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SEMI-KELLEY COMPACTIFICATIONS OF $(0,1]$

BY

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Abstract. We characterize the semi-Kelley compactifications of $(0,1]$ with remainder being an arc or a simple closed curve. We also prove that there are no semi-Kelley compactifications of $(0,1]$ with remainder being a triod. Finally, we prove that if X is a semi-Kelley compactification of $(0,1]$ with remainder being a Peano continuum G, then G is an arc or a simple closed curve.

1. Introduction. A *continuum* is a compact connected metric space with more than one point. A *subcontinuum* of a continuum X is a nonempty compact connected subset of X, so one point subsets of X are subcontinua of X. Given a continuum X, we consider the hyperspace $C(X)$ of subcontinua of X with the Hausdorff metric H (see [9, Definition 2.1, p. 11]).

A continuum X is said to be a *Kelley continuum* provided that for each point $p \in X$, for each subcontinuum K of X containing p, and for each sequence $\{p_n\}_{n=1}^{\infty}$ in X converging to p, there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of X converging to K such that $p_n \in K_n$ for every $n \in \mathbb{N}$.

Let K be a subcontinuum of a continuum X. A continuum $M \subset K$ is called a *maximal limit continuum of* K *in* X if there is a sequence $\{M_n\}_{n=1}^{\infty}$ of subcontinua of X converging to M such that for each convergent sequence $\{M'_n\}_{n=1}^{\infty}$ of subcontinua of X with $M_n \subset M'_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} M'_n = M' \subset K$, we have $M' = M$. A continuum X is said to be a *semi-Kelley continuum* if for each subcontinuum K of X and for any two maximal limit continua L and M of K in X, either $L \subset M$ or $M \subset L$.

Kelley continua were introduced by J. L. Kelley [11] and they have been useful in the study of contractibility of hyperspaces and in the study of homogeneous continua. Semi-Kelley continua were introduced by J. J. Chara-

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tonik and W. J. Charatonik [5] as a weaker version of Kelley continua. The authors of [5] generalized several results known for Kelley continua to semi-Kelley continua concerning products, hyperspaces, and mapping properties. For instance, they proved that if a Cartesian product of two nondegenerate continua is semi-Kelley, then each factor continuum is Kelley [5, Theorem 4.1, p. 80], but the converse does not hold [5, Example 4.3, p. 81]. In [3], E. Castañeda-Alvarado and I. Vidal-Escobar answered questions posed by J. J. Charatonik, W. J. Charatonik, and A. Illanes by constructing a Kelley continuum X such that neither $X \times [0,1]$ nor $C(X)$ nor small Whitney levels in $C(X)$ are semi-Kelley continua. Recently, in [8], A. Illanes presented an equivalent definition of semi-Kelley continua and he used it to generalize some previous results and to obtain new ones. For more information about semi-Kelley continua, we refer the reader to [8] and [6].

A Kelley continuum X is a *Kelley compactification* if it is a compactification of $(0,1]$. A semi-Kelley continuum X is a *semi-Kelley compactification* if it is a compactification of $(0,1]$. A continuum X is a *Kelley remainder*, respectively *semi-Kelley remainder*, if it is the remainder of a Kelley compactification, respectively semi-Kelley compactification. Kelley compactifications were studied in [1, Section 6] and [14, Corollary 7.2, p. 673]; Kelley remainders were studied in [2] and [4]. G. Acosta and A. Illanes showed that if X is a Kelley compactification then X is atriodic and each subcontinuum of X is a Kelley continuum [1, Theorems 6.2 and 6.3]. P. Pellicer-Covarrubias [14] proved that a continuum X is hereditarily indecomposable if and only if for each compactification Z of $(0,1]$ with remainder X, Z is a Kelley continuum. R. A. Beane and W. J. Charatonik showed that arc-like Kelley continua and Kelley arc continua are Kelley remainders [2, Theorems 2.3 and 3.1]. M. E. Chacón-Tirado proved that circle-like Kelley continua are Kelley remainders [4, Theorem 1, p. 170]. An interesting problem in this area is to determine which known results for Kelley remainders can be extended to semi-Kelley remainders.

In this paper we characterize the semi-Kelley compactifications with remainder being an arc or a simple closed curve. A continuum X is called a *triod* if there is a subcontinuum Z of X such that $X \setminus Z$ is the union of three nonempty sets any two of which are mutually separated in X [13, Definition 11.22, p. 208]. We prove that triods are not semi-Kelley remainders. We also prove that if a Peano continuum G is a semi-Kelley remainder, then G is an arc or a simple closed curve.

2. Preliminaries. A *map* is a continuous function. Given a continuum X with metric d, a point $p \in X$, a nonempty subset A of X, and a positive real number ε, we define $B(\varepsilon,p) = \{x \in X : d(p,x) < \varepsilon\}$, $N(\varepsilon,A) = \bigcup\{B(\varepsilon,x) : x \in A\}$, and $d(p,A) = \inf\{d(p,x) : x \in A\}$.

THEOREM 2.1 ([8, Theorem 2.1]). *Let X be a continuum. Then X is not a semi-Kelley continuum if and only if there exist a subcontinuum K of X, an open subset U of X, and sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of components of* $\mathrm{cl}(U)$ *converging to respective subcontinua A and B of X such that*

(1) *$K \subset U$,*
(2) *$(A \cap K) \setminus B \neq \emptyset$ and $(B \cap K) \setminus A \neq \emptyset$.*

REMARK 2.2. *Let X be a continuum. Let A, B, K, U, A_n, and B_n be subsets of X for every $n \in \mathbb{N}$, as given in Theorem* 2.1.

(1) *If D is the component of* $\mathrm{cl}(U)$ *that contains K, then $K \not\subset \mathrm{int}(D)$.*
(2) *If V is an open subset of X such that $K \subset V \subset U$, then there exist $E, F \in C(X)$ and two sequences $\{E_n\}_{n=1}^{\infty}$, $\{F_n\}_{n=1}^{\infty}$ of components of* $\mathrm{cl}(V)$ *converging to E and F, respectively, such that $E \cap K \not\subset F$ and $F \cap K \not\subset E$.*

Proof. (1) Assume $K \subset \mathrm{int}(D)$ and $a \in A \cap K$. Since $\lim_{n\to\infty} A_n = A$, there exists $a_n \in A_n$, for every $n \in \mathbb{N}$, such that $\lim_{n\to\infty} a_n = a$. As $a \in \mathrm{int}(D)$, there exists $m \in \mathbb{N}$ such that $a_n \in \mathrm{int}(D)$ for all $n \geq m$. As A_n and D are components of $\mathrm{cl}(U)$ that contain a_n, we have $A_n = D$ for all $n \geq m$. Hence, $A = D$. In a similar way, $B = D$. So $(A \cap K) \setminus B = \emptyset$, which is a contradiction. Therefore, $K \not\subset \mathrm{int}(D)$.

