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Finitely generated Whitney mappings ☆

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ABSTRACT

For a metric continuum X, we consider the hyperspace of subcontinua $C(X)$ of X, with the Hausdorff metric. A Whitney mapping is a continuous function $\mu : C(X) \to [0, \infty)$ such that: (a) for each $p \in X$, $\mu(p) = 0$, and (b) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$. The Whitney mapping μ is finitely generated if there exist a finite number of continuous functions $f_1, \ldots, f_n : X \to [0, 1]$ such that for each $A \in C(X)$, $\mu(A) = \mathit{length}(f_1(A)) + \cdots + \mathit{length}(f_n(A))$. In this paper we study the continua X for which there exist finitely generated Whitney mappings. In particular, when X is a tree, we find relations among the number of necessary mappings to generate a Whitney mapping with: the number of necessary arcs for covering X; the number of end-points of X; the disconnection number of X; the dimension of $C(X)$ and the number n for which X is an n-od.

1. Introduction

A *continuum* is a nonempty non-degenerate compact connected metric space. A *subcontinuum* of a continuum X is a nonempty closed connected subspace of X, so one-point sets are subcontinua. A *mapping* is a continuous function. We consider the *hyperspace of subcontinua* $C(X)$ of X with the Hausdorff metric [3, Definition 2.1].

A Whitney mapping is a continuous function $\mu : C(X) \to [0, \infty)$ such that:
(a) for each $p \in X$, $\mu(p) = 0$ and,
(b) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

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The Whitney mapping μ *is generated by* the n *mappings* $f_1, \ldots, f_n : X \to [0,1]$ such that for every $A \in C(X)$

$$\mu(A) = length(f_1(A)) + \cdots + length(f_n(A)).$$

Define the *Whitney generated degree, $wgd(X)$, of X* by

$$wgd(X) = min(\{n \in \mathbb{N} : \text{there exists a Whitney mapping } \mu : C(X) \to [0, \infty) \text{ that can be generated by } n \text{ mappings}\} \cup \{\infty\}).$$

Note that for each subcontinuum A of the continuum X, $wgd(A) \leq wgd(X)$.

Whitney mappings where defined by H. Whitney in [7, p. 275]. Since then Whitney mappings have been a very useful tool for the study of the structure of hyperspaces. There are only a few explicit formulas for them (see [5, 0.50.1, 0.50.2, 0.50.3] and [1]) and they are defined not only for the hyperspace $C(X)$ but for the hyperspace 2^X of all nonempty closed subsets of X. There are some few continua X for which is easy to define Whitney mappings. For example, for finite graphs (continua which are a finite union of arcs, called edges, such that each pair of them intersect only in a subset of their end-points), it is possible to define a Whitney mapping for $C(X)$ by taking, for each $A \in C(X)$ the sum of the lengths of the intersection of A with the edges of X. Another simple example is the continuum Y defined as the closure of the graph of the mapping $sin(\frac{1}{x})$, taking $x \in (0,1]$. Then the mapping $\mu : C(Y) \to [0,3]$ given by $\mu(A) = length(\pi_1(A)) + length(\pi_2(A))$ is a Whitney mapping (π_1 and π_2 are the natural projections from the plane onto the real line). This paper is motivated by the question: for which continua X it is possible to define a simple Whitney mapping for $C(X)$? (see [2, Problem 2]).

We study this question, we find some partial general results, but our most important results are in trees (finite graphs without simple closed curves). For a tree X, we are able to relate $wgd(X)$ with some other interesting properties of X, namely (see Definition 2.1): the number of necessary arcs for covering X; the number of end-points of X; the disconnection number of X; the dimension of $C(X)$ and the number n for which X is an n-od, see Corollary 3.7.

2. General results

Definition 2.1. Given an integer $n \geq 2$, a continuum X is an *n-od* provided that there exists a subcontinuum A of X such that $X \setminus A$ has at least n components. Define

$$O(X) = sup\{n \geq 2 : X \text{ is an } n\text{-od}\}.$$

The continuum X is *irreducible* with respect to its subset S, provided that no proper subcontinuum of X contains S.

A cardinal number $n \leq \aleph_0$ is called a *disconnection number* for the continuum X provided that whenever $A \subset X$ is such that A has exactly n points, we have that $X \setminus A$ is not connected. We write $D(X) \leq \aleph_0$ to mean there is a disconnection number for X. When $D(X) \leq \aleph_0$, we let $D^s(X)$ denote the smallest disconnection number for X.

A point p in a tree X is an *end-point of X* if $X \setminus \{p\}$ is connected, and p is a *ramification point of X* if $X \setminus \{p\}$ has at least three components. The set of end-points of X is denoted by $E(X)$ and the cardinality of $E(X)$ is denoted by $e(X)$. The set of ramification points of X is denoted by $R(X)$.

