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Mauricio Chacón-Tirado and María de J. López

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
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THE PROPERTY OF SEMI-KELLEY FOR HAUSDORFF CONTINUA

MAURICIO CHACÓN-TIRADO AND MARÍA DE J. LÓPEZ

Abstract. In this paper we introduce the property of semi-Kelley for Hausdorff continua. We use this notion to characterize Hausdorff continua with the property of Kelley. We prove that if a product of Hausdorff continua has the property of semi-Kelley, then each factor continuum has the property of Kelley. Concerning hyperspaces, we prove that if either $C(X)$, $C_n(X)$, or 2^X has the property of semi-Kelley, then X has the property of Kelley.

1. Introduction

The property of Kelley is introduced by J. L. Kelley [12, p. 26, property 3.2] to study contractibility of hyperspaces of metric continua. In 1998, Janusz J. Charatonik and Włodzimierz J. Charatonik [3, Definition 3.16], introduce the property of semi-Kelley for metric continua; in this paper, they prove that the property of semi-Kelley is a weaker property than the property of Kelley and generalize several results known for metric continua with the property of Kelley to metric continua with the property of semi-Kelley.

In 1999, W. J. Charatonik [7, Definition 2.1] and Władyslaw Makuchowski [16, p. 124] extend, independently, the property of Kelley for Hausdorff continua; in particular, Charatonik shows an example of a homogeneous continuum that does not have the property of Kelley, and Makuchowski uses the property of Kelley to show that several definitions of local connectivity are equivalent in the hyperspace $C(X)$ of a continuum X with the property of Kelley. In 2006, J. J. Charatonik and Alejandro

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Illanes [6] study local connectedness, local arcwise connectedness, strong local connectedness, and strong local arcwise connectedness at a point for the hyperspaces of a compact Hausdorff space; moreover, they study relationships between those variants when the space has the property of Kelley. In 2017, Sergio Macías [14] studies the property of Kelley and the property of Kelley weakly for Hausdorff continua. Hausdorff continua with the property of Kelley are also studied in [5], [10], [15], and [17].

The aim of this paper is to extend the property of semi-Kelley to the class of Hausdorff continua and to provide proofs of some results of [3] in this new setting. Following this introduction, there are four sections. In section 2, we recall preliminary results about the Vietoris topology and limit superior of nets of subcontinua of a Hausdorff continuum. In section 3, we define the notion of Hausdorff maximal limit continuum (Definition 3.1), and, using this notion, we define the property of semi-Kelley for Hausdorff continua (Definition 3.15). We use these concepts to characterize Hausdorff continua with the property of Kelley (Theorem 3.12), and we also characterize Hausdorff continua with the property of semi-Kelley (Theorem 3.18). In section 4, we study the property of Kelley for Cartesian products and hyperspaces. We prove that if the Cartesian product of two Hausdorff continua has the property of semi-Kelley, then each factor continuum has the property of Kelley (Theorem 4.1). Concerning hyperspaces, we show that if X is a Hausdorff continuum which does not have the property of Kelley and $\mathcal{H}(X)$ is a continuum such that $C(X) \subset \mathcal{H}(X) \subset 2^X$, then $\mathcal{H}(X)$ does not have the property of semi-Kelley (Theorem 4.2). In particular, we obtain that if the n-fold hyperspace $C_n(X)$ has the property of semi-Kelley, then X has the property of Kelley (Corollary 4.3). On the other hand, we prove that if X has the property of semi-Kelley and $A, B \in C(X)$ are such that $A \subsetneq B$ and $C(X)$ is connected im kleinen at A, then $C(X)$ is connected im kleinen at B as well (Theorem 4.9). In the last section, we show that the property of semi-Kelley is preserved under retractions (Theorem 5.2), and, moreover, semi-confluent images of continua with the property of Kelley have the property of semi-Kelley (Theorem 5.4). We also pose some open questions.

2. Preliminaries

A *continuum* is a compact, connected Hausdorff space with more than one point. A *metric continuum* is a continuum with a metric d that generates its topology. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of natural numbers. Given a subset A of a topological space X, the interior of A, the closure of A, and the boundary of A, are denoted by $int(A)$, $cl(A)$, and $bd(A)$, respectively. A space is *degenerate* if it contains only one point.

For a continuum X, let 2^X be the collection of all nonempty closed subsets of X, called the *hyperspace of closed subsets of* X, endowed with the Vietoris topology [18, Definition 1.7]. It is known that if X is a continuum, then 2^X is a continuum [18, Theorem 4.9.6 and Theorem 4.10].

We will need the following lemma.

Lemma 2.1. *Let X be a continuum. If β is a basis of the topology of X, then the family of all sets of the form $\langle U_1, \dots, U_n \rangle$ with $n \in \mathbb{N}$ and $U_1, \dots, U_n \in \beta$ is a basis for the Vietoris topology of the hyperspace 2^X.*

Proof. Let β be a basis of the topology of X. Let $\langle V_1, \dots, V_m \rangle$ be a basic open subset of 2^X and let $A \in \langle V_1, \dots, V_m \rangle$. For each $i \in \{1, \dots, m\}$, choose $U_i \in \beta$ such that $A \cap U_i \neq \emptyset$ and $U_i \subset V_i$. Since β is a basis, the collection $\{U \in \beta : U \subset \bigcup_{i=1}^{n} V_i\}$ is an open cover of A; by compactness of A, we can find a finite subcover $\{U_{m+1}, \dots, U_n\}$ such that $A \cap U_i \neq \emptyset$ for each $i \in \{m+1, \dots, n\}$. By [18, Lemma 2.3.1], $\langle U_1, \dots, U_n \rangle \subset \langle V_1, \dots, V_n \rangle$. $\square$

We consider $C(X) = \{A \in 2^X : A \text{ is connected}\}$ as a subspace of 2^X. The elements of $C(X)$ are called *subcontinua of* X and $C(X)$ is called the *hyperspace of subcontinua of* X. Given $B \in C(X)$, let $C(B, X) = \{A \in C(X) : B \subset A \subset X\}$. Notice that $C(B, X) = (2^X - \bigcup_{b \in B} \langle X - \{b\} \rangle) \cap C(X)$; thus, $C(B, X)$ is a closed subset of $C(X)$.

Given a continuum X, if $A \in 2^X \setminus C(X)$, there exist U and V open and disjoint subsets of X such that $A \subset U \cup V$ and $A \cap U \neq \emptyset \neq A \cap V$. Notice that $A \in \langle U, V \rangle \subset 2^X \setminus C(X)$; therefore, $C(X)$ is a closed subset of 2^X. By [19, Corollary 2.6], $C(X)$ is connected. Hence, $C(X)$ is a continuum. Similarly, sets of the form $C(B, X)$ are continua if B and X are continua and $B \subset X$.

For each $n \in \mathbb{N}$, the *n-fold hyperspace of* X is defined as $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$, considered as a subspace of 2^X.

For $\mathcal{A} \subset 2^X$ or $\mathcal{A} \subset C(X)$, we denote $\bigcup \mathcal{A} = \{x \in X : \text{there exists } A \in \mathcal{A} \text{ such that } x \in A\}$.

Given a continuum X and $A, B \in C(X)$ with $A \subset B$, an *order arc in* $C(X)$ *from A to B* is a subcontinuum $\mathcal{A}$ of $C(X)$ such that A and B are elements of $\mathcal{A}$, any two elements of $\mathcal{A}$ are comparable by inclusion, and for each $C \in \mathcal{A}$, $A \subset C \subset B$; see [19, p. 286]. The main theorem about order arcs is the following.

Theorem 2.2. *Let X be a continuum. If A and B are subcontinua of X and $A \subset B$, then there exists an order arc in $C(X)$ from A to B.*

Proof. Let A and B be subcontinua of X such that $A \subset B$. By [13, p. 173, Theorem 5], $C(X)$ is order-dense (that is, for any $R, T \in C(X)$ with $R \subsetneq T$, there exists $S \in C(X)$ such that $R \subsetneq S \subsetneq T$). Since $A, B \in C(X)$ and $C(X)$ is a compact Hausdorff order-dense subspace of 2^X, by [19, Theorem 2.4], there is an order arc (in 2^X) from A to B, which can be taken in $C(X)$ by [19, Theorem 2.3]. □

A *mapping* is a continuous function. Given a function $f : X \to Y$ between continua X and Y and subsets $A \subset X$ and $B \subset Y$, let $f[A] = \{f(a) : a \in A\}$ denote the image of A under f, and let $f^{-1}[B] = \{x \in X : f(x) \in B\}$ denote the inverse image of B under f. The induced function $C(f) : C(X) \to C(Y)$ is defined by $C(f)(A) = f[A]$ for each $A \in C(X)$. The induced function of a mapping f is also a mapping [18, Theorem 5.10.1].

A *net* in a space X consists of a directed set D and a function $f : D \to X$. We usually denote a net in X by $\{x_d\}_{d\in D}$, where $x_d = f(d)$ for each $d \in D$.

Given a continuum X and a net $\{A_d\}_{d\in D}$ in 2^X, S. Mrówka [20, p. 237] defines the *limit superior* of $\{A_d\}_{d\in D}$ as follows: $\limsup\{A_d\}_{d\in D} = \{x \in X :$ for each open subset U of X with $x \in U$ and for each $d \in D$, there exists $m \in D$ with $d \leq m$ and $U \cap A_m \neq \emptyset\}$.

Mrówka [20, p. 238, 4.] proves that $\limsup\{A_d\}_{d\in D}$ is a nonempty closed subset of X.

We will use the following results concerning the limit superior. The proofs are left to the reader.