(2) Let $a \in (A \cap K) \setminus B$ and $b \in (B \cap K) \setminus A$. Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, there exist $a_n \in A_n$ and $b_n \in B_n$, for every $n \in \mathbb{N}$, such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. We can clearly assume that $a_n, b_n \in V$ for all $n \in \mathbb{N}$. Let E_n and F_n be the components of $\mathrm{cl}(V)$ that contain a_n and b_n, respectively. Then $E_n \subset A_n$ and $F_n \subset B_n$. Without loss of generality suppose that $\lim_{n\to\infty} E_n = E$ and $\lim_{n\to\infty} F_n = F$ for some $E, F \in C(X)$. Notice that $E \subset A$, $F \subset B$, $a \in E \cap K$, and $b \in F \cap K$. Hence, $E \cap K \not\subset F$ and $F \cap K \not\subset E$. ∎

LEMMA 2.3. *Let $X = (0,1] \cup Y$ be a compactification of $(0,1]$ with remainder Y, K be a subcontinuum of Y, and M be a maximal limit continuum of K in Y. Then M is a maximal limit continuum of K in X.*

Proof. Since M is a maximal limit continuum of K in Y, there exists a sequence $\{M_n\}_{n=1}^{\infty}$ of subcontinua of Y such that

(1) $\lim_{n\to\infty} M_n = M$,
(2) for every convergent sequence $\{M'_n\}_{n=1}^{\infty}$ of subcontinua of Y with $M_n \subset M'_n$, for each $n \in \mathbb{N}$ we have
$$\text{if } \lim_{n\to\infty} M'_n = M' \subset K \text{ then } M = M'.$$

Let $\{M''_n\}_{n=1}^{\infty}$ be a convergent sequence of subcontinua of X with $M_n \subset M''_n$, for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} M''_n = M'' \subset K$. We shall prove that $M'' = M$.

Case 1. *There exist* $n_1 < n_2 < \cdots$ *such that* $M''_{n_k} \cap (0,1] \neq \emptyset$ *for each* $k \in \mathbb{N}$. In this case $Y \subset M''_{n_k}$ for every $k \in \mathbb{N}$, and so $Y \subset \lim_{k\to\infty} M''_{n_k} = M'' \subset K \subset Y$. Hence, $Y = K = M''$. Since Y is the unique maximal limit continuum of Y in Y, we have $Y = M$ and so $M = M''$.

Case 2. *There is* $N \in \mathbb{N}$ *such that* $M''_n \cap (0,1] = \emptyset$ *for each* $n \geq N$. In this case, we obtain $M''_n \subset Y$ for every $n \geq N$. Since M is a maximal limit continuum of K in Y, it follows that $M = M''$.

In both cases we obtain $M = M''$. Therefore M is a maximal limit continuum of K in X. ∎

Corollary 2.4. *Let* $X = (0,1] \cup Y$ *be a compactification of* $(0,1]$ *with remainder* Y. *If* X *is a semi-Kelley continuum, then so is* Y.

3. Characterization of semi-Kelley compactifications with remainder being an arc. A continuum X is called an *E-continuum* if X is a compactification of $(0,1]$ with remainder being an arc.

Remark 3.1. If X is an E-continuum, then by [12, (3.1) Lemma, p. 330], we can consider X embedded in the plane in such a way that the remainder is $\{0\} \times [0,1]$ and the rest of the continuum is the graph of a continuous function f_X from $(0,1]$ to $[0,1]$.

For the rest of this paper, an E-continuum X and a map $f_X : (0,1] \to [0,1]$ will always be considered as in Remark 3.1. Given $i, n \in \mathbb{N}$, let $A^i_n = [(i-1)/n, i/n]$.

Theorem 3.2. *Let* X *be an* E*-continuum. Then the following statements are equivalent*:

(1) X *is not a semi-Kelley continuum.*
(2) *There exist* $n, i, j, k, l \in \mathbb{N}$, *with* $n > 4$, $2 < j, k < n-1$, $i < \min\{j,k\} - 1$, $l > \max\{j,k\} + 1$ *such that for each* $\varepsilon \in (0,1]$,
    (a) *there exist* $0 < u < v < w < \varepsilon$ *such that* $f_X(u), f_X(w) \in A^l_n$, $f_X(v) \in A^j_n$ *and* $f_X([u,w]) \subset \bigcup_{m=j}^{l} A^m_n$, *and*
    (b) *there exist* $0 < x < y < z < \varepsilon$ *such that* $f_X(x), f_X(z) \in A^i_n$, $f_X(y) \in A^k_n$ *and* $f_X([x,z]) \subset \bigcup_{m=i}^{k} A^m_n$.

Proof. Assume (2) holds and take n, i, j, k, l as given in (2). Let $\varepsilon \in (0,1]$. Define

$$K = \{0\} \times \left[\frac{\min\{j,k\} - 1}{n} - \frac{1}{4n}, \frac{\max\{j,k\}}{n} + \frac{1}{4n}\right],$$
$$U = \left([0,1] \times \left(\frac{i}{n} + \frac{1}{2n}, \frac{l-1}{n} - \frac{1}{2n}\right)\right) \cap X.$$

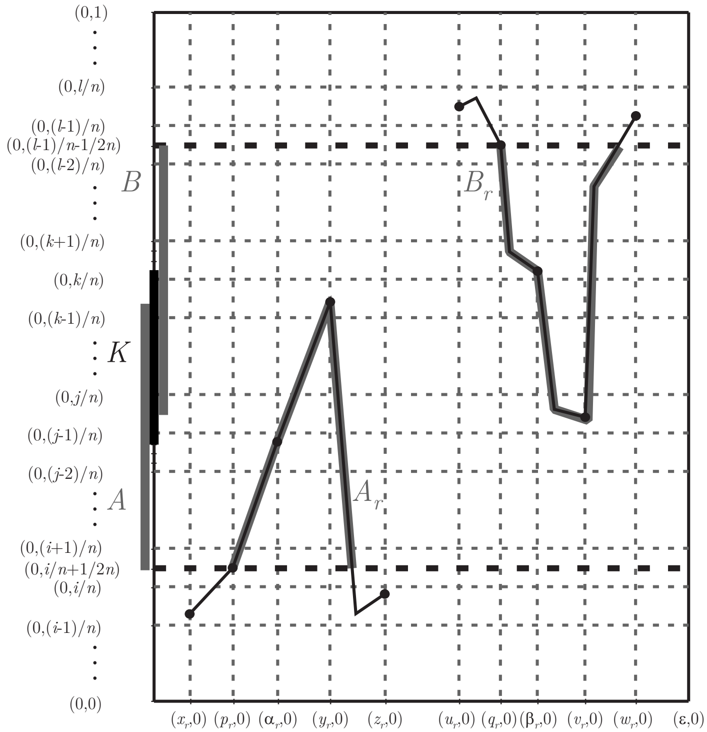

Fig. 1. Sets used in (2)⇒(1), assuming $j < k$

Notice that $K \subset U$. By hypothesis, we can define six sequences of points in $(0,\varepsilon)$, $\{u_r\}_{r=1}^{\infty}$, $\{v_r\}_{r=1}^{\infty}$, $\{w_r\}_{r=1}^{\infty}$, $\{x_r\}_{r=1}^{\infty}$, $\{y_r\}_{r=1}^{\infty}$, and $\{z_r\}_{r=1}^{\infty}$ converging to 0 such that $w_{r+1} < x_r < y_r < z_r < u_r < v_r < w_r$ for each $r \in \mathbb{N}$. Further u_r, v_r, and w_r satisfy (a), and x_r, y_r, and z_r satisfy (b). For all $r \in \mathbb{N}$, let B_r and A_r be the components of $\mathrm{cl}(U)$ that contain $(v_r, f_X(v_r))$ and $(y_r, f_X(y_r))$, respectively (see Figure 1). The following claim is clear.

Claim 1. *If $D \in C(X)$, $0 < p \le q \le 1$, and $(p, f_X(p)), (q, f_X(q)) \in D$, then $\{(t, f_X(t)) : t \in [p,q]\} \subset D$.*

We prove the following claim.