Theorem 2.2. *Let X be a tree and $n \geq 2$, then the following are equivalent.*
(a) $O(X) = n$,

(b) $D^s(X) = n + 1$,
(c) $e(X) = n$,
(d) $dim[C(X)] = n$.

Proof. By Corollary 3.7 of [4], $O(X) = D^s(X) - 1$. By Corollary 3.8 of [4], $O(X) = e(X)$. By Theorem 4.1 of [4], $dim[C(X)] = \omega(X)$ (the symbol $\omega(X)$ was defined in p. 537 of [4]), and by Theorem 4.3 of [4], $\omega(X) = O(X)$. Thus, $O(X) = dim[C(X)]$. □

Proposition 2.3. *Let X be a continuum such that $n = wgd(X)$ is a positive integer. Then X does not contain $(2n + 1)$-ods and there exists a subset S of X such that S contains at most $2n$ points and X is irreducible with respect to S.*

Proof. Let $\mu : C(X) \to [0, \infty)$ be a Whitney mapping generated by the mappings $f_1, \dots, f_n : X \to [0, 1]$. For each $i \in \{1, \dots, n\}$, let $p_i, q_i \in X$ be such that $f_i(p_i)$ and $f_i(q_i)$ are the respective minimum and maximum of the set $f_i(X)$. Let $S = \{p_1, \dots, p_n, q_1, \dots, q_n\}$. Let A be a subcontinuum of X such that $S \subset A$. Then for each $i \in \{1, \dots, n\}$, $f_i(A) = [f_i(p_i), f_i(q_i)] = f_i(X)$. This implies that $\mu(A) = \mu(X)$. Since μ is a Whitney mapping, $A = X$. Therefore X is irreducible with respect to S.

In order to show that X does not contain $(2n + 1)$-ods, first we show that X is not an $(2n + 1)$-od. Suppose to the contrary that there exists a subcontinuum A of X and nonempty pairwise separated subsets $C_1, \dots, C_{2n+1}$ of X such that $X \setminus A = C_1 \cup \dots \cup C_{2n+1}$. Since S contains at most $2n$ points, there exists $i \in \{1, \dots, 2n+1\}$ such that $S \cap C_i = \emptyset$. Then S is contained in the proper subcontinuum $A \cup (\bigcup \{C_j : j \neq i\})$ of X, which contradicts the irreducibility of X with respect to S.

Now we show that X does not contain $(2n + 1)$-ods. Let B be a subcontinuum of X. Let $m = wgd(B)$. Then $m \leq n$, so B is not a $(2m + 1)$-od. This implies that B is not an $(2n + 1)$-od. □

Theorem 2.4. *Let Z be a compactification of the ray $[0, 1)$ with remainder X such that $wgd(X)$ is finite. Suppose that $wgd(Z) = n$. Let $\nu : C(Z) \to [0, \infty)$ be a Whitney mapping generated by the mappings $f_1, \dots, f_n : Z \to [0, 1]$. Then*

$$X \subset \bigcup \{f_i^{-1}(\{min(f_i(X)), max(f_i(X))\}) : i \in \{1, \dots, n\}\}.$$

Proof. Suppose to the contrary that there exists a point w in the set $W = X \setminus (\bigcup \{f_i^{-1}(\{min(f_i(X)), max(f_i(X))\}) : i \in \{1, \dots, n\}\})$. Let d be a metric for Z. Given $i \in \{1, \dots, n\}$, let $x_i, y_i \in X$ be such that $f_i(x_i) = min(f_i(X))$ and $f_i(y_i) = max(f_i(X))$. Then $f_i(x_i) < f_i(w) < f_i(y_i)$. Thus there exists $\epsilon > 0$ such that for each $i \in \{1, \dots, n\}$, $3\epsilon < min\{f_i(y_i) - f_i(w), f_i(w) - f_i(x_i)\}$. Let $\delta > 0$ be such that if $p, q \in Z$ and $d(p, q) < 2\delta$, then for each $i \in \{1, \dots, n\}$, $|f_i(p) - f_i(q)| < \epsilon$.

By the density of $[0, 1)$ in Z, for each $i \in \{1, \dots, n\}$ we can fix points $p_i, q_i \in [0, 1)$ such that $d(p_i, x_i) < \delta$ and $d(q_i, y_i) < \delta$. Let

$$t = max\{p_1, \dots, p_n, q_1, \dots, q_n\}.$$

Since $t < 1$, there exists $w_0 \in [t, 1)$ such that $d(w, w_0) < \delta$. Take $w_1 \in (w_0, 1)$ such that $diameter([w_0, w_1]) < \delta$ (this diameter is taken in Z). Let $A = [0, w_0]$ and $B = [0, w_1]$. Then A and B are subcontinua of Z such that $A \subsetneq B$.