Lemma 2.3. *Let X be a continuum, let $Y \in 2^X$, and let $\{A_d\}_{d\in D}$ be a net in 2^X. If for each $d \in D$, there exists $m \in D$ such that $d \leq m$ and $Y \cap A_m \neq \emptyset$, then $Y \cap \limsup\{A_d\}_{d\in D} \neq \emptyset$.*

Lemma 2.4. *Let $f : X \to Y$ be a mapping between continua X and Y. If $\{A_d\}_{d\in D}$ is a net in 2^X, then $f[\limsup\{A_d\}_{d\in D}] = \limsup\{f[A_d]\}_{d\in D}$.*

We prove the following lemma.

Lemma 2.5. *Let X be a continuum and let A be a subcontinuum of X. If $\{A_d\}_{d\in D}$ is a net in $C(X)$ converging to A, then A is an element of $\limsup\{C(A_d, X)\}_{d\in D}$, and for each $B \in \limsup\{C(A_d, X)\}_{d\in D}$, $A \subset B$.*

Proof. Let $\{A_d\}_{d\in D}$ be a net in $C(X)$ converging to A. By [2, Lemma 3.2], $A \in \limsup\{C(A_d, X)\}_{d\in D}$.

On the other hand, let $B \in \limsup\{C(A_d, X)\}_{d\in D}$. Suppose that there exists a point $a \in A - B$. We can choose two open subsets U and V of X such that $B \subset U$, $a \in V$, and $U \cap V = \emptyset$. Since $\langle V, X\rangle \cap C(X)$ is an open subset of $C(X)$ and $A \in \langle V, X\rangle \cap C(X)$, we can choose $d_0 \in D$ such that $A_m \in \langle V, X\rangle \cap C(X)$ for each $m \in D$ with $d_0 \leq m$. Since $\langle U\rangle \cap C(X)$ is an open subset of $C(X)$, $B \in \langle U\rangle \cap C(X)$, and $B \in \limsup\{C(A_d, X)\}_{d\in D}$ for $d_0 \in D$, there exists $m \in D$ with $d_0 \leq m$ and $\langle U\rangle \cap C(X) \cap C(A_m, X) \neq \emptyset$. If $B_m \in \langle U\rangle \cap C(X) \cap C(A_m, X)$, then $A_m \subset B_m \subset U$ and $A_m \cap V \neq \emptyset$. Hence, $U \cap V \neq \emptyset$, a contradiction. Therefore, $A \subset B$. □

We recall the Maximum-Minimum Theorem (for the metric case, see [11, p. 110] and [22, p. 68]).

Theorem 2.6 (Maximum-Minimum Theorem)**.** *Let X be a continuum. If $\mathcal{A}$ is a nonempty closed subset of $C(X)$, then there exists a maximal element in $\mathcal{A}$, with respect to inclusion, and there exists a minimal element in $\mathcal{A}$, with respect to inclusion.*

Proof. Let $\mathcal{A}$ be a nonempty closed subset of $C(X)$. We use the Kuratowski–Zorn lemma. Take a chain $\mathcal{C} \subset \mathcal{A}$, that is, every pair of elements of $\mathcal{C}$ are comparable by inclusion. Since $\mathcal{A}$ is a closed subset of $C(X)$ and $\mathcal{C}$ is a chain, we have that $cl(\bigcup \mathcal{C}) \in \mathcal{A}$. Also, $C \subset cl(\bigcup \mathcal{C})$ for each $C \in \mathcal{C}$; we have that $\mathcal{C}$ has an upper bound in $\mathcal{A}$. By the Kuratowski–Zorn lemma $\mathcal{A}$ has a maximal element, with respect to inclusion. The existence of the minimal element is similar, taking $\bigcap \mathcal{C} \in \mathcal{A}$ as a lower bound of the chain $\mathcal{C}$. □

The following lemma is similar to [12, Lemma 1.2]; the proof is also similar and we include it for the convenience of the reader.

Lemma 2.7. *Let X be a continuum. If $\mathcal{K}$ is a connected subset of $C(X)$, then $\bigcup \mathcal{K}$ is a connected subset of X.*

Proof. Let $\mathcal{K}$ be a connected subset of $C(X)$. Suppose, to the contrary, that $\bigcup \mathcal{K}$ is not connected; thus, there exist U_1 and U_2 nonempty disjoint open subsets of $\bigcup \mathcal{K}$ such that $\bigcup \mathcal{K} = U_1 \cup U_2$. Let W_1 and W_2 be open subsets of X such that $U_1 = W_1 \cap \bigcup \mathcal{K}$ and $U_2 = W_2 \cap \bigcup \mathcal{K}$.

Notice that $\langle W_1\rangle \cap \mathcal{K}$ and $\langle W_2\rangle \cap \mathcal{K}$ are open subsets of $\mathcal{K}$, disjoint and nonempty, and $\mathcal{K} = (\langle W_1\rangle \cap \mathcal{K}) \cup (\langle W_2\rangle \cap \mathcal{K})$. Thus, $\mathcal{K}$ is not connected, a contradiction to the hypothesis; therefore, $\bigcup \mathcal{K}$ is connected. □

3. The Property of Semi-Kelley

In 1998, J. J. Charatonik and W. J. Charatonik [3, Definition 3.2] introduce the concept of maximal limit continuum for metric continua.

We extend this concept for Hausdorff continua as follows.

Given a continuum X and $\mathcal{U} \subset C(X)$, we define the collection

$$F(\mathcal{U}) = \{B \in C(X) : C(B, X) \cap \mathcal{U} \neq \emptyset\}.$$

Definition 3.1 ([2, Definition 4.5]). Let X be a continuum and let M and K be subcontinua of X with $M \subset K$. We say that M is a *Hausdorff maximal limit continuum in* K provided that, for each L subcontinuum of X with $M \subsetneq L \subset K$, there is an open subset $\mathcal{U}$ of $C(X)$ such that $L \in \mathcal{U}$ and the collection $F(\mathcal{U})$ is not a neighborhood of M.

The following lemma follows directly from the definition.

Lemma 3.2. *Let X be a continuum and let M, K, and L be subcontinua of X with $M \subset K \subset L$. If M is a Hausdorff maximal limit continuum in L, then M is a Hausdorff maximal limit continuum in K.*

Lemma 3.3 ([2, Lemma 4.7]). *Let X be a continuum. If K is a subcontinuum of X, then K is a Hausdorff maximal limit continuum in K.*

In 1999, W. J. Charatonik [7, Definition 2.1] and Makuchowski [16, p. 124] introduce, independently, the concept of the property of Kelley for continua as follows.

Definition 3.4. Let X be a continuum and let $p \in X$. We say that X has the *property of Kelley at* p if, for each $K \in C(\{p\}, X)$ and for each open subset $\mathcal{U}$ f $C(X)$ with $K \in \mathcal{U}$, there exists an open subset U of X with $p \in U$ such that if $q \in U$, then there exists $L \in C(\{q\}, X) \cap \mathcal{U}$. We say that X has the *property of Kelley* provided that it has the property of Kelley at each of its points.

We will use the following theorem.

Theorem 3.5 ([2, Theorem 4.3]). *The following statements are equivalent for a continuum X:*

(1) *X has the property of Kelley;*
(2) *for each open subset $\mathcal{U}$ of $C(X)$, the union $\bigcup \mathcal{U}$ is an open subset of X;*
(3) *the function $f : X \to 2^{C(X)}$, defined by $f(p) = C(\{p\}, X)$ for each $p \in X$, is a mapping.*

Lemma 3.6 ([2, Lemma 4.16]). *Let X be a continuum with the property of Kelley and let M and K be subcontinua of X. If $M \subsetneq K$, then M is not a Hausdorff maximal limit continuum in K.*

In 1998, J. J. Charatonik and W. J. Charatonik [3, Definition 3.3] introduce the concept of strong maximal limit continuum for metric continua. Recently, in [2, Definition 4.10] the authors extend this notion to a weaker concept for continua. Now we present an alternative definition, which is equivalent to the original definition in the metric case. With this new definition, we are able to extend some results in the literature and to find new ones.

Definition 3.7. Let X be a continuum and let M and K be subcontinua of X with $M \subset K$. We say that M is a *Hausdorff strong maximal limit continuum in* K provided that there exists a net $\{M_d\}_{d\in D}$ in $C(X)$ converging to M such that $C(K) \cap \limsup\{C(M_d, X)\}_{d\in D} = \{M\}$.

The following lemma is immediate from the definition.

Lemma 3.8. *Let X be a continuum. If K is a subcontinuum of X, then K is a Hausdorff strong maximal limit continuum in K.*

Lemma 3.9. *Let X be a continuum and let M and K be subcontinua of X with $M \subset K$. If M is a Hausdorff strong maximal limit continuum in K, then M is a Hausdorff maximal limit continuum in K.*

Proof. Let M be a Hausdorff strong maximal limit continuum in K and choose a net $\{M_d\}_{d\in D}$ in $C(X)$ converging to M such that $C(K) \cap \limsup\{C(M_d, X)\}_{d\in D} = \{M\}$. Let $L \in C(X)$ be such that $M \subsetneq L \subset K$. Since $L \in C(K)$ and $L \neq M$, we have that $L \notin \limsup\{C(M_d, X)\}_{d\in D}$. Hence, there exist an open subset $\mathcal{U}$ of $C(X)$ and $d_0 \in D$ such that $L \in \mathcal{U}$ and $\mathcal{U} \cap C(M_d, X) = \emptyset$ for each $d \in D$ with $d_0 \leq d$.

Suppose that $F(\mathcal{U}) = \{B \in C(X) : C(B, X)\cap\mathcal{U} \neq \emptyset\}$ is a neighborhood of M in $C(X)$. Since $\{M_d\}_{d\in D}$ converges to M, there exists $n_0 \in D$ such that $M_d \in F(\mathcal{U})$ for all $d \in D$ with $n_0 \leq d$. Therefore, $C(M_d, X) \cap \mathcal{U} \neq \emptyset$ for all $d \in D$ with $n_0 \leq d$. Let $d \in D$ be such that $d_0 \leq d$ and $n_0 \leq d$; then $\mathcal{U} \cap C(M_d, X) = \emptyset$ and $C(M_d, X) \cap \mathcal{U} \neq \emptyset$, a contradiction. So M is a Hausdorff maximal limit continuum in K. □

By Lemma 3.6, Lemma 3.8, and Lemma 3.9, we have the following corollary.