Claim 2. *If $(t, f_X(t)) \in A_r$, then $t \in [x_r, z_r]$.*

Let $t \in (0,1]$ be such that $(t, f_X(t)) \in A_r$. Assume that $t > z_r$. By the Intermediate Value Theorem, there exists $\gamma \in [y_r, z_r]$ such that $f_X(\gamma) = i/n$. By Claim 1, $(\gamma, i/n) \in A_r$. We have the required contradiction, since $A_r \subset \mathrm{cl}(U) \subset [0,1] \times \left[\frac{i}{n} + \frac{1}{2n}, \frac{l-1}{n} - \frac{1}{2n}\right]$. In a similar way, we find a contradiction if $t < x_r$.

By Claim 2, $A_r \subset \{(t, f_X(t)) : t \in [x_r, z_r]\} \subset [x_r, z_r] \times \left[\frac{i}{n} + \frac{1}{2n}, \frac{k}{n}\right]$. In a similar way, $B_r \subset [u_r, w_r] \times \left[\frac{j-1}{n}, \frac{l-1}{n} - \frac{1}{2n}\right]$. By the Boundary Bumping Theorem, there exist $p_r \in [x_r, z_r]$ and $q_r \in [u_r, w_r]$ such that $\left(p_r, \frac{i}{n} + \frac{1}{2n}\right) \in A_r$ and $\left(q_r, \frac{l-1}{n} - \frac{1}{2n}\right) \in B_r$ for each $r \in \mathbb{N}$. By the Intermediate Value Theorem, there exist $\alpha_r \in [x_r, z_r]$ and $\beta_r \in [u_r, w_r]$ such that $f_X(\alpha_r) = \frac{\min\{j,k\}-1}{n} - \frac{1}{4n}$ and $f_X(\beta_r) = \frac{\max\{j,k\}}{n} + \frac{1}{4n}$ for each $r \in \mathbb{N}$.

We can clearly assume that $\lim_{r\to\infty} A_r = A$ and $\lim_{r\to\infty} B_r = B$ for some $A, B \in C(X)$. Notice that $A \subset \{0\} \times \left[\frac{i}{n} + \frac{1}{2n}, \frac{k}{n}\right]$ and $B \subset \{0\} \times \left[\frac{j-1}{n}, \frac{l-1}{n} - \frac{1}{2n}\right]$. Moreover $\left(0, \frac{\min\{j,k\}-1}{n} - \frac{1}{4n}\right) \in (A \cap K) \setminus B$ and $\left(0, \frac{\max\{j,k\}}{n} + \frac{1}{4n}\right) \in (B \cap K) \setminus A$. By Theorem 2.1, X is not a semi-Kelley continuum.

Now, assume (1) holds. By Theorem 2.1, there exist subcontinua A, B, and K of X, an open subset U of X, and sequences $\{A_r\}_{r=1}^{\infty}$, $\{B_r\}_{r=1}^{\infty}$ of components of $\mathrm{cl}(U)$ converging to A and B, respectively, such that $K \subset U$, $A \cap K \not\subset B$, and $B \cap K \not\subset A$. Notice that $K \neq X$. Let $\mathrm{graph}(f_X) = \{(a, f_X(a)) \in \mathbb{R}^2 : a \in (0,1]\}$. Consider the following two cases.

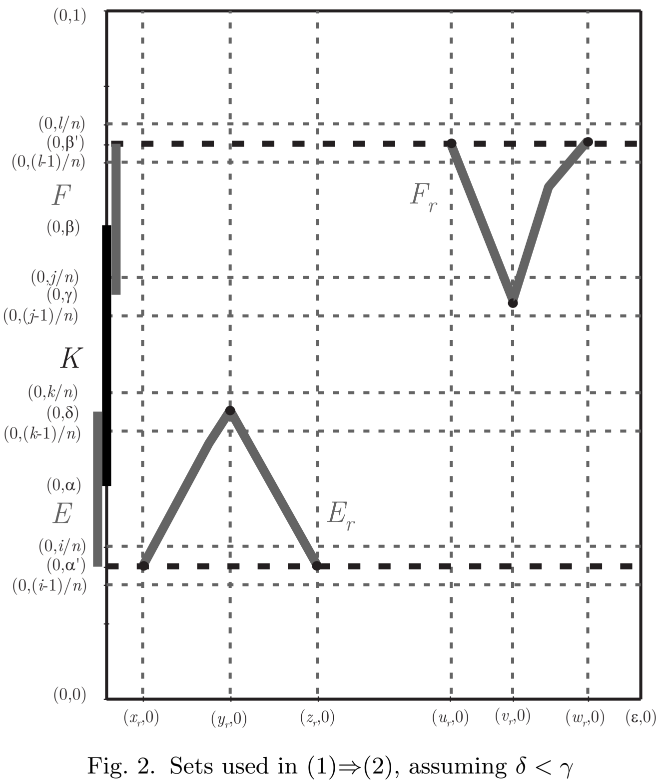

Fig. 2. Sets used in (1)⇒(2), assuming $\delta < \gamma$

CASE 1: $K \subset \mathrm{graph}(f_X)$. Since the points of K are points of local connectedness of X, there exists an open connected subset V of X such that $K \subset V \subset U$. By Remark 2.2, there exist $E, F \in C(X)$ and two sequences $\{E_r\}_{r=1}^{\infty}$, $\{F_r\}_{r=1}^{\infty}$ of components of $\mathrm{cl}(V)$ converging to E and F, respectively, such that $E \cap K \not\subset F$ and $F \cap K \not\subset E$ (see Figure 2). Since $\mathrm{cl}(V)$ is a connected set, $E_r = \mathrm{cl}(V) = F_r$ for all $r \in \mathbb{N}$. Hence, $E = \mathrm{cl}(V) = F$. This contradicts the fact that $F \cap K \not\subset E$.

CASE 2: $\{0\} \times [0,1] \subset K$. Since $K \neq X$, there exists $0 < \varepsilon < 1$ such that $K \subset ([0,\varepsilon) \times [0,1]) \cap X \subset U$. Let $V = ([0,\varepsilon) \times [0,1]) \cap X$. Notice that V is an open connected subset of X such that $K \subset V \subset U$. Arguing as in Case 1, we reach a contradiction.

By Cases 1 and 2, we find that $K \subsetneq \{0\} \times [0,1]$.

We now prove that $(0,0) \notin K$. Assume that $(0,0) \in K$. Then $K = \{0\} \times [0,\alpha]$ with $\alpha < 1$. Let $\varepsilon > 0$ be such that $\alpha + \varepsilon < 1$ and $V = ([0,\varepsilon) \times [0,\alpha+\varepsilon)) \cap X \subset U$.

The following claim is easy to prove.

CLAIM 3. *If C is a component of* $\mathrm{cl}(V)$*, then* $C = \{0\} \times [0,\alpha+\varepsilon]$ *or* $C = \{(t, f_X(t)) : t \in [u,v]\}$ *for some* $0 < u \le v \le \varepsilon$.

We prove the following claim.

CLAIM 4. *If $\{C_r\}_{r=1}^{\infty}$ is a sequence of components of* $\mathrm{cl}(V)$ *such that* $\lim_{r\to\infty} C_r = C$ *and* $C \cap K \neq \emptyset$*, then* $(0,\alpha) \in C$.