We claim that $\mu(A) = \mu(B)$. In order to prove this, let $i \in \{1, \dots, n\}$. The facts that $d(p_i, x_i) < \delta$ and $d(q_i, y_i) < \delta$ imply that $|f_i(p_i) - f_i(x_i)| < \epsilon$ and $|f_i(q_i) - f_i(y_i)| < \epsilon$. Given $p \in [w_0, w_1]$, $d(p, w_0) < \delta$, so $d(p, w) < 2\delta$ and $|f_i(p) - f_i(w)| < \epsilon$. Since $f_i(x_i) < f_i(w) - 3\epsilon < f_i(w) + 3\epsilon < f_i(y_i)$, we have that $f_i(p_i) < f_i(p) < f_i(q_i)$. Since $f_i(p_i), f_i(q_i) \in f_i(A)$, we conclude that $f_i(p) \in f_i(A)$. We have shown that $f_i([w_0, w_1]) \subset f_i(A)$. This implies that $f_i(B) \subset f_i(A)$. Thus $f_i(A) = f_i(B)$. Since this equality holds for each $i \in \{1, \dots, n\}$, we conclude that $\mu(A) = \mu(B)$. This contradicts the fact that μ is a Whitney mapping and completes the proof of the theorem. □

Theorem 2.5. *Let Z be a continuum and let $h : Z \to [0,1]$ be a mapping satisfying:*
(a) for each $t \in [0,1]$, $h^{-1}(t)$ is connected,
(b) there exists a finite set $F \subset [0,1]$ such that $h^{-1}(t)$ is non-degenerate if and only if $t \in F$,
(c) for every subcontinuum A of Z and every $t \in [0,1]$, either $A \subset h^{-1}(t)$ or $h^{-1}(t) \subset A$ or $A \cap h^{-1}(t) = \emptyset$.
Then $wgd(Z) \leq max\{wgd(h^{-1}(t)) : t \in F\} + 1$.

Proof. We may assume that $wgd(h^{-1}(t))$ is finite for each $t \in F$. Set $r = max\{wgd(h^{-1}(t)) : t \in F\}$. For each $t \in F$, let $B_t = h^{-1}(t)$ and let $f_1^{(t)}, \dots, f_r^{(t)} : B_t \to [0,1]$ be mappings that generate a Whitney mapping $\mu_t : C(B_t) \to [0, \infty)$ (if $wgd(B_t) < r$, we can repeat some mappings $f_j^{(t)}$).

For each $j \in \{1, \dots, r\}$, consider the mapping $g_j : h^{-1}(F) \to [0,1]$ given by $g_j(p) = f_j^{(t)}(p)$, if $p \in h^{-1}(t)$ (where $t \in F$). Since g_j is defined in the union of a finite family of pairwise disjoint closed subsets and it is continuous on each of these sets, we have that g_j is continuous. By Tietze's Extension Theorem, there exists a continuous extension G_j of g_j defined on the continuum Z.

Given a subcontinuum A of Z, define $\mu : C(Z) \to [0, \infty)$ by

$$\mu(A) = length(h(A)) + length(G_1(A)) + \cdots + length(G_r(A)).$$

In order to check that μ is a Whitney mapping, take subcontinua A and B of Z such that $A \subsetneq B$. First, we consider the case that $B \subset h^{-1}(t)$ for some $t \in F$. Given $b \in B$ and $j \in \{1, \dots, r\}$, $G_j(b) = g_j(b) = f_j^{(t)}(b)$. Then $\mu(B) = length(h(B)) + length(G_1(B)) + \cdots + length(G_r(B)) = length(h(B)) + length(f_1^{(t)}(B)) + \cdots + length(f_r^{(t)}(B)) = length(h(B)) + \mu_t(B) > length(h(A)) + \mu_t(A) = \mu(A)$. Thus $\mu(B) > \mu(A)$. Now, we consider the case that B is not contained in any set of the form $h^{-1}(t)$ ($t \in F$). Then $h(B)$ is non-degenerate and by (c), $B = h^{-1}(h(B))$. If $h(A)$ is non-degenerate, then $A = h^{-1}(h(A))$. Hence $h^{-1}(h(A)) \subsetneq h^{-1}(h(B))$. This implies that $h(A) \subsetneq h(B)$ and $length(h(A)) < length(h(B))$. Thus $\mu(A) < \mu(B)$. Therefore, μ is a Whitney mapping.