Corollary 3.10. *Let X be a continuum with the property of Kelley. If K is a subcontinuum of X, then K is the only Hausdorff strong maximal limit continuum in K.*

Proposition 3.11. *Let X be a continuum, let K be a subcontinuum of X, and let $\{A_d\}_{d\in D}$ be a net in $C(X)$ converging to some subcontinuum A of K. If M is a maximal element, with respect to inclusion, of the set $C(K) \cap \limsup\{C(A_d, X)\}_{d\in D}$, then M is a Hausdorff strong maximal limit continuum in K.*

Proof. Let M be a maximal element, with respect to inclusion, of the set $C(K)\cap\limsup\{C(A_d, X)\}_{d\in D}$. Let $\mathfrak{U} = \{\mathcal{U} \subset C(X) : \mathcal{U}$ is an open subset of $C(X)$ and $M \in \mathcal{U}\}$. Order the set $D\times\mathfrak{U}$ as follows: If $(d_1,\mathcal{U}_1), (d_2,\mathcal{U}_2) \in D\times\mathfrak{U}$, we say that $(d_1,\mathcal{U}_1) \leq (d_2,\mathcal{U}_2)$ if and only if $d_1 \leq d_2$ and $\mathcal{U}_2 \subset \mathcal{U}_1$, so $D\times\mathfrak{U}$ is a directed set. For each $(d,\mathcal{U}) \in D\times\mathfrak{U}$, define an element $M_{(d,\mathcal{U})}$ in $C(X)$ as follows: Since $M \in \limsup\{C(A_d, X)\}_{d\in D}$, let $n \in D$ be such that $d \leq n$ and $\mathcal{U} \cap C(A_n, X) \neq \emptyset$, and choose $M_{(d,\mathcal{U})} \in \mathcal{U} \cap C(A_n, X)$. Since $M_{(d,\mathcal{U})} \in \mathcal{U}$ for each $(d, \mathcal{U}) \in D \times \mathfrak{U}$, the net $\{M_{(d,\mathcal{U})}\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}}$ converges to M in $C(X)$.

We show that $C(K) \cap \limsup\{C(M_{(d,\mathcal{U})}, X)\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}} = \{M\}$. Let $B \in C(K) \cap \limsup\{C(M_{(d,\mathcal{U})}, X)\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}}$. By Lemma 2.5, since $\{M_{(d,\mathcal{U})}\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}}$ converges to M in $C(X)$,

$$M \in \limsup\{C(M_{(d,\mathcal{U})}, X)\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}}$$

and $M \subset B$. Now, we prove that $B \in \limsup\{C(A_d, X)\}_{d \in D}$. Let $d \in D$ and let $\mathcal{V}$ be an open subset of $C(X)$ such that $B \in \mathcal{V}$. Since $B \in \limsup\{C(M_{(d,\mathcal{U})}, X)\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}}$ and $(d, C(X)) \in D \times \mathfrak{U}$, there exists $(d_1, \mathcal{U}_1) \in D \times \mathfrak{U}$ such that $(d, C(X)) \leq (d_1, \mathcal{U}_1)$ and $\mathcal{V} \cap C(M_{(d_1,\mathcal{U}_1)}, X) \neq \emptyset$. Let $d_2 \in D$ be such that $d_1 \leq d_2$ and $M_{(d_1,\mathcal{U}_1)} \in \mathcal{U}_1 \cap C(A_{d_2}, X)$. Thus, $A_{d_2} \subset M_{(d_1,\mathcal{U}_1)}$ and $C(M_{(d_1,\mathcal{U}_1)}, X) \subset C(A_{d_2}, X)$. Therefore, $\mathcal{V} \cap C(A_{d_2}, X) \neq \emptyset$ and $d \leq d_2$. Hence, $B \in C(K) \cap \limsup\{C(A_d, X)\}_{d \in D}$. By the maximality of M, it follows that $M = B$. So

$$C(K) \cap \limsup\{C(M_{(d,\mathcal{U})}, X)\}_{(d,\mathcal{U}) \in D \times \mathfrak{U}} = \{M\}.$$

It follows that M is a Hausdorff strong maximal limit continuum in K. $\square$

The next theorem generalizes [3, Theorem 3.11].

Theorem 3.12. *The following statements are equivalent for a continuum X:*

(1) *X has the property of Kelley;*
(2) *for each K subcontinuum of X, K is the only Hausdorff maximal limit continuum in K;*
(3) *for each K subcontinuum of X, K is the only Hausdorff strong maximal limit continuum in K.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 3.3 and Lemma 3.6.

The implication (2) $\Rightarrow$ (3) follows from Lemma 3.8 and Lemma 3.9.

We show the implication (3) $\Rightarrow$ (1). Let $\mathcal{U}$ be an open subset of $C(X)$ and suppose that $\bigcup \mathcal{U}$ is not an open subset of X. Take a point $x \in \bigcup \mathcal{U} - int(\bigcup \mathcal{U})$ and choose $K \in \mathcal{U}$ such that $x \in K$. Let $\mathcal{D} = \{U \subset X : U$ is an open subset of X and $x \in U\}$ be a directed set with $U_1 \leq U_2$ if and only if $U_2 \subset U_1$ for each $U_1, U_2 \in \mathcal{D}$.

Now, for each $U \in \mathcal{D}$, take a point $x_U \in U - \bigcup \mathcal{U}$. Notice that $\{x_U\}_{U \in \mathcal{D}}$ is a net in X converging to x and $\{\{x_U\}\}_{U \in \mathcal{D}}$ is a net in $C(X)$ converging to $\{x\}$. By Lemma 2.5, $\{x\} \in C(K) \cap \limsup\{C(\{x_U\}, X)\}_{U \in \mathcal{D}}$; by Theorem 2.6, take a maximal element M, with respect to inclusion, of the set $C(K) \cap \limsup\{C(\{x_U\}, X)\}_{U \in \mathcal{D}}$. By Proposition 3.11, M is a

Hausdorff strong maximal limit continuum in K and, by hypothesis, $M = K$. Hence, K is an element of $\limsup\{C(\{x_U\}, X)\}_{U\in\mathcal{D}}$. Since $K \in \mathcal{U}$ and $X \in \mathcal{D}$, there exists $V \in \mathcal{D}$ such that $X \leq V$ and $\mathcal{U}\cap C(\{x_V\}, X) \neq \emptyset$. Let $B_V \in \mathcal{U} \cap C(\{x_V\}, X)$, so $x_V \in B_V \in \mathcal{U}$. Therefore, $x_V \in \bigcup\mathcal{U}$, a contradiction to the definition of x_V. Consequently, $\bigcup\mathcal{U}$ is an open subset of X. By Theorem 3.5, X has the property of Kelley. $\square$

The following theorem generalizes [3, Proposition 3.15]; the proof in the metric cases uses Whitney mappings. It is worth mentioning that Whitney mappings do not exist in general for continua [4, Observation 3].

Theorem 3.13. *If a continuum X does not have the property of Kelley, then there exist M and K subcontinua of X such that M is a nondegenerate Hausdorff strong maximal limit continuum in K and $M \subsetneq K$.*

Proof. Let X be a continuum without the property of Kelley. By Theorem 3.12, let M and K be subcontinua of X such that $M \subsetneq K$ and M is a Hausdorff strong maximal limit continuum in K. If M contains more than one point, the proof is complete.

Suppose that M is a set consisting of one point and let $p \in X$ be such that $M = \{p\}$. Since M is a Hausdorff strong maximal limit continuum in K, there exists a net $\{M_d\}_{d\in D}$ in $C(X)$ converging to M such that $\{M\} = C(K) \cap \limsup\{C(M_d, X)\}_{d\in D}$. Also, since $M \neq K$, $K \notin \limsup\{C(M_d, X)\}_{d\in D}$; thus, there exists an open subset $\mathcal{V}$ of $C(X)$ and $m \in D$ such that $K \in \mathcal{V}$ and $\mathcal{V} \cap C(M_n, X) = \emptyset$ for each $n \in D$ with $m \leq n$. Let $V_1, \ldots, V_r$ be open subsets of X such that $K \in \langle V_1, \ldots, V_r\rangle \cap C(X) \subset \mathcal{V}$.

Claim 1. There exists a closed subset W of X such that $p \in int(W)$ and $K \cup A \in \mathcal{V}$ for each $A \in \langle W\rangle \cap C(X)$.

Proof of Claim 1. Since $p \in K$, without loss of generality, suppose that $p \in V_1$. Let U be an open subset of X such that $p \in U$ and $cl(U) \subset V_1$. Define $W = cl(U)$. Notice that for each $A \in \langle W\rangle\cap C(X)$, $A \subset cl(U) \subset V_1$. Hence, $K \cup A \in \langle V_1, \ldots, V_r\rangle \cap C(X) \subset \mathcal{V}$. This completes the proof of Claim 1.

Claim 2. There exists a nondegenerate subcontinuum B of X such that $M \subset B \subset W$ and $B \in \limsup\{C(M_d, X)\}_{d\in D}$.