Let $(0,c) \in C \cap K$. By the Boundary Bumping Theorem, $C_r \cap \mathrm{Bd}(V) \neq \emptyset$ for every $r \in \mathbb{N}$. Since $\mathrm{Bd}(V) \subset ((\{\varepsilon\} \times [0,\alpha+\varepsilon]) \cup ([0,\varepsilon] \times \{\alpha+\varepsilon\})) \cap X$, by Claim 3, without loss of generality we can assume that $C_r \cap \mathrm{Bd}(V) \subset [0,\varepsilon] \times \{\alpha+\varepsilon\}$ for all $r \in \mathbb{N}$. Moreover, we can assume that $C_r = \{(t, f_X(t)) : t \in [u_r, v_r]\}$ for some $u_r < v_r \le \varepsilon$ such that $f_X(u_r) = f_X(v_r) = \alpha+\varepsilon$ and $\lim_{r\to\infty} v_r = 0$. Hence, $(0,\alpha+\varepsilon) \in C$. On the other hand, since $(0,c) \in C$, we have $\{0\} \times [c,\alpha+\varepsilon] \subset C$. Therefore, $(0,\alpha) \in C$ and Claim 4 is proved.

By Remark 2.2, there exist $E, F \in C(X)$ and two sequences $\{E_r\}_{r=1}^{\infty}$, $\{F_r\}_{r=1}^{\infty}$ of components of $\mathrm{cl}(V)$ converging to E and F, respectively, such that $E \cap K \not\subset F$ and $F \cap K \not\subset E$. By Claim 4, $(0,\alpha) \in E$ and $(0,\alpha) \in F$. By Claim 3, $E, F \subset \{0\} \times [0,\alpha+\varepsilon]$. Since $K = \{0\} \times [0,\alpha]$, $E \cap K$ and $F \cap K$ are subcontinua of K containing $(0,\alpha)$. So, $E \cap K \subset F \cap K$ or $F \cap K \subset E \cap K$. This is a contradiction. Therefore $(0,0) \notin K$.

In a similar way, $(0,1) \notin K$.

Since $(0,0) \notin K$ and $(0,1) \notin K$, there are $0 < \alpha \le \beta < 1$ such that $K = \{0\} \times [\alpha,\beta]$. Define $V = ([0,\varepsilon) \times (\alpha',\beta')) \cap X$ for some $\varepsilon > 0$, $\alpha' \in (0,\alpha)$ and $\beta' \in (\beta,1)$, so that $K \subset V \subset U$. By Remark 2.2, there exist $E, F \in C(X)$ and two sequences $\{E_r\}_{r=1}^{\infty}$, $\{F_r\}_{r=1}^{\infty}$ of components of $\mathrm{cl}(V)$ converging to E and F, respectively, such that $E \cap K \not\subset F$ and $F \cap K \not\subset E$. As in Claim 3, without loss of generality we can assume that $E_r = \{(t, f_X(t)) : t \in [x_r, z_r]\}$ for some $x_r < z_r$, and $F_r = \{(t, f_X(t)) : t \in [u_r, w_r]\}$ for some $u_r < w_r$, with $\lim_{r\to\infty} z_r = 0$ and $\lim_{r\to\infty} w_r = 0$. Moreover, as in the proof of Claim 4, we can also assume that $f_X(u_r) = f_X(w_r) = \beta'$, $f_X(x_r) = f_X(z_r) = \alpha'$,

$E = \{0\} \times [\alpha', \delta]$, and $F = \{0\} \times [\gamma, \beta']$ for some $\delta \in [\alpha, \beta)$ and $\gamma \in (\alpha, \beta]$. Hence, $(0, \alpha) \in E$ and $(0, \beta) \in F$.

Consider $n \in \mathbb{N}$ such that $5/n < \min\{\alpha', 1 - \beta', \alpha - \alpha', \beta' - \beta\}$ and $\alpha' n$, $\beta' n$, δn, γn are not integers. Define $i = \min\{m \in \mathbb{N} : m/n > \alpha'\}$, $k = \min\{m \in \mathbb{N} : m/n > \delta\}$, $j = \min\{m \in \mathbb{N} : m/n > \gamma\}$, and $l = \min\{m \in \mathbb{N} : m/n > \beta'\}$. Notice that $n > 5$, $5 < j, k < n - 5$, $i < \min\{j, k\} - 4$, and $l > \max\{j, k\} + 4$.

Since E_r and F_r are components of $\mathrm{cl}(V)$ and $[\alpha', \beta'] \subset \left(\frac{i-1}{n}, \frac{l}{n}\right)$, we see that $f_X([x_r, z_r]), f_X([u_r, w_r]) \subset \left(\frac{i-1}{n}, \frac{l}{n}\right)$. Further, as $\lim_{r\to\infty} E_r = E = \{0\} \times [\alpha', \delta] \subset \{0\} \times \left(\frac{i-1}{n}, \frac{k}{n}\right)$ and $\lim_{r\to\infty} F_r = F = \{0\} \times [\gamma, \beta'] \subset \{0\} \times \left(\frac{j-1}{n}, \frac{l}{n}\right)$, we have $\lim_{r\to\infty} f_X([x_r, z_r]) = [\alpha', \delta]$ and $\lim_{r\to\infty} f_X([u_r, w_r]) = [\gamma, \beta']$. So, without loss of generality we can suppose that $f_X([x_r, z_r]) \subset \left(\frac{i-1}{n}, \frac{k}{n}\right)$ and $f_X([u_r, w_r]) \subset \left(\frac{j-1}{n}, \frac{l}{n}\right)$.

For each $r \in \mathbb{N}$ define $y_r \in [x_r, z_r]$ and $v_r \in [u_r, w_r]$ such that $f_X(y_r) = \max\{f_X(t) : t \in [x_r, z_r]\}$ and $f_X(v_r) = \min\{f_X(t) : t \in [u_r, w_r]\}$. Hence, $f_X([x_r, z_r]) = [\alpha', f_X(y_r)]$ and $f_X([u_r, w_r]) = [f_X(v_r), \beta']$. So,

$$\lim_{r\to\infty} f_X(y_r) = \delta \quad \text{and} \quad \lim_{r\to\infty} f_X(v_r) = \gamma.$$

We can conclude that for each $a \in (0, 1]$, there exists $r \in \mathbb{N}$ such that

(1) $0 < u_r < v_r < w_r < a$, $f_X(u_r), f_X(w_r) \in A_n^l$, $f_X(v_r) \in A_n^j$, $f_X([u_r, w_r]) \subset \bigcup_{m=j}^{l} A_n^m$, and

(2) $0 < x_r < y_r < z_r < a$, $f_X(x_r), f_X(z_r) \in A_n^i$, $f_X(y_r) \in A_n^k$ and $f_X([x_r, z_r]) \subset \bigcup_{m=i}^{k} A_n^m$. ∎

4. Characterization of semi-Kelley compactifications with remainder being a simple closed curve. Let S^1 be the standard unit circle in $\mathbb{R}^2$. Following Nadler [12, p. 321], let $(SP)_1 = S^1 \cup \{(1 + 1/t)e^{it} : t \geq 1\}$. A continuum X is called a *Σ-continuum* if X is a compactification of $(0, 1]$ with remainder being a simple closed curve.

We start this section with a lemma (compare to [12, (3.1) Lemma]).

