\) such that \(Y = X \cup_p D\) is a locally connected continuum not homeomorphic to \(X\).

Let \(E = C_d(\varepsilon, p)\). By Lemma 5.1, we have that \(F_1(E)\) is a \(Z\)-set of \(C_n(X, E)\) and \(C_n(Y, E \cup D)\). Using [6, Theorem 22, Claim 2], we have that \(\text{bd}_{C_n(X)}(C_n(X, E)) \cup F_1(E)\) is a \(Z\)-set of \(C_n(X, E)\) and \(\text{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_1(E)\) is a \(Z\)-set of \(C_n(Y, E \cup D)\). Moreover, by [6, Lemma 19], we have that \(\text{bd}_{C_n(X)}(C_n(X, E)) \cup F_1(E) = \text{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_1(E)\). Hence, the identity map

\[
id : \text{bd}_{C_n(X)}(C_n(X, E)) \cup F_1(E) \rightarrow \text{bd}_{C_n(Y)}(C_n(Y, E \cup D)) \cup F_1(E)
\]

is a well-defined homeomorphism. By [6, Theorem 16], we know that \(C_n(X, E)\) and \(C_n(Y, E \cup D)\) are Hilbert cubes. Thus, by Anderson’s homogeneity theorem (Theorem 5.2), the identity map can be extended to a homeomorphism \(h_1 : C_n(X, E) \rightarrow C_n(Y, E \cup D)\).

We define \(h : C_n(X) \rightarrow C_n(Y)\) by

\[
h(A) = \begin{cases} 
h_1(A) & \text{if } A \in C_n(X, E), \\
A & \text{if } A \in C_n(X) - C_n(X, E).
\end{cases}
\]

Notice \(h\) is a homeomorphism such that \(h(F_1(X)) = F_1(X)\).
Let \( q^n_{X,Y} : C_n(Y) \to C_n(Y)/F_1(X) \) be the quotient function and \( q^n_{X,Y}(F_1(X)) = \{ F^n_{X,Y} \} \). Since \( q^n_{X,Y} \big|_{C_n(X) - F_1(X)} \), \( h \big|_{C_n(X) - F_1(X)} \) and \( q^n_{X,Y} \big|_{C_n(Y) - F_1(X)} \) are homeomorphisms, then \( PHS_n(X) - \{ F^n_{X,Y} \} \) is homeomorphic to \( C_n(Y)/F_1(X) - \{ F^n_{X,Y} \} \). Thus, \( PHS_n(X) \) is homeomorphic to \( C_n(Y)/F_1(X) \).

We will prove that \( C_n(Y)/F_1(X) \) is homeomorphic to \( PHS_n(Y) \). First, we are going to prove that \( q^n_Y((C_n(Y,E \cup D))\) and \( q^n_{X,Y}((C_n(Y,E \cup D)) \) are Hilbert cubes. Notice that \( q^n_Y(C_n(Y,E \cup D)) \) is homeomorphic to \( C_n(Y,D)/F_1(Y,E \cup D) \) and \( q^n_{X,Y}(C_n(Y,E \cup D)) \) is homeomorphic to \( C_n(Y,E \cup D)/F_1(Y,E) \). By [3, Theorem 1.2 (21)], we know that \( D \) is contractible. Thus, \( E \cup_n D \) is contractible. Hence, \( F_1(Y,E \cup D) \) and \( F_1(Y,E) \) are contractible. Since \( Y \) is locally connected, by Lemma 5.1, we have that \( F_1(Y,E \cup D) \) and \( F_1(Y,E) \) are \( Z \)-sets of \( C_n(Y,E \cup D) \). By [10, Corollary 2.7], we have that \( C_n(Y,E \cup D)/F_1(Y,E \cup D) \) and \( C_n(Y,E \cup D)/F_1(Y,E) \) are Hilbert cubes. Therefore, \( q^n_{X,Y}(C_n(Y,E \cup D)) \) and \( q^n_{X,Y}(C_n(Y,E \cup D)) \) are Hilbert cubes.

Similar to the Claim from Theorem 5.3 was proved, the following Claim can be shown.

**Claim.** The space \( \text{bd}_{PHS_n(Y)}(q^n_Y(C_n(Y,E \cup D))) \) is a \( Z \)-set of \( q^n_Y(C_n(Y,E \cup D)) \) and the set \( \text{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y,E \cup D))) \) is a \( Z \)-set of \( q^n_{X,Y}(C_n(Y,E \cup D)) \).

Using [10, Lemma 2.9(b)], the function \( f : q^n_{X,Y}(C_n(X)) \to q^n_Y(C_n(X)) \) defined by \( f(q^n_{X,Y}(A)) = q^n_Y(A) \), for each \( A \in C_n(X) \), is a homeomorphism. Thus,

\[
f(q^n_{X,Y}(\text{bd}_{C_n(Y)}(C_n(Y,E \cup D)))) = q^n_Y(\text{bd}_{C_n(Y)}(C_n(Y,E \cup D)))
\]

and therefore,

\[
f(\text{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y,E \cup D)))) = \text{bd}_{PHS_n(Y)}(q^n_Y(C_n(Y,E \cup D))).
\]

Hence, \( f(\text{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y,E \cup D)))) \) is a homeomorphism between \( Z \)-sets \( \text{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y,E \cup D))) \) and \( \text{bd}_{PHS_n(Y)}(q^n_Y(C_n(Y,E \cup D))) \), by Anderson’s homogeneity theorem (Theorem 5.2) there exists a homeomorphism \( g : q^n_{X,Y}(C_n(Y,E \cup D)) \to q^n_Y(C_n(Y,E \cup D)) \) such that \( g(A) = f(A) \), for each \( A \in \text{bd}_{C_n(Y)/F_1(X)}(q^n_{X,Y}(C_n(Y,E \cup D))) \).

Let \( \overline{h} : C_n(Y)/F_1(X) \to PHS_n(Y) \) be given by

\[
\overline{h}(A) = \begin{cases} 
  f(A) & \text{if } A \in C_n(Y)/F_1(X) - q^n_{X,Y}(C_n(Y,E \cup D)), \\
  g(A) & \text{if } A \in q^n_{X,Y}(C_n(Y,E \cup D)).
\end{cases}
\]

Then, \( \overline{h} \) is a homeomorphism. Therefore, \( X \) does not have unique hyperspace \( PHS_n(X) \). □

**Question 5.6.** Is Theorem 5.3 still true if we remove the assumption that \( R \) is contractible?

Regarding to Theorem 5.5, we ask:

**Question 5.7.** Let \( X \) be a locally connected continuum such that \( X \) is not almost meshed and let \( n \in \mathbb{N} \). Does \( X \) have unique hyperspace \( PHS_n(X) \)?

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References