Proof of Claim 2. Let $n \in D$. Since $\{M_d\}_{d\in D}$ converges to M and $M \in \langle int(W)\rangle$, there is an element $s \in D$ with $n \leq s$ such that $M_s \in \langle int(W)\rangle$. Let B_s be the component of W that contains M_s. By the Boundary Bumping Theorem [11, p. 101, Theorem 12.10], $B_s \cap bd(W) \neq \emptyset$, so $B_s \in \langle W, bd(W)\rangle \cap C(M_s, X)$. Hence, $\langle W, bd(W)\rangle \cap C(M_s, X) \neq \emptyset$. By Lemma 2.3, $\langle W, bd(W)\rangle \cap \limsup\{C(M_d, X)\}_{d\in D} \neq \emptyset$. Pick $B \in$

$\langle W, bd(W)\rangle \cap \limsup\{C(M_d, X)\}_{d\in D}$. By Lemma 2.5, $M \subset B$. Since $M \subset int(W)$ and $B \cap bd(W) \neq \emptyset$, B is a nondegenerate subcontinuum of X. This ends the proof of Claim 2.

Notice that $B \in C(K \cup B) \cap \limsup\{C(M_d, X)\}_{d\in D}$, so let B' be a maximal element in $C(K \cup B) \cap \limsup\{C(M_d, X)\}_{d\in D}$, with respect to inclusion, such that $B \subset B'$. By Claim 1, $K \cup B \in \mathcal{V}$, so that $K \cup B \notin \limsup\{C(M_d, X)\}_{d\in D}$. Therefore, $B' \neq K \cup B$. By Proposition 3.11, B' is a Hausdorff strong maximal limit continuum in $K \cup B$ and, moreover, B' contains more than one point. The proof the theorem is complete. $\square$

Corollary 3.14. *If a continuum X does not have the property of Kelley, then there exist M and K subcontinua of X such that M is a nondegenerate Hausdorff maximal limit continuum in K and $M \subsetneq K$.*

The property of semi-Kelley for metric continua is introduced by J. J. Charatonik and W. J. Charatonik in [3, Definition 3.16] using the notion of maximal limit continuum. We extend this concept to continua using the notion of Hausdorff maximal limit continuum as follows.

Definition 3.15. Let X be a continuum. We say that X has the *property of semi-Kelley* provided that for each subcontinuum K of X, if M_1 and M_2 are Hausdorff maximal limit continua in K, then $M_1 \subset M_2$ or $M_2 \subset M_1$.

By Theorem 3.12, we obtain the following remark. In the case of metric continua, the same property holds (see [3, Statement 3.17]).

Remark 3.16. If X is a continuum with the property of Kelley, then X is a continuum with the property of semi-Kelley.

Lemma 3.17. *Let X be a continuum and let M and K be subcontinua of X such that M is a Hausdorff maximal limit continuum in K. If $M \neq K$, then there exists S a Hausdorff strong maximal limit continuum in K such that $M \subset S \subsetneq K$.*

Proof. Suppose that $M \neq K$. Since M is a Hausdorff maximal limit continuum in K, there exists an open subset $\mathcal{U}$ of $C(X)$ such that $K \in \mathcal{U}$ and the set $F(\mathcal{U})$ is not a neighborhood of M in $C(X)$. Let $\mathfrak{D} = \{\mathcal{V} \subset C(X) : \mathcal{V}$ is an open subset of $C(X)$ and $M \in \mathcal{V}\}$ be a directed set with $\mathcal{V}_1 \leq \mathcal{V}_2$ if and only if $\mathcal{V}_2 \subset \mathcal{V}_1$ for each $\mathcal{V}_1, \mathcal{V}_2 \in \mathfrak{D}$. For each $\mathcal{V} \in \mathfrak{D}$ take $M_{\mathcal{V}} \in \mathcal{V}$ such that $M_{\mathcal{V}} \notin F(\mathcal{U})$. Notice that the net $\{M_{\mathcal{V}}\}_{\mathcal{V}\in\mathfrak{D}}$ converges to M, and $C(M_{\mathcal{V}}, X) \cap \mathcal{U} = \emptyset$ for each $\mathcal{V} \in \mathfrak{D}$. Hence, $K \notin \limsup\{C(M_{\mathcal{V}}, X)\}_{\mathcal{V}\in\mathfrak{D}}$. Since $C(M, K) \cap \limsup\{C(M_{\mathcal{V}}, X)\}_{\mathcal{V}\in\mathfrak{D}}$ is closed and nonempty (M is an element of it), by Theorem 2.6, there is a maximal element $S \in C(M, K) \cap \limsup\{C(M_{\mathcal{V}}, X)\}_{\mathcal{V}\in\mathfrak{D}}$. Notice that $S \subsetneq K$ and S is a maximal element of $S \in C(K) \cap \limsup\{C(M_{\mathcal{V}}, X)\}_{\mathcal{V}\in\mathfrak{D}}$.

By Proposition 3.11, S is a Hausdorff strong maximal limit continuum in K. □

The next theorem generalizes [3, Theorem 3.18].

Theorem 3.18. *The following statements are equivalent for a continuum X:*

(1) *X has the property of semi-Kelley;*
(2) *for each K subcontinuum of X, if M_1 and M_2 are Hausdorff strong maximal limit continua in K, then $M_1 \subset M_2$ or $M_2 \subset M_1$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the definition of the semi-Kelley property and Lemma 3.9.

We prove (2) $\Rightarrow$ (1). Suppose that X does not have the property of semi-Kelley; thus, there exist M_1, M_2, and $L \in C(X)$ such that M_1 and M_2 are Hausdorff maximal limit continua in L, with M_1 and M_2 not comparable by inclusion. Notice that $\{B \in C(L) : M_1, M_2 \subset B\}$ is a nonempty closed subset of $C(X)$. By Theorem 2.6, choose a minimal element K of $\{B \in C(L) : M_1, M_2 \subset B\}$, with respect to inclusion. By Lemma 3.8, K is a Hausdorff strong maximal limit continuum in K.

Claim. At least one of the following statements holds:

- if S is a Hausdorff strong maximal limit continuum in K containing M_1, then $S = K$;
- if S is a Hausdorff strong maximal limit continuum in K containing M_2, then $S = K$.

Proof of Claim. Let S_1 and S_2 be Hausdorff strong maximal limit continua in K such that $M_1 \subset S_1$ and $M_2 \subset S_2$. By (2), $S_1 \subset S_2$ or $S_2 \subset S_1$; thus, $S_1 \cup S_2 = S_2$ or $S_1 \cup S_2 = S_1$. In either case, $S_1 \cup S_2 \subset K$ and $S_1 \cup S_2$ contains M_1 and M_2. By the choice of K, $S_1 \cup S_2 = K$. So $S_1 = K$ or $S_2 = K$, and the claim is proved.

Without loss of generality, suppose that K is the only Hausdorff strong maximal limit continuum in K containing M_1. By Lemma 3.2, since M_1 is a Hausdorff maximal limit continuum in L, M_1 is a Hausdorff maximal limit continuum in K. Since K contains M_2, $M_1 \subsetneq K$. By Lemma 3.17, there exists a Hausdorff strong maximal limit continuum S in K such that $M_1 \subset S \subsetneq K$, a contradiction. Therefore, X has the property of semi-Kelley. The proof of the theorem is complete. □

4. Products and Hyperspaces

We start this section with a result related to products. In 1997, Roger W. Wardle [23, Corollary 4.6] shows that if the Cartesian product of two metric continua has the property of Kelley, then each factor continuum has the property of Kelley. This result can be strengthened by assuming that the product is a metric continuum with the property of semi-Kelley [3, Theorem 4.1]. We generalize those results below.

Theorem 4.1. *Let X and Y be continua. If $X \times Y$ has the property of semi-Kelley, then X and Y have the property of Kelley.*

Proof. Suppose that $X \times Y$ has the property of semi-Kelley and that X does not have the property of Kelley. By Theorem 3.12, there exist $M, K \in C(X)$ such that $M \subsetneq K$ and M is a Hausdorff maximal limit continuum in K. Let p and q be distinct points in Y and let $a \in K - M$. Define $\mathcal{K} = (K \times \{p, q\}) \cup (\{a\} \times Y)$. Notice that $\mathcal{K}$ is a subcontinuum in $X \times Y$.

Claim. The set $M \times \{p\}$ is a Hausdorff maximal limit continuum in $\mathcal{K}$.

Proof of Claim. Let $\mathcal{L} \in C(X \times Y)$ be such that $M \times \{p\} \subsetneq \mathcal{L} \subset \mathcal{K}$. We show that there exists an open subset $\mathfrak{U}$ of $C(X \times Y)$ such that $\mathcal{L} \in \mathfrak{U}$ and the set $F(\mathfrak{U}) = \{B \in C(X \times Y) : C(B, X \times Y) \cap \mathfrak{U} \neq \emptyset\}$ is not a neighborhood of $M \times \{p\}$ in $C(X \times Y)$.

Denote by $\pi_1 : X \times Y \to X$ the projection on the first coordinate. Since $M \times \{p\} \subsetneq \mathcal{L}$, we have that $M \subset \pi_1[\mathcal{L}]$. Suppose that $M = \pi_1[\mathcal{L}]$. Observe that $\mathcal{L} \subset \pi_1^{-1}[M] \cap \mathcal{K} = M \times \{p, q\}$. Since $\mathcal{L}$ is connected, it follows that $\mathcal{L} \subset M \times \{p\}$. Therefore, $\mathcal{L} = M \times \{p\}$, which contradicts $M \times \{p\} \subsetneq \mathcal{L}$. So we have that $M \subsetneq \pi_1[\mathcal{L}]$.

Note that $\pi_1[\mathcal{L}]$ is a subcontinuum in X and $\pi_1[\mathcal{L}] \subset K$. Since M is a Hausdorff maximal limit continuum in K, there exists an open subset $\mathcal{U}$ of $C(X)$ such that $\pi_1[\mathcal{L}] \in \mathcal{U}$ and the set $F(\mathcal{U}) = \{B \in C(X) : C(B, X) \cap \mathcal{U} \neq \emptyset\}$ is not a neighborhood in M. Let $C(\pi_1) : C(X \times Y) \to C(X)$ be the induced mapping and let $\mathfrak{U} = (C(\pi_1))^{-1}[\mathcal{U}]$. Since $C(\pi_1)$ is a mapping, $\mathfrak{U}$ is an open subset of $C(X \times Y)$. Recall that $\mathcal{L} \in \mathfrak{U}$.