LEMMA 4.1. *Let X be a Σ-continuum. Then X can be embedded in $\mathbb{R}^2$ in such a way that the remainder is S^1 and $X \setminus S^1 = \{(1 + t)g_X(t) : t \in (0, 1]\}$ for some continuous function $g_X : (0, 1] \to S^1$.*

Proof. By [12, (4.4) Lemma, p. 336], we may assume X is embedded in $\mathbb{R}^2$ with remainder S^1, and moreover X does not contain the origin. Let $h : H \to (0, 1]$ be a homeomorphism where $H = X \setminus S^1 \subset \mathbb{R}^2$. Define a function $g : X \to \mathbb{R}^2$ by

$$g(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in S^1, \\ (1 + h(x, y)) \frac{(x, y)}{\|(x, y)\|} & \text{if } (x, y) \in H. \end{cases}$$

We now prove that g is injective. Let $(a,b),(x,y) \in X$ be such that $g(a,b) = g(x,y)$. If $(a,b) \in S^1$ and $(x,y) \in H$, then $h(x,y) = 0$, which contradicts the definition of h, so this case is impossible. If $(a,b),(x,y) \in S^1$, then clearly $(a,b) = (x,y)$. If $(a,b),(x,y) \in H$, then $h(a,b) = h(x,y)$; since h is injective, $(a,b) = (x,y)$. Hence, g is injective.

Since the domain of g is compact, g is a homeomorphism onto its image. Define $g_X : (0,1] \to S^1$ by $g_X(t) = \frac{h^{-1}(t)}{\|h^{-1}(t)\|}$. Since h is a homeomorphism and H does not contain the origin, the function g_X is well-defined and continuous. Notice that $g(X) \setminus S^1 = \big\{(1+h(x,y))\frac{(x,y)}{\|(x,y)\|} : (x,y) \in H\big\} = \{(1+t)g_X(t) : t \in (0,1]\}$. Hence, $g(X)$ is the required embedding and g_X is the required function. ∎

For the rest of this paper, a Σ-continuum X and a map $g_X : (0,1] \to S^1$ will always be considered as in Lemma 4.1. Notice that since X is a compactification of $(0,1]$, for each $p \in S^1$ there exists a decreasing sequence $\{t_m\}_{m=1}^{\infty}$ in $(0,1]$ converging to 0 such that $\lim_{m\to\infty} g_X(t_m) = p$. Given $k,n \in \mathbb{N}$, let $B_n^k = \big\{e^{it} : t \in \big[\frac{k-1}{2^n}2\pi, \frac{k}{2^n}2\pi\big]\big\}$ and $D_n = \big\{B_n^i : i \in \{1,\dots,2^n\}\big\}$.

The following result generalizes [2, Theorem 4.1, p. 107].

THEOREM 4.2. *Let X be a Σ-continuum. Then the following statements are equivalent*:

(1) *For each $n \in \mathbb{N}$, there exists $r(n) \in (0,1]$ such that for each $k \in \{1,\dots,2^n\}$, if $s \le t \le r(n)$ and $g_X(s), g_X(t) \in B_n^k$, then $g_X([s,t]) \cap B_n^j$ is nonempty for all $j \in \{1,\dots,2^n\}$ or $g_X([s,t])$ is contained in the union of three elements of D_n.*
(2) *X is homeomorphic to $(SP)_1$.*
(3) *X is a semi-Kelley continuum.*

Proof. (1)⇒(2) is a straightforward application of [12, (4.5) Lemma, p. 336], and (2)⇒(3) is clear.

We prove (3)⇒(1). Assume (3) holds and that (1) does not hold. Hence, there exists $n \in \mathbb{N}$ such that for each $r \in (0,1]$ there exist $k \in \{1,\dots,2^n\}$ and $s \le t \le r$ with $g_X(s), g_X(t) \in B_n^k$, $g_X([s,t]) \cap B_n^j = \emptyset$ for some $j \in \{1,\dots,2^n\}$ and $g_X([s,t])$ is not contained in the union of three elements of D_n. Therefore, for each $m \in \mathbb{N}$, there exist $k(m), j(m) \in \{1,\dots,2^n\}$ and there exist $0 < s(m) \le t(m) \le 1/m$ such that $g_X(s(m)), g_X(t(m)) \in B_n^{k(m)}$, $g_X([s(m),t(m)]) \cap B_n^{j(m)} = \emptyset$ and $g_X([s(m),t(m)])$ is not contained in the union of three elements of D_n. By passing to a subsequence if necessary, we may assume that $t(m+1) < s(m)$ and $k(m)$, $j(m)$ are constant sequences such that $k(m) = k$ and $j(m) = j$ for every $m \in \mathbb{N}$. Without loss of generality we may also assume that $k = 1$. Moreover, assume that $\lim_{m\to\infty} g_X(s(m)) = p$ and $\lim_{m\to\infty} g_X(t(m)) = q$ for some points $p,q \in B_n^1$.

Since $g_X(s(m)) \in B_n^1$ and $g_X([s(m), t(m)])$ is connected and not contained in the union of three elements of D_n, we find that $g_X([s(m), t(m)]) \cap B_n^i \neq \emptyset$ for $i = 2, 3$ or for $i = 2^n - 1, 2^n$. We can clearly assume that $g_X([s(m), t(m)]) \cap B_n^2 \neq \emptyset$ and $g_X([s(m), t(m)]) \cap B_n^3 \neq \emptyset$ for each $m \in \mathbb{N}$.

Since $g_X : (0, 1] \to S^1$ is continuous and $(0, 1]$ is contractible, the function g_X is homotopic to a constant function $\kappa : (0, 1] \to S^1$. Since κ has a lifting, by [7, Proposition 1.30] there is a lifting $G : (0, 1] \to \mathbb{R}$ of g_X, that is, $g_X(t) = e^{iG(t)}$ for each $t \in (0, 1]$.

Let $m \in \mathbb{N}$. Since $g_X([s(m), t(m)]) \cap B_n^j = \emptyset$, we have $|G(s(m)) - G(t(m))| \leq 2\pi/2^n$.

Let $u(m) \in [s(m), t(m)]$ be such that $G(u(m)) = \max G([s(m), t(m)])$ and let $v(m) \in [u(m + 1), u(m)]$ with $G(v(m)) = \min G([u(m + 1), u(m)])$.

As $g_X([s(m), t(m)]) \cap B_n^3 \neq \emptyset$, we find that $G(u(m)) - G(s(m)) \geq 2\pi/2^n$ and $G(u(m)) - G(t(m)) \geq 2\pi/2^n$. Notice that $G(u(m)) - G(v(m)) \geq 2\pi/2^n$ and $G(u(m + 1)) - G(v(m)) \geq 2\pi/2^n$.

We can assume that $\lim_{m\to\infty} g_X(u(m)) = x$ and $\lim_{m\to\infty} g_X(v(m)) = y$ for some points $x, y \in S^1$. To end the proof, consider the following two cases:

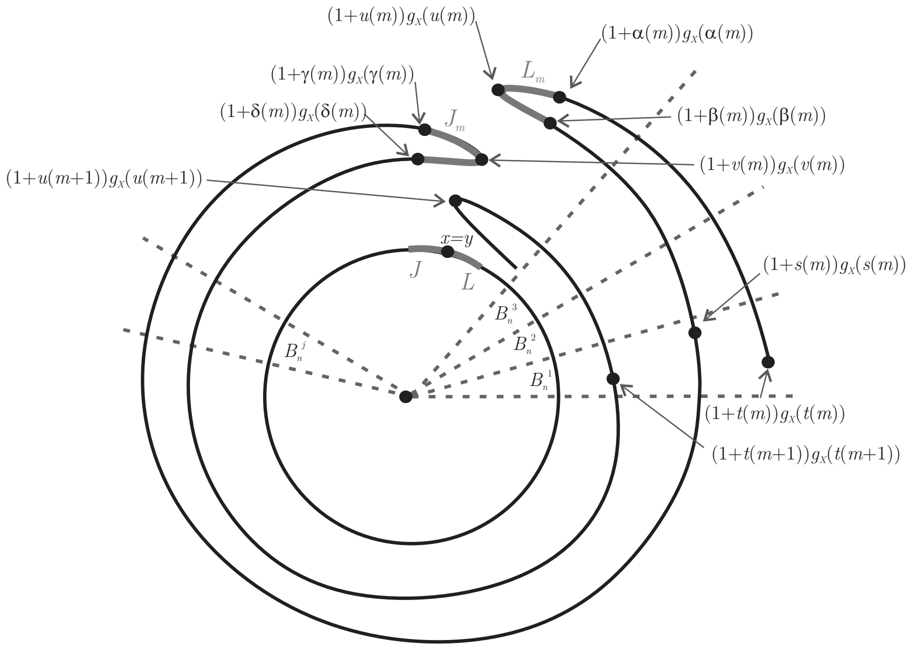

Fig. 3. Sets used in Case 1

CASE 1: $x = y$. For each $m \in \mathbb{N}$, define points $\alpha(m)$, $\beta(m)$, $\gamma(m)$, and $\delta(m)$ as follows:

$$\alpha(m) = \min \{t \in [u(m), t(m)] : G(t) = G(u(m)) - 2\pi/2^{n+1}\},$$

$$\beta(m) = \max\{t \in [s(m), u(m)] : G(t) = G(u(m)) - 2\pi/2^{n+1}\},$$
$$\gamma(m) = \min\{t \in [v(m), u(m)] : G(t) = G(v(m)) + 2\pi/2^{n+1}\},$$
$$\delta(m) = \max\{t \in [u(m+1), v(m)] : G(t) = G(v(m)) + 2\pi/2^{n+1}\}.$$

Define $L_m = \{(1+t)g_X(t) : t \in [\beta(m), \alpha(m)]\} \subset X$ and $J_m = \{(1+t)g_X(t) : t \in [\delta(m), \gamma(m)]\}$ (see Figure 3). By the definitions, $\lim_{m\to\infty} L_m = L$ for some subarc $L \subset S^1$ of arc length $2\pi/2^{n+1}$ with endpoint x, $\lim_{m\to\infty} J_m = J$ for some subarc $J \subset S^1$ of arc length $2\pi/2^{n+1}$ with endpoint x, and $L \cap J = \{x\}$. Let $K = L \cup J$. Then L and J are incomparable maximal limit continua of K in X.

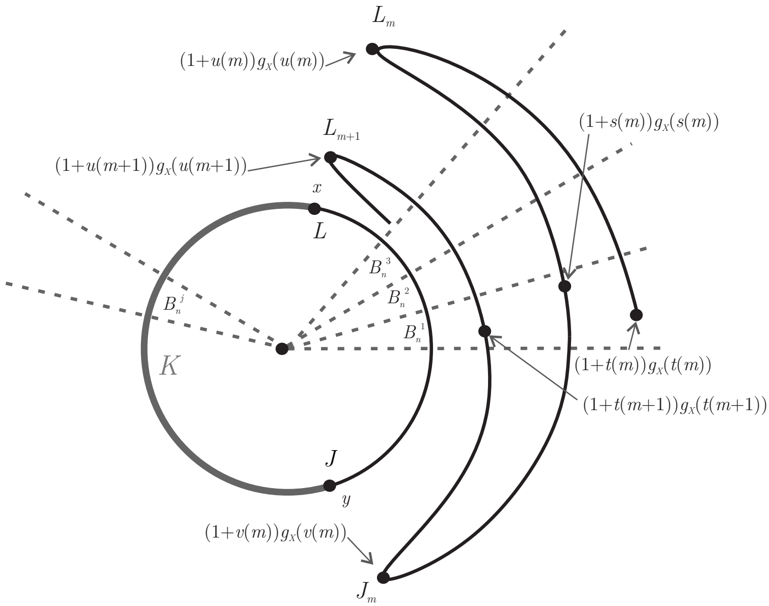

Fig. 4. Sets used in Case 2

CASE 2: $x \neq y$. For each $m \in \mathbb{N}$, put $L_m = \{(1+u(m))g_X(u(m))\} \subset X$, $J_m = \{(1+v(m))g_X(v(m))\} \subset X$, $L = \{x\}$ and $J = \{y\}$. Let $t_x, t_y \in \mathbb{R}$ be such that $x = e^{it_x}$, $y = e^{it_y}$ and $t_y \in [t_x, t_x + 2\pi]$. Finally, we define $K = \{e^{it} \in S^1 : t \in [t_x, t_y]\}$ (see Figure 4). By definitions, $\lim_{m\to\infty} L_m = L$, $\lim_{m\to\infty} J_m = J$, $L, J \in C(K)$, and L, J are incomparable maximal limit continua of K in X.

In both cases X is not a semi-Kelley continuum. ∎

PROBLEM 4.3. *Let X be a solenoid. Is it true that there is a unique semi-Kelley compactification with remainder X?*

PROBLEM 4.4. *For which semi-Kelley continua X there is a unique semi-Kelley compactification with remainder X?*

5. There is no semi-Kelley compactification with remainder being a triod. A continuum T is called a *triod* if there is a subcontinuum H of T such that $T \setminus H$ is the union of three nonempty sets any two of which are mutually separated in T [13, Definition 11.22, p. 208].

THEOREM 5.1. *Let X be a continuum. Assume that there exist an open set $V \subset X$, a triod $T \subset V$, a sequence $\{E_n\}_{n=1}^{\infty}$ of components of* $\mathrm{cl}(V)$, *and a sequence $\{T_n\}_{n=1}^{\infty}$ of pairwise disjoint arcs converging to T such that $T_n \subset E_n$ and each point of T_n disconnects E_n. Then X is not a semi-Kelley continuum.*

Proof. We will use Theorem 2.1. Let W be an open subset of X such that $T \subset W \subset \mathrm{cl}(W) \subset V$. Since T is a triod, there exists a subcontinuum H of T such that $T \setminus H$ is the union of three nonempty sets I, J, L, any two of which are mutually separated in T.

Fix points $a \in I$, $b \in J$, $c \in L$. Put

$$\varepsilon = \min\{d(a, H \cup J \cup L), d(b, H \cup I \cup L), d(c, H \cup I \cup J)\}/4.$$

Choose $k \in \mathbb{N}$ so that $\frac{1}{k} < \varepsilon$ and $\mathrm{cl}\big(B\big(\frac{1}{k}, a\big)\big) \cup \mathrm{cl}\big(B\big(\frac{1}{k}, b\big)\big) \cup \mathrm{cl}\big(B\big(\frac{1}{k}, c\big)\big) \subset W$. Since $\lim_{n\to\infty} T_n = T$, we may assume $B\big(\frac{1}{k+6}, a\big) \cap T_n \neq \emptyset$, $B\big(\frac{1}{k+6}, b\big) \cap T_n \neq \emptyset$, and $B\big(\frac{1}{k+6}, c\big) \cap T_n \neq \emptyset$ for every $n \in \mathbb{N}$.