We prove that the set $F(\mathfrak{U})$ is not a neighborhood of $M \times \{p\}$ in $C(X \times Y)$. Suppose, on the contrary, that $F(\mathfrak{U})$ is a neighborhood of $M \times \{p\}$ in $C(X \times Y)$. By Lemma 2.1, since $\{U \times V : U$ is an open subset of X and V is an open subset of $Y\}$ is a basis for the topology of $X \times Y$, there exist open subsets $U_1, \ldots, U_n$ of X and open subsets $V_1, \ldots, V_n$ of Y such that

$$M \times \{p\} \in \langle U_1 \times V_1, \ldots, U_n \times V_n \rangle \cap C(X \times Y) \subset F(\mathfrak{U}).$$

Notice that $M \in \langle U_1, \ldots, U_n \rangle \cap C(X)$. Hence, there exists an element $B \in \langle U_1, \ldots, U_n \rangle \cap C(X)$ such that $B \notin F(\mathcal{U})$; that is, $C(B, X) \cap \mathcal{U} = \emptyset$. Since $B \times \{p\} \in \langle U_1 \times V_1, \ldots, U_n \times V_n \rangle \cap C(X)$, we have that $C(B \times \{p\}, C(X \times Y)) \cap \mathfrak{U} \neq \emptyset$. Let $D \in C(B \times \{p\}, C(X \times Y)) \cap \mathfrak{U}$; notice

$B \subset \pi_1[D]$ and $\pi_1[D] \in \mathcal{U}$. Hence, $\pi_1[D] \in C(B, X) \cap \mathcal{U}$, a contradiction. The proof of the claim is complete.

Similarly, the set $M \times \{q\}$ is a Hausdorff maximal limit continuum in $\mathcal{K}$. Since $M \times \{p\}$ and $M \times \{q\}$ are Hausdorff maximal limit continua in $\mathcal{K}$ which are non-comparable, $X \times Y$ does not have the property of semi-Kelley. Therefore, X must have the property of Kelley. Likewise, Y has the property of Kelley. This completes the proof of the theorem. $\square$

The following theorem generalizes [3, Theorem 4.5 and Theorem 4.7].

Theorem 4.2. *Let X be a continuum which does not have the property of Kelley. If $\mathcal{H}(X)$ is a continuum such that $C(X) \subset \mathcal{H}(X) \subset 2^X$, then $\mathcal{H}(X)$ does not have the property of semi-Kelley.*

Proof. Assume that $\mathcal{H}(X)$ is a continuum such that $C(X) \subset \mathcal{H}(X) \subset 2^X$.

Since X does not have the property of Kelley, by Corollary 3.14, there exist $M, K \in C(X)$ such that M is a nondegenerate Hausdorff maximal limit continuum in K and $M \subsetneq K$.

Let $a \in K - M$; by Theorem 2.2, let $\mathcal{A}$ be an order arc in $C(X)$ from $\{a\}$ to K and let $\mathcal{M}$ be an order arc in $C(X)$ from M to K. Define $\mathcal{K} = F_1(K) \cup \mathcal{A} \cup \mathcal{M}$. Then $\mathcal{K}$ is a continuum and $\mathcal{K} \subset C(X) \subset \mathcal{H}(X)$.

Claim 1. The set $F_1(M)$ is a Hausdorff maximal limit continuum in $\mathcal{K}$.

Proof of Claim 1. Let $\mathcal{L}$ be a continuum such that $F_1(M) \subsetneq \mathcal{L} \subset \mathcal{K}$. We show that there exists an open subset $\mathfrak{U}$ of $C(\mathcal{H}(X))$ with $\mathcal{L} \in \mathfrak{U}$ such that the set $\{\mathcal{B} \in C(\mathcal{H}(X)) : C(\mathcal{B}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset\}$ is not a neighborhood of $F_1(M)$ in $C(\mathcal{H}(X))$.

Notice that $M = \bigcup F_1(M) \subset \bigcup \mathcal{L}$. Suppose that $M = \bigcup \mathcal{L}$. For each $L \in \mathcal{L}$, $L \subset \bigcup F_1(M) = M$, so $L \in C(M)$. Hence, $\mathcal{L} \subset C(M) \cap \mathcal{K} = \{M\} \cup F_1(M)$. Since $\mathcal{L}$ is connected and $F_1(M) \subset \mathcal{L}$, $\mathcal{L} = F_1(M)$, a contradiction. Therefore, $M = \bigcup F_1(M) \subsetneq \bigcup \mathcal{L} \subset \bigcup \mathcal{K} = K$. Since M is a Hausdorff maximal limit continuum in K, there is an open subset $\mathcal{U}$ of $C(X)$ such that $\bigcup \mathcal{L} \in \mathcal{U}$ and such that the set

$$\{B \in C(X) : C(B, X) \cap \mathcal{U} \neq \emptyset\} \tag{4.1}$$

is not a neighborhood of M in $C(X)$.

Let $U_1, \ldots, U_n$ be open subsets of X such that $\bigcup \mathcal{L} \in \langle U_1, \ldots, U_n \rangle \cap C(X) \subset \mathcal{U}$.

For ease of notation, given a finite family $\mathcal{V} = \{V_1, \ldots, V_m\}$ of open subsets of X, define the set

$$\langle \mathcal{V} \rangle = \langle V_1, \ldots, V_m \rangle;$$

given a finite family $\mathfrak{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_m\}$ of open subsets of $\mathcal{H}(X)$, define the set

$$\langle \mathfrak{V} \rangle_{2^{\mathcal{H}(X)}} = \langle \mathcal{V}_1, \ldots, \mathcal{V}_m \rangle_{2^{\mathcal{H}(X)}},$$

a basic open set of the Vietoris topology of $2^{\mathcal{H}(X)}$.

For each $L \in \mathcal{L}$, $L \subset \bigcup \mathcal{L} \subset U_1 \cup \cdots \cup U_n$. Consider the finite family $\mathcal{U}_L = \{U \in \{U_1, \ldots, U_n\} : L \cap U \neq \emptyset\}$ of open subsets of X. Hence, $L \in \langle \mathcal{U}_L \rangle \cap \mathcal{H}(X)$. Observe that $\{\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X) : L \in \mathcal{L}\}$ is a finite family of open subsets of $\mathcal{H}(X)$. Let $\mathfrak{U} = \langle \{\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X) : L \in \mathcal{L}\} \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X))$. We prove that $\mathcal{L} \in \mathfrak{U}$. For each $L \in \mathcal{L}$, $L \in \langle \mathcal{U}_L \rangle \cap \mathcal{H}(X)$. Therefore, $\mathcal{L} \subset \bigcup_{L \in \mathcal{L}} (\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X))$. Moreover, $\mathcal{L} \cap (\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X)) \neq \emptyset$ since L is an element of it. So we have that $\mathcal{L} \in \mathfrak{U}$.

Finally, we prove that the set

$$\{\mathcal{B} \in C(\mathcal{H}(X)) : C(\mathcal{B}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset\} \tag{4.2}$$

is not a neighborhood of $F_1(M)$ in $C(\mathcal{H}(X))$.

Suppose, on the contrary, that (4.2) is a neighborhood of $F_1(M)$ in $C(\mathcal{H}(X))$. Let $\mathcal{W}_1, \ldots, \mathcal{W}_m$ be open subsets of $\mathcal{H}(X)$ such that $F_1(M)$ is an element of $\langle \mathcal{W}_1, \ldots, \mathcal{W}_m \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X))$ and

$$\langle \mathcal{W}_1, \ldots, \mathcal{W}_m \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X)) \subset \{\mathcal{B} \in C(\mathcal{H}(X)) : C(\mathcal{B}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset\}.$$

Notice that the function $p : F_1(X) \to X$, defined by $p(\{x\}) = x$ for each $\{x\} \in F_1(X)$, is a homeomorphism. For each $i \in \{1, \ldots, m\}$, define $W_i = \{x \in X : \{x\} \in \mathcal{W}_i \cap F_1(X)\} = p[\mathcal{W}_i \cap F_1(X)]$. Since $\mathcal{W}_i \cap F_1(X)$ is an open subset of $F_1(X)$, W_i is an open subset of X for each $i \in \{1, \ldots, m\}$. We show that $\langle W_1, \ldots, W_m \rangle$ contains M. Since $F_1(M) \subset \bigcup_{i=1}^m \mathcal{W}_i$, it follows that $F_1(M) \subset \bigcup_{i=1}^m (\mathcal{W}_i \cap F_1(X))$. Hence, $M = p[F_1(M)] \subset p[\bigcup_{i=1}^m (\mathcal{W}_i \cap F_1(X))] = \bigcup_{i=1}^m p[\mathcal{W}_i \cap F_1(X)] = \bigcup_{i=1}^m W_i$. For each $i \in \{1, \ldots, m\}$, $F_1(M) \cap \mathcal{W}_i \neq \emptyset$, so $p[F_1(M)] \cap p[\mathcal{W}_i \cap F_1(X)] \neq \emptyset$; that is, $M \cap W_i \neq \emptyset$. Thus, $\langle W_1, \ldots, W_m \rangle \cap C(X)$ is a neighborhood of M in $C(X)$. By (4.1), there exists an element

$$B \in \langle W_1, \ldots, W_m \rangle \cap C(X) \text{ such that } C(B, X) \cap \mathcal{U} = \emptyset. \tag{4.3}$$

Note that $F_1(B) \in \langle \mathcal{W}_1, \ldots, \mathcal{W}_m \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X))$, so $C(F_1(B), \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset$. Take $\mathcal{D} \in C(F_1(B), \mathcal{H}(X)) \cap \mathfrak{U}$. Since $F_1(B) \subset \mathcal{D} \subset \mathcal{H}(X)$, $B = \bigcup F_1(B) \subset \bigcup \mathcal{D} \subset X$. By [6, Lemma 2.2], since $\mathcal{D} \subset 2^X$ is a continuum and $\mathcal{D} \cap C(X) \neq \emptyset$, $\bigcup \mathcal{D}$ is a continuum. Therefore, $\bigcup \mathcal{D} \in C(B, X)$.