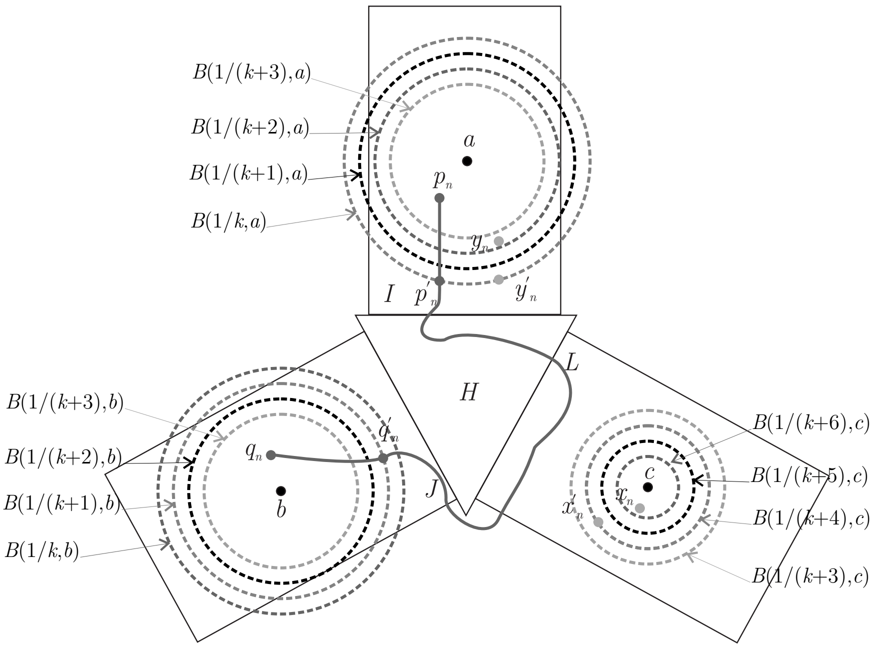

Fig. 5. Sets used in Theorem 5.1

Given $p, q \in T_n$, let pq denote the arc in T_n with endpoints p and q.

For each $n \in \mathbb{N}$, choose $p_n \in T_{2n} \cap B\big(\frac{1}{k+3}, a\big)$ and $q_n \in T_{2n} \cap \big(B\big(\frac{1}{k+3}, b\big) \cup B\big(\frac{1}{k+3}, c\big)\big)$ such that p_nq_n intersects exactly one of $B\big(\frac{1}{k+3}, b\big)$ and $B\big(\frac{1}{k+3}, c\big)$. We may assume $q_n \in B\big(\frac{1}{k+3}, b\big)$ and $p_nq_n \cap B\big(\frac{1}{k+3}, c\big) = \emptyset$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, choose $x_n \in T_{2n+1} \cap B\big(\frac{1}{k+6}, c\big)$ and $y_n \in T_{2n+1} \cap \big(B\big(\frac{1}{k+2}, a\big) \cup B\big(\frac{1}{k}, b\big)\big)$ such that x_ny_n intersects exactly one of $B\big(\frac{1}{k+2}, a\big)$ and $B\big(\frac{1}{k}, b\big)$ (see Figure 5). We may assume $y_n \in B\big(\frac{1}{k+2}, a\big)$ and $x_ny_n \cap B\big(\frac{1}{k}, b\big) = \emptyset$ for each $n \in \mathbb{N}$.

Define K as the component of $\mathrm{cl}(W) \setminus \big(B\big(\frac{1}{k}, a\big) \cup B\big(\frac{1}{k+1}, b\big) \cup B\big(\frac{1}{k+4}, c\big)\big)$ that contains H and also define $U = V \setminus \big(\mathrm{cl}\big(B\big(\frac{1}{k+1}, a\big)\big) \cup \mathrm{cl}\big(B\big(\frac{1}{k+2}, b\big)\big) \cup \mathrm{cl}\big(B\big(\frac{1}{k+5}, c\big)\big)\big)$. By definition, U is an open set of X so that $K \subset U$. Now, we will construct convergent sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of components of $\mathrm{cl}(U)$ such that $\lim_{n\to\infty} A_n = A$, $\lim_{n\to\infty} B_n = B$, $A \cap K \not\subset B$, and $B \cap K \not\subset A$.

For each $n \in \mathbb{N}$, define the natural order $\leq$ of the arc T_n for which $p_n \leq q_n$ and $x_n \leq y_n$.

Define $q_n' = \min\big\{t \in p_nq_n : t \in \mathrm{cl}\big(B\big(\frac{1}{k+1}, b\big)\big)\big\}$ and $p_n' = \max\big\{t \in p_nq_n' : t \in \mathrm{cl}\big(B\big(\frac{1}{k}, a\big)\big)\big\}$. Thus, $p_n'q_n' \subset \mathrm{cl}(V) \setminus \big(B\big(\frac{1}{k}, a\big) \cup B\big(\frac{1}{k+1}, b\big) \cup B\big(\frac{1}{k+4}, c\big)\big) \subset \mathrm{cl}(U)$. Hence, $p_n'q_n' \subset \mathrm{cl}(U)$. Let A_n be the component of $\mathrm{cl}(U)$ such that $p_n'q_n' \subset A_n$. Since A_n is a connected subset of $\mathrm{cl}(V)$ and $A_n \cap E_{2n} \neq \emptyset$, we have $A_n \subset E_{2n}$. Since p_n, q_n disconnect E_{2n} and $p_n, q_n \notin \mathrm{cl}(U)$, we see that $A_n \subset p_nq_n$. Then $A_n \cap B\big(\frac{1}{k+3}, c\big) = \emptyset$. Assume that $\lim_{n\to\infty} A_n = A$, for some $A \in C(X)$. Since $A_n \cap B\big(\frac{1}{k+3}, c\big) = \emptyset$ for each $n \in \mathbb{N}$, we have $A \cap B\big(\frac{1}{k+3}, c\big) = \emptyset$. Thus, $A \cap \mathrm{cl}\big(B\big(\frac{1}{k+4}, c\big)\big) = \emptyset$. Since $\lim_{n\to\infty} T_n = T$ and $A_n \subset p_nq_n \subset T_{2n}$ for each $n \in \mathbb{N}$, we have $A \subset T$. Assume that $\lim_{n\to\infty} p_n' = p'$, $\lim_{n\to\infty} q_n' = q'$, and $\lim_{n\to\infty} p_n'q_n' = A'$, for some $p', q' \in A$ and $A' \in C(A)$. Notice that $p' \in T \cap \mathrm{cl}\big(B\big(\frac{1}{k}, a\big)\big)$ and $q' \in T \cap \mathrm{cl}\big(B\big(\frac{1}{k+1}, b\big)\big)$. Then $p' \in I$ and $q' \in J$. Hence, $A' \cap H \neq \emptyset$. Therefore, $A' \subset K$, $p', q' \in A \cap K$, and $A \cap \mathrm{cl}\big(B\big(\frac{1}{k+4}, c\big)\big) = \emptyset$.

Now, define $y_n' = \min\big\{t \in x_ny_n : t \in \mathrm{cl}\big(B\big(\frac{1}{k}, a\big)\big)\big\}$ and $x_n' = \max\big\{t \in x_ny_n' : t \in \mathrm{cl}\big(B\big(\frac{1}{k+4}, c\big)\big)\big\}$. Hence, $x_n'y_n' \subset \mathrm{cl}(U)$. Let B_n be the component of $\mathrm{cl}(U)$ such that $x_n'y_n' \subset B_n$. As in the previous paragraph, we can assume that $\lim_{n\to\infty} B_n = B$, for some $B \subset T$. Further, $\lim_{n\to\infty} x_n' = x'$ and $\lim_{n\to\infty} y_n' = y'$ with $x' \in T \cap \mathrm{cl}\big(B\big(\frac{1}{k+4}, c\big)\big)$ and $y' \in T \cap \mathrm{cl}\big(B\big(\frac{1}{k}, a\big)\big)$. Moreover, $x', y' \in B \cap K$ and $B \cap \mathrm{cl}\big(B\big(\frac{1}{k+1}, b\big)\big) = \emptyset$.