Now we prove that $\bigcup \mathcal{D} \in \langle U_1, \ldots, U_n \rangle \cap C(X)$. Given $x \in \bigcup \mathcal{D}$, there exists an element $E \in \mathcal{D}$ such that $x \in E$. Since $\mathcal{D} \in \mathfrak{U}$, $\mathcal{D} \subset \bigcup_{L \in \mathcal{L}} (\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X))$. Hence, there exists an element $L \in \mathcal{L}$ such that $E \in \langle \mathcal{U}_L \rangle \cap \mathcal{H}(X)$, and $x \in E \subset \bigcup \mathcal{U}_L \subset \bigcup_{i=1}^n U_i$. Therefore, $\bigcup \mathcal{D} \subset \bigcup_{i=1}^n U_i$. On the other hand, let $k \in \{1, \ldots, n\}$; since $\bigcup \mathcal{L} \in \langle U_1, \ldots, U_n \rangle \cap C(X)$, consider a point $x \in (\bigcup \mathcal{L}) \cap U_k$. Now, there exists an element $L \in \mathcal{L}$ such that

$x \in L \cap U_k$; since $\mathcal{D} \in \mathfrak{U}$, $\mathcal{D} \cap (\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X)) \neq \emptyset$. Consider $E \in \mathcal{D}$ such that $E \in \langle \mathcal{U}_L \rangle \cap \mathcal{H}(X)$. Since $U_k \in \mathcal{U}_L$, $E \cap U_k \neq \emptyset$. As $E \subset \bigcup \mathcal{D}$, it follows that $(\bigcup \mathcal{D}) \cap U_k \neq \emptyset$. Therefore, $\bigcup \mathcal{D} \in C(B, X) \cap (\langle U_1, \ldots, U_n \rangle \cap C(X))$. Hence, $\bigcup \mathcal{D} \in C(B, X) \cap \mathcal{U}$, contradicting (4.3). The proof of Claim 1 is complete.

CLAIM 2. The set $\{M\}$ is a Hausdorff maximal limit continuum in $\mathcal{K}$.

Proof of Claim 2. Let $\mathcal{L}$ be a continuum such that $\{M\} \subsetneq \mathcal{L} \subset \mathcal{K}$. We show that there exists an open subset $\mathfrak{U}$ of $C(\mathcal{H}(X))$ such that $\mathcal{L} \in \mathfrak{U}$ and the set $\{\mathcal{B} \in C(\mathcal{H}(X)) : C(\mathcal{B}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset\}$ is not a neighborhood of $\{M\}$ in $C(\mathcal{H}(X))$. As in Claim 1, $M \subsetneq \bigcup \mathcal{L} \subset \bigcup \mathcal{K} = K$ and $\bigcup \mathcal{L}$ is a subcontinuum in X. Since M is a Hausdorff maximal limit continuum in K, there exists an open subset $\mathcal{U}$ of $C(X)$ such that $\bigcup \mathcal{L} \in \mathcal{U}$ and the set

$$\{B \in C(X) : C(B, X) \cap \mathcal{U} \neq \emptyset\} \tag{4.4}$$

is not a neighborhood of M in $C(X)$.

Let $U_1, \ldots, U_n$ be open subsets of X such that $\bigcup \mathcal{L} \in \langle U_1, \ldots, U_n \rangle \cap C(X) \subset \mathcal{U}$. We consider the sets $\langle \mathcal{V} \rangle$ and $\mathcal{U}_L$ as in the proof of Claim 1.

Define $\mathfrak{U} = \langle \{\langle \mathcal{U}_L \rangle \cap \mathcal{H}(X) : L \in \mathcal{L}\} \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X))$. As in Claim 1, $\mathfrak{U}$ is an open subset of $C(\mathcal{H}(X))$ and $\mathcal{L} \in \mathfrak{U}$.

We prove that the set

$$\{\mathcal{B} \in C(\mathcal{H}(X)) : C(\mathcal{B}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset\} \tag{4.5}$$

is not a neighborhood of $\{M\}$ in $C(\mathcal{H}(X))$.

Suppose that (4.5) is a neighborhood of $\{M\}$ in $C(\mathcal{H}(X))$. We consider an open subset $\mathcal{W}$ of $\mathcal{H}(X)$ such that $\{M\} \in \langle \mathcal{W} \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X)) \subset \{\mathcal{B} \in C(\mathcal{H}(X)) : C(\mathcal{B}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset\}$. Hence, $\{M\} \subset \mathcal{W}$ and it follows that $M \in \mathcal{W} \cap C(X)$. By (4.4), there exists an element

$$B \in \mathcal{W} \cap C(X) \text{ such that } C(B, X) \cap \mathcal{U} = \emptyset. \tag{4.6}$$

Therefore, $C(B, X) \cap (\langle U_1, \ldots, U_n \rangle \cap C(X)) = \emptyset$. Note that $\{B\} \in \langle \mathcal{W} \rangle_{2^{\mathcal{H}(X)}} \cap C(\mathcal{H}(X))$ and $C(\{B\}, \mathcal{H}(X)) \cap \mathfrak{U} \neq \emptyset$. Consider an element $\mathcal{D} \in C(\{B\}, \mathcal{H}(X)) \cap \mathfrak{U}$. Since $\{B\} \subset \mathcal{D} \subset \mathcal{H}(X)$, $B \subset \bigcup \mathcal{D} \subset X$. By [6, Lemma 2.2], since $\mathcal{D} \subset 2^X$ is a continuum which intersects $C(X)$, $\bigcup \mathcal{D}$ is a continuum. Hence, $\bigcup \mathcal{D} \in C(B, X)$. As in the proof of Claim 1, $\bigcup \mathcal{D} \in \langle U_1, \ldots, U_n \rangle \cap C(X)$. Therefore, $\bigcup \mathcal{D} \in C(B, X) \cap \mathcal{U}$, contradicting (4.6). The proof of Claim 2 is complete.

By claims 1 and 2, $F_1(M)$ and $\{M\}$ are Hausdorff maximal limit continua in $\mathcal{K}$; since $F_1(M)$ and $\{M\}$ are non-comparable, we obtain that $\mathcal{H}(X)$ does not have the property of semi-Kelley. The proof of the theorem is complete. $\square$

As a consequence of Theorem 4.2, we obtain the following result concerning the n-fold hyperspace of a continuum.

Corollary 4.3. *Let X be a continuum and let $n \in \mathbb{N}$. If $C_n(X)$ has the property of semi-Kelley, then X has the property of Kelley.*

Question 4.4. Let X be a continuum with the property of Kelley. Does there exist a continuum $\mathcal{H}(X)$ with the property of semi-Kelley such that $C(X) \subset \mathcal{H}(X) \subset 2^X$?

The following question of Sam B. Nadler, Jr., is still unsolved (see [21, p. 558, Question (16.37)]).

Question 4.5. Let X be a continuum. If 2^X has the property of Kelley, then does $C(X)$ have the property of Kelley?

With relation to the previous question, we pose the following question.

Question 4.6. Let X be a continuum. If 2^X has the property of semi-Kelley, then does $C(X)$ have the property of semi-Kelley?

The continuum X given in [1, Example 2.1] has the property of Kelley, while $C(X)$ does not have the property of semi-Kelley. The following question is natural and is related to Question 4.6.

Question 4.7. Let X be the continuum given in [1, Example 2.1]. Does 2^X have the property of semi-Kelley?

Given a continuum X and $x \in X$, we say that X *is connected im kleinen at* x provided that for each open subset U of X with $x \in U$, there exists a continuum K such that $K \subset U$ and $x \in int(K)$.

The following result is a generalization of [9, Theorem 2].

Lemma 4.8. *The following statements are equivalent for a continuum X and a subcontinuum A of X:*

(1) *$C(X)$ is connected im kleinen at A;*
(2) *for each open subset U of X with $A \subset U$, there exists an open subset $\mathcal{U}$ of $C(X)$ such that $A \in \mathcal{U} \subset \langle U\rangle \cap C(X)$, and, for each $A' \in \mathcal{U}$, A and A' are contained in the same component of U.*

Proof. We prove (1) $\Rightarrow$ (2). Let U be an open subset of X such that $A \subset U$. Since $A \in \langle U\rangle \cap C(X)$ and $C(X)$ is connected im kleinen at A, there exists a continuum $\mathcal{K} \subset \langle U\rangle \cap C(X)$ such that $A \in int(\mathcal{K})$. Let $\mathcal{U} = int(\mathcal{K})$ and consider $A' \in \mathcal{U}$. By Lemma 2.7, $\bigcup \mathcal{K}$ is connected; since $A, A' \subset \bigcup \mathcal{K} \subset U$, A and A' are contained in the same component of U.

Now, we prove (2) $\Rightarrow$ (1). Let $\mathcal{W}$ be an open subset of $C(X)$ such that $A \in \mathcal{W}$ and let $\mathcal{U}$ be an open subset of $C(X)$ such that $cl(\mathcal{U}) \subset \mathcal{W}$.

Consider open subsets $U_1, \ldots, U_n$ of X such that $A \in \langle U_1, \ldots, U_n \rangle \cap C(X) \subset \mathcal{U}$. Let $U = U_1 \cup \cdots \cup U_n$, and let V be an open subset of X such that $A \subset V \subset cl(V) \subset U$. By (2), there exists an open subset $\mathcal{V}$ of $C(X)$ such that $A \in \mathcal{V} \subset \langle V \rangle \cap C(X)$, and, for each $A' \in \mathcal{V}$, A and A' are contained in the same component of V. Denote by K_A the closure of the component of V that contains A. Consider the set $\mathcal{K} = \{B \in C(X) : B \subset K_A$ and there exists $D \in \mathcal{V} \cap \langle U_1, \ldots, U_n \rangle$ such that $D \subset B\}$.

Notice $A \in \mathcal{V} \cap \langle U_1, \ldots, U_n \rangle$. We prove that $\mathcal{V} \cap \langle U_1, \ldots, U_n \rangle \subset \mathcal{K}$: Let $B \in \mathcal{V} \cap \langle U_1, \ldots, U_n \rangle$; since $B \in \mathcal{V}$, it follows that $B \subset K_A$. Taking $D = B$, we obtain $B \in \mathcal{K}$. Therefore, $\mathcal{V} \cap \langle U_1, \ldots, U_n \rangle \subset \mathcal{K}$. Now we prove that $\mathcal{K}$ is a connected set. Take elements $B_1, B_2 \in \mathcal{K}$; notice that $B_1, B_2 \subset K_A$. Since K_A is a continuum, by Theorem 2.2, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two order arcs in $C(X)$ from B_1 to K_A and from B_2 to K_A, respectively. Notice that $\mathcal{B}_1 \subset \mathcal{K}$ and $\mathcal{B}_2 \subset \mathcal{K}$. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2$ is a connected set in $\mathcal{K}$ containing B_1 and B_2. Therefore, $\mathcal{K}$ is connected.

We prove that $\mathcal{K} \subset \mathcal{U}$. Let $B \in \mathcal{K}$ and let $D \in \mathcal{V} \cap \langle U_1, \ldots, U_n \rangle$ such that $D \subset B \subset K_A \subset cl(V) \subset U$. Moreover, since $D \in \langle U_1, \ldots, U_n \rangle$, then $D \cap U_i \neq \emptyset$ for each $i \in \{1, \ldots, n\}$. Hence, $B \cap U_i \neq \emptyset$ for each $i \in \{1, \ldots, n\}$. Since $B \subset U$, $U = U_1 \cup \cdots \cup U_n$, and $\langle U_1, \ldots, U_n \rangle \cap C(X) \subset \mathcal{U}$, it follows that $B \in \mathcal{U}$. Therefore, $\mathcal{K} \subset \mathcal{U}$.

Since $\mathcal{K} \subset \mathcal{U}$, $cl(\mathcal{K}) \subset cl(\mathcal{U}) \subset \mathcal{W}$. We have that $cl(\mathcal{K})$ is a continuum contained in $\mathcal{W}$ that has A in its interior. Therefore, $C(X)$ is connected im kleinen at A. $\square$

The following theorem generalizes [3, Theorem 4.9].

Theorem 4.9. *Let X be a continuum with the property of semi-Kelley and let A and B be subcontinua of X with $A \subsetneq B$. If $C(X)$ is connected im kleinen at A, then $C(X)$ is connected im kleinen at B.*

Proof. Assume that $C(X)$ is not connected im kleinen at B. We will prove that $C(X)$ is not connected im kleinen at A.

By Lemma 4.8, there is an open subset U of X with the following properties: (a) $B \subset U$ and (b) for each open subset $\mathcal{U}$ of $C(X)$ with $B \in \mathcal{U} \subset \langle U \rangle \cap C(X)$, there exists an element $B' \in \mathcal{U}$ such that B and B' are contained in distinct components of U.

Claim. For each $a \in A$, there exists a Hausdorff maximal limit continuum A' in A with the following properties: (c) $a \in A'$ and (d) for each open subset $\mathcal{U}$ of $C(X)$ with $A' \in \mathcal{U} \subset \langle U \rangle \cap C(X)$, there exists an element $J \in \mathcal{U}$ such that B and J are contained in distinct components of U.

Proof of Claim. Let $a \in A$ and let

$$\mathcal{D} = \{W \subset U : W \text{ is an open subset of } X \text{ and } a \in W\}$$

be a directed set with $W_1 \leq W_2$ if and only if $W_2 \subset W_1$, for each $W_1, W_2 \in \mathcal{D}$.

Given $W \in \mathcal{D}$, notice that $a \in B \cap W$, $a \in B \cap U$, and $B \subset U = W \cup U$. Hence, $B \in \langle W, U\rangle \cap C(X) \subset \langle U\rangle \cap C(X)$. Take an element $B(W) \in \langle W, U\rangle \cap C(X)$ such that B and $B(W)$ are contained in distinct components of U, and choose a point $a(W) \in B(W) \cap W$. Notice that the net $\{a(W)\}_{W\in\mathcal{D}}$ converges to a. By Lemma 2.5,

$$\{a\} \in C(A) \cap \limsup\{C(\{a(W)\}, X)\}_{W\in\mathcal{D}}.$$

By Theorem 2.6, take a maximal element A' of $C(A) \cap \limsup \{C(\{a(W)\}, X)\}_{W\in\mathcal{D}}$, with respect to inclusion. By Lemma 3.9 and Proposition 3.11, A' is a Hausdorff maximal limit continuum in A. By Lemma 2.5, $a \in A'$.

Now we prove (d). Let $\mathcal{U}$ be an open subset of $C(X)$ with $A' \in \mathcal{U} \subset \langle U\rangle \cap C(X)$. Since $A' \in \limsup\{C(\{a(W)\}, X)\}_{W\in\mathcal{D}}$ and $U \in \mathcal{D}$, then there exists an element $W \in \mathcal{D}$ such that $U \leq W$ and $C(\{a(W)\}, X) \cap \mathcal{U} \neq \emptyset$. Now, let $J \in C(\{a(W)\}, X) \cap \mathcal{U}$, so $a(W) \in J \subset U$. Therefore, J and $B(W)$ are contained in the same component of U, and J and B are contained in distinct components of U. The proof the claim is complete.

Now we use Lemma 4.8 to prove that $C(X)$ is not connected im kleinen at A. Let $\mathcal{U}$ be an open subset of $C(X)$ such that $A \in \mathcal{U} \subset \langle U\rangle \cap C(X)$. We show an element $A' \in \mathcal{U}$ such that A and A' are contained in distinct components of U. Let $U_1, \dots, U_n$ be open subsets of X such that $A \in \langle U_1, \dots, U_n\rangle \cap C(X) \subset \mathcal{U}$. For each $i \in \{1, \dots, n\}$, choose $a_i \in A \cap U_i$, and let A_i be a Hausdorff maximal limit continuum in A, as given by the claim applied to the point a_i. Since X is a continuum with the property of semi-Kelley, there exists $j \in \{1, \dots, n\}$ such that $A_i \subset A_j$ for each $i \in \{1, \dots, n\}$. Note that $A_j \in \langle U_1, \dots, U_n\rangle \cap C(X) \subset \langle U\rangle \cap C(X)$ and, by the definition of A_j, there exists an element $A' \in \langle U_1, \dots, U_n\rangle \cap C(X) \subset \mathcal{U}$ such that A' and B are contained in distinct components of U. Since $A_j \subset A \subset B$, A' and A are contained in distinct components of U. By Lemma 4.8, $C(X)$ is not connected im kleinen at A. The proof of the theorem is complete. $\square$

To finish this section, we give an example of a metric continuum X without the property of semi-Kelley that satisfies Theorem 4.9.

Example 4.10. Given $p, q \in \mathbb{R}^2$, let pq denote the convex straight line segment from p to q. For each $n \in \mathbb{N}$, define $a_n = (0, \frac{1}{n})$, $b_n = (0, -\frac{1}{n})$, $c = (-1, 0)$, $d = (1, 0)$, and $e = (0, 0)$. Set $X = cd \cup \bigcup\{ca_n \cup db_n : n \in \mathbb{N}\}$. We prove that X does not have the property of semi-Kelley. Let $K = [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$, $M_1 = [-\frac{1}{2}, 0] \times \{0\}$, and $M_2 = [0, \frac{1}{2}] \times \{0\}$; notice that M_1

and M_2 are Hausdorff maximal limit continua in K and the sets M_1 and M_2 are not comparable.

Let $A \in C(X)$; we prove that $C(X)$ is connected im kleinen at A if and only if at least one of following two statements holds: (i) A contains a point x such that X is connected im kleinen at x or (ii) $A - ce \neq \emptyset$ and $A - ed \neq \emptyset$.

Suppose that $C(X)$ is connected im kleinen at A and that A does not contain a point x such that X is connected im kleinen at x. There exist $p, q \in (-1, 1)$ such that $p \leq q$ and $A = [p, q] \times \{0\}$. Suppose $q \leq 0$. We contradict Lemma 4.8 as follows: Let $U = X - \{c, d\}$; notice that the component of U that contains A is $cd - \{c, d\}$. For each $n \in \mathbb{N}$, let $p_n = (p, (1+p)\frac{1}{n})$ and let $q_n = (q, (1+q)\frac{1}{n})$ be points in ca_n. Notice that p_nq_n and pq are contained in distinct components of U and that $\{p_nq_n\}_{n=1}^{\infty}$ converges in $C(X)$ to pq. This contradicts Lemma 4.8. Therefore, $q > 0$. In a similar way, we prove that $p < 0$. Hence, $A - ce \neq \emptyset$ and $A - ed \neq \emptyset$.

Now suppose that either (i) or (ii) holds. If (i) holds, by [17, Corollary 4], $C(X)$ is connected im kleinen at A. If (ii) holds and (i) does not hold, there exist $-1 < p < 0$ and $0 < q < 1$ such that $A = [p, q] \times \{0\}$. We use Lemma 4.8; let U be an open subset of X such that $A \subset U$; let V and W be the two distinct components of $X - \{e\}$; notice that V and W are open in X and $V \cap U \neq \emptyset \neq W \cap U$. Define $\mathcal{U} = \langle U, V \cap U, W \cap U \rangle \cap C(X)$; notice that $\mathcal{U} \subset \langle U \rangle \cap C(X)$ and, for each $A' \in \mathcal{U}$, $e \in A'$. Since $e \in A$, A and A' are contained in the same component of U. By Lemma 4.8, $C(X)$ is connected im kleinen at A.