By definitions, $q' \in (A \cap K) \setminus B$ and $x' \in (B \cap K) \setminus A$. Hence, by Theorem 2.1, X is not semi-Kelley. ■

Corollary 5.2. *Let $X = Y \cup (0,1]$ be a compactification of $(0,1]$ with remainder Y. If T is a triod contained in Y and there exists a sequence*

$\{T_n\}_{n=1}^{\infty}$ of pairwise disjoint arcs in $(0,1]$ converging to T, then X is not a semi-Kelley continuum.

Proof. By Theorem 5.1 for $V = X$ and $E_n = X$ for every $n \in \mathbb{N}$, X is not a semi-Kelley continuum. ∎

COROLLARY 5.3. *Let T be a triod and $X = T \cup (0,1]$ be a compactification of $(0,1]$ with remainder T. Then X is not a semi-Kelley continuum.*

6. Semi-Kelley compactifications with remainder being a Peano continuum. In this section we study the semi-Kelley compactifications with remainder being a Peano continuum. First, we recall the following theorem which characterizes the Peano continua that are not triods.

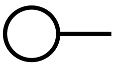

Fig. 6. Noose

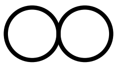

Fig. 7. Eight

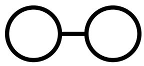

Fig. 8. Dumbbell

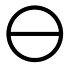

Fig. 9. Theta

THEOREM 6.1 ([10, Theorem 3.10, p. 536]). *If G is a Peano continuum that is not a triod, then G is one of the following objects: an arc, a simple closed curve, a noose (Figure 6), a figure eight (Figure 7), a dumbbell (Figure 8), or a theta curve (Figure 9).*

THEOREM 6.2. *Let G be a Peano continuum and $X = G \cup (0,1]$ be a compactification of $(0,1]$ with remainder G. If X is a semi-Kelley continuum, then G is an arc or a simple closed curve.*

Proof. Assume that $X = G \cup (0,1]$ is a semi-Kelley continuum. By Corollary 5.3, G is not a triod. By Theorem 6.1, G is one of the following objects: an arc, a simple closed curve, a noose, a figure eight, a dumbbell, or a letter theta. We will prove that G is not a noose, a figure eight, a dumbbell, or a letter theta.

CASE 1: *G is a noose.* Suppose that $G = S \cup L$, where S is a simple closed curve, L is an arc with endpoints u and v, and $S \cap L = \{v\}$. Let Y be the continuum obtained by identifying all points of L to a single point and let $\pi : X \to Y$ be the quotient map. Notice that Y is a Σ-continuum. Since π is monotone, by [6, Theorem 8, p. 311], Y is a semi-Kelley continuum. Hence, by Theorem 4.2, Y is homeomorphic to $(SP)_1$, so Y is a Kelley continuum. Let $S^1 \subset Y$ be the remainder of Y and let $(0,1]_Y = Y \setminus S^1$. Let $A \subset S^1$ be an arc such that $\pi(v) \in A$ and $\pi(v)$ is not an endpoint of A. Moreover, let $\{u_n\}_{n=1}^{\infty}$ be a sequence of points in $(0,1]$ such that $\lim_{n\to\infty} u_n = u$. Since Y is a Kelley continuum and $\lim_{n\to\infty} \pi(u_n) = \pi(u) = \pi(v)$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of arcs in $(0,1]_Y$ such that $\lim_{n\to\infty} A_n = A$ and $\pi(u_n) \in A_n$ for each $n \in \mathbb{N}$. By definition of π, we find that $\{\pi^{-1}(A_n)\}_{n=1}^{\infty}$ is a sequence of arcs in $(0,1]$. Without loss of generality assume that $\lim_{n\to\infty} \pi^{-1}(A_n) = T$

for some subcontinuum T of G. Notice that $\pi(T) = A$ and $u, v \in T$. Hence, T is a triod. By Corollary 5.2, X is not a semi-Kelley continuum, which contradicts our assumption.

CASE 2: *G is a figure eight.* Suppose that $G = S \cup R$, where S and R are simple closed curves such that $S \cap R = \{v\}$. Let $u \in R$ be such that $u \neq v$ and let $\{u_n\}_{n=1}^{\infty}$ be a sequence of points in $(0,1]$ such that $\lim_{n\to\infty} u_n = u$. Let Y be the continuum obtained by identifying all points of R to a single point and let $\pi : X \to Y$ be the quotient map. As in Case 1, we find that Y is homeomorphic to $(SP)_1 = S^1 \cup (0,1]_Y$, so Y is a Kelley continuum. Let $A \subset S^1$ be an arc such that $\pi(v) \in A$ and $\pi(v)$ is not an endpoint of A. Since Y is a Kelley continuum and $\lim_{n\to\infty} \pi(u_n) = \pi(u) = \pi(v)$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of arcs in $(0,1]_Y$ such that $\lim_{n\to\infty} A_n = A$ and $\pi(u_n) \in A_n$ for each $n \in \mathbb{N}$. By definition of π, $\{\pi^{-1}(A_n)\}_{n=1}^{\infty}$ is a sequence of arcs in $(0,1]$. Assume that $\lim_{n\to\infty} \pi^{-1}(A_n) = T$ for some subcontinuum T of G. Notice that $\pi(T) = A$ and $u, v \in T$. Hence, T is a triod. By Corollary 5.2, X is not a semi-Kelley continuum, which contradicts our assumption.

CASE 3: *G is a dumbbell.* Suppose that $G = S \cup L \cup R$, where S and R are disjoint simple closed curves, L is an arc with end points u and v, $S \cap L = \{u\}$, and $R \cap L = \{v\}$. Let Y be the continuum obtained by identifying all points of R to a single point and let $\pi : X \to Y$ be the quotient map. Notice that Y is a compactification of $(0,1]$ with remainder being a noose. By [6, Theorem 8, p. 311], Y is a semi-Kelley continuum, in contradiction with Case 1.

CASE 4: *G is a letter theta.* Suppose that $G = L \cup J \cup K$, where L, J and K are arcs with endpoints u and v such that $L \cap J = L \cap K = J \cap K = \{u, v\}$. Let Y be the continuum obtained by identifying all points of J to a single point and let $\pi : X \to Y$ be the quotient map. Notice that Y is a compactification of $(0,1]$ with remainder being a figure eight. By [6, Theorem 8, p. 311], Y is a semi-Kelley continuum, in contradiction with Case 2.

The proof concludes by observing that G is an arc or a simple closed curve. ∎

R. Beane and W. J. Charatonik proved that if X is an arc-like Kelley continuum then X is a Kelley remainder [2, Theorem 2.3, p. 105]. M. E. Chacón-Tirado proved that if X is a circle-like Kelley continuum then X is a Kelley remainder [4, Theorem 1, p. 170]. The following question is natural.

PROBLEM 6.3. *Let X be an arc-like or circle-like semi-Kelley continuum. Is X a semi-Kelley remainder?*

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