If $A \subset B$ and A satisfies (i) or (ii), then B satisfies (i) or (ii). Hence, $C(X)$ is connected im kleinen at B if it is connected im kleinen at A.

5. Mappings

A surjective mapping $f : X \to Y$ between continua is said to be

- a *retraction*, provided that $Y \subset X$ and $f(y) = y$ for each $y \in Y$; in this case, we say that Y is a *retract of* X;
- *open*, provided that for each open subset of X, its image under f is an open subset of Y;
- *monotone*, provided that the point-inverse $f^{-1}(y)$ is connected for each point $y \in Y$;
- *confluent*, provided that for each subcontinuum Q of Y, each component of the inverse image $f^{-1}[Q]$ is mapped onto Q under f;
- *weakly confluent*, provided that for each subcontinuum Q of Y, there is a component of the inverse image $f^{-1}[Q]$ which is mapped onto Q under f;

- *semi-confluent*, provided that for each subcontinuum Q of Y and for each two components C_1 and C_2 of the inverse image $f^{-1}[Q]$, either $f[C_1] \subset f[C_2]$ or $f[C_2] \subset f[C_1]$.

In this section, we prove that the property of semi-Kelley is preserved under retractions (Theorem 5.2) and, moreover, semi-confluent images of continua with the property of Kelley have the property of semi-Kelley (Theorem 5.4).

We start with a lemma, generalizing [3, Lemma 5.1].

Lemma 5.1. *Let X and Y be continua such that Y is a retract of X, and let K be a subcontinuum of Y. If M is a Hausdorff maximal limit continuum in K when K is considered as a subcontinuum of Y, then M is a Hausdorff maximal limit continuum in K when K is considered as a subcontinuum of X.*

Proof. Suppose that M is a Hausdorff maximal limit continuum in K when K is considered as a subcontinuum of Y. Let $r : X \to Y$ be a retraction. Choose a subcontinuum L of X with $M \subsetneq L \subset K$. Since M is a Hausdorff maximal limit continuum in K when K is considered as a subcontinuum of Y, there exists an open subset $\mathcal{U}$ of $C(Y)$ such that $L \in \mathcal{U}$ and the set $\{B \in C(Y) : C(B, Y) \cap \mathcal{U} \neq \emptyset\}$ is not a neighborhood of M in $C(Y)$. Let $C(r) : C(X) \to C(Y)$ be the induced mapping and let $\mathcal{V} = C(r)^{-1}[\mathcal{U}]$; then $\mathcal{V}$ is an open subset of $C(X)$ such that $L \in \mathcal{U} \subset \mathcal{V}$. Observe that $\{B \in C(X) : C(B, X) \cap \mathcal{V} \neq \emptyset\} \cap C(Y) = \{B \in C(Y) : C(B, Y) \cap \mathcal{U} \neq \emptyset\}$. Hence, $\{B \in C(X) : C(B, X) \cap \mathcal{V} \neq \emptyset\}$ is not a neighborhood of M in $C(X)$. □

The following theorem follows from Lemma 5.1.

Theorem 5.2. *Let X and Y be continua and let $r : X \to Y \subset X$ be a retraction. If X has the property of semi-Kelley, then Y has the property of semi-Kelley.*

The following result generalizes [3, Lemma 5.3].

Lemma 5.3. *Let X and Y be continua such that X has the property of Kelley, let $f : X \to Y$ be a weakly confluent mapping, and let K be a subcontinuum of Y. If A is a Hausdorff strong maximal limit continuum in K, then there exists a component E of $f^{-1}[K]$ such that $f[E] = A$.*

Proof. Suppose that A is a Hausdorff strong maximal limit continuum in K. Let $\{A_d\}_{d \in D}$ be a net in $C(Y)$ converging to A such that

$$C(K) \cap \limsup\{C(A_d, Y)\}_{d \in D} = \{A\}.$$

Since f is weakly confluent for each $d \in D$, choose a component B_d of $f^{-1}[A_d]$ such that $f[B_d] = A_d$, so $B_d \in C(X)$. Since $\{B_d\} \subset C(X)$ for each $d \in D$, $\limsup\{\{B_d\}\}_{d\in D} \subset C(X)$. Let $B \in \limsup\{\{B_d\}\}_{d\in D}$.

Claim 1. $f[B] = A$.

Proof of Claim 1. Consider the induced mapping $C(f) : C(X) \to C(Y)$. Notice that $C(f)[\{B_d\}] = \{A_d\}$ for each $d \in D$. By Lemma 2.4, $C(f)[\limsup\{\{B_d\}\}_{d\in D}] = \limsup\{C(f)[\{B_d\}]\}_{d\in D} = \limsup\{\{A_d\}\}_{d\in D}$ $= \{A\}$. Hence, $C(f)(B) \in \{A\}$. This finishes the proof of Claim 1.

Let E be the component of $f^{-1}[K]$ such that $B \subset E$. It follows that $A = f[B] \subset f[E] \subset K$.

Claim 2. $f[E] \subset A$.

Proof of Claim 2. Suppose that $A \subsetneq f[E] \subset K$; since A is a Hausdorff strong maximal limit continuum in K, we have that $f[E]$ is not an element of $\limsup\{C(A_d, Y)\}_{d\in D}$. Hence, there exists an open subset $\mathcal{U}$ of $C(Y)$ such that $f[E] \in \mathcal{U}$, and there exists an element $d_0 \in D$ such that $\mathcal{U} \cap C(A_d, Y) = \emptyset$ for each $d \in D$ with $d_0 \leq d$. Without loss of generality, we may choose $U_1, \ldots, U_n$ open subsets of Y and $\mathcal{U} = \langle U_1, \ldots, U_n\rangle \cap C(Y)$ for some $n \in \mathbb{N}$. Since $A \subset f(E) \in \mathcal{U}$, renumbering the sets $U_1, \ldots, U_n$, if necessary, we may suppose that $\{1, \ldots, m\} = \{i \in \{1, \ldots, n\} : U_i \cap A \neq \emptyset\}$ for some $m \in \mathbb{N}$. Since $\langle U_1, \ldots, U_m\rangle \cap C(Y)$ is an open subset of $C(X)$ that contains A, choose $d_1 \in D$ such that $A_d \in \langle U_1, \ldots, U_m\rangle \cap C(Y)$ for each $d \in D$ with $d_1 \leq d$.

Observe that $C(f)^{-1}[\mathcal{U}]$ is an open subset of $C(X)$ and $E \in C(f)^{-1}[\mathcal{U}]$. Let $U = \bigcup C(f)^{-1}[\mathcal{U}]$. Since X has the property of Kelley, by Theorem 3.5, U is an open subset of X. Notice that $B \subset E \subset U$, so $B \in \langle U\rangle$.

Since $d_0 \in D$, $\langle U\rangle \cap C(X)$ is an open subset of $C(X)$ containing B and $B \in \limsup\{\{B_d\}\}_{d\in D}$, there exists an element $d_2 \in D$ with $d_0 \leq d_2$, $d_1 \leq d_2$, and $\langle U\rangle \cap C(X) \cap \{B_{d_2}\} \neq \emptyset$. Therefore, $B_{d_2} \in \langle U\rangle \cap C(X)$ and $B_{d_2} \subset U$. For $d_2 \in D$, take a point $b_{d_2} \in B_{d_2} \subset U$; then there exists an element $V_{d_2} \in C(f)^{-1}[\mathcal{U}]$ such that $b_{d_2} \in V_{d_2}$. Notice that $C(f)[V_{d_2}] \in \mathcal{U}$. Now, $f[V_{d_2} \cup B_{d_2}] = f[V_{d_2}] \cup f[B_{d_2}] = f[V_{d_2}] \cup A_{d_2} \in C(A_{d_2}, Y)$, which implies $f[V_{d_2}] \cup A_{d_2} \notin \mathcal{U}$ as $d_0 \leq d_2$. Since $f[V_{d_2}] \in \langle U_1, \ldots, U_n\rangle \cap C(Y)$ and $A_{d_2} \in \langle U_1, \ldots, U_m\rangle \cap C(Y)$ by $d_1 \leq d_2$, it follows that $f[V_{d_2}] \cup A_{d_2} \in \langle U_1, \ldots, U_n\rangle \cap C(Y)$, a contradiction. This proves Claim 2.

By claims 1 and 2, E is a component of $f^{-1}[K]$ such that $f[E] = A$. $\square$

As a corollary of Lemma 5.3, we obtain the following theorem, whose proof is the same as the one given in [3, Theorem 5.5] for metric continua.

Theorem 5.4. *Let X and Y be continua. If $f : X \to Y$ is a semi-confluent mapping and X has the property of Kelley, then Y has the property of semi-Kelley.*

The property of semi-Kelley is not preserved by confluent mappings, even in the metric case ([3, Example 5.8]). On the other hand, it was shown very recently that the property of semi-Kelley is preserved under open mappings and under monotone mappings of metric continua [8, Theorem 7 and Theorem 8]. In connection with these results, the following question is interesting and natural.

Question 5.5. Is the property of semi-Kelley preserved under (a) monotone, (b) open mappings?

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(Chacón-Tirado and de J. López) Facultad de Ciencias Físico Matemáticas; Benemérita Universidad Autónoma de Puebla; Avenida San Claudio y 18 Sur; Colonia San Manuel. Edificio FM1-101B; Ciudad Universitaria, C.P. 72570, Puebla, Mexico

Email address, Chacón-Tirado: `mauricio.chacon@correo.buap.mx`

Email address, de J. López: `mjlopez@fcfm.buap.mx`