This completes the proof that $wgd(Z) \leq r + 1$. □

Corollary 2.6. *Let Z be a compactification of the ray $[0, \infty)$ with remainder X. Then $wgd(X) \leq wgd(Z) \leq wgd(X) + 1$.*

Corollary 2.7. *Let Z be a compactification of the real line $\mathbb{R}$ with disconnected remainder $X_1 \cup X_2$, where X_1 and X_2 are continua. Then*

$$max\{wgd(X_1), wgd(X_2)\} \leq wgd(Z) \leq max\{wgd(X_1), wgd(X_2)\} + 1.$$

Problem 2.8. Are there a continuum X and a compactification Z of the ray $[0, 1)$ with remainder X such that $wgd(X) = wgd(Z)$?

With respect to Problem 2.8, in the next section (Theorem 3.9), we prove that in the case that X is a tree, we have that $wgd(Z) = wgd(X) + 1$.

3. Trees

Given points p and q in a tree X, let pq denote the unique arc joining p and q in X, if $p \neq q$, and $pq = \{p\}$, if $p = q$.

Theorem 3.1. *Let X be an arcwise connected continuum such that $wgd(X)$ is finite. Then X is a tree.*

Proof. By Proposition 2.3, X is irreducible with respect to a finite set S. Since X is arcwise connected, it is possible to construct a tree T in X containing S. Thus $X = T$ is a tree. □

Corollary 3.2. *Let X be a continuum such that $wgd(X)$ is finite. If A is an arcwise connected subcontinuum of X, then A is a tree. In particular, X does not contain simple closed curves.*

Theorem 3.3. *Let X be a tree such that $e(X) \leq 2n$. Then X is the union of n arcs.*

Proof. We prove this theorem by induction. If $e(X) \leq 2$, then X is an arc and we are done.

Suppose that the assertion holds for n. Suppose that X is a tree and $3 \leq e(X) \leq 2(n+1)$. Then $2 \leq n+1$. Fix two distinct end-points e_1 and e_2 of X. Suppose that a_1 and a_2 are ramification points of X such that a_1e_1 and a_2e_2 are edges of X. Consider the tree $T = X \setminus ((a_1e_1 \setminus \{a_1\}) \cup (a_2e_2 \setminus \{a_2\}))$.

Claim 1. *If $e \in E(T)$, then either $e \in E(X) \setminus \{e_1, e_2\}$ or $a_1 = e = a_2$.*

We prove this claim. Suppose to the contrary that either $e \notin E(X) \setminus \{e_1, e_2\}$ and $a_1 \neq a_2$ or $a_1 = a_2 \neq e$. Since $e_1, e_2 \notin T$, we have that $e \notin E(X)$. If $e \notin R(X)$, then there exists an edge $L = b_1b_2$ in X such that $e \in L \setminus \{b_1, b_2\}$. Since $L \notin \{a_1e_1, a_2e_2\}$, we have that $L \subset T$ and $e \notin E(T)$, a contradiction. Thus $e \in R(X)$. If $e \notin \{a_1, a_2\}$, then $e \in R(T)$ which is also a contradiction. Then we may assume that $e = a_1$. This implies that $a_1 \neq a_2$. Since at least three edges of X contain a_1, and only one of them is removed to obtain T, we have that $e = a_1 \notin E(T)$, this contradiction completes the proof of the claim.

We consider two cases.

Case 1. $E(T) \subset E(X) \setminus \{e_1, e_2\}$.

In this case, $E(T)$ has at most $2n$ elements. By the induction hypothesis, there exist n subarcs $\alpha_1, \ldots, \alpha_n$ of T such that $T = \alpha_1 \cup \ldots \cup \alpha_n$. Let α_{n+1} be the unique arc in X joining e_1 to e_2. Then $a_1e_1 \cup a_2e_2 \subset \alpha_{n+1}$. Thus $X = \alpha_1 \cup \ldots \cup \alpha_{n+1}$. This completes this case.

Case 2. There exists an end-point e of T such that $e \notin E(X) \setminus \{e_1, e_2\}$.

Since $e \in T$, $e \notin \{e_1, e_2\}$, so $e \notin E(X)$. By Claim 1, $e = a_1 = a_2$. Let a_3 be a vertex of T such that ea_3 is an edge of T. Let $S = T \setminus (ea_3 \setminus \{a_3\})$. Then S is either a one-point set or S is a tree. In the case that S is a one-point set, we have that X is a simple triod. Then X can be covered by 2 arcs and $2 \leq n + 1$ and we are done. So we suppose that S is a tree. In this case, a_3 is not an end-point of S. This implies that $E(T) = E(S) \cup \{e\}$. By Claim 1, $E(S) \subset E(X) \setminus \{e_1, e_2\}$. By the induction hypothesis, there exist n subarcs $\alpha_1, \ldots, \alpha_n$ of S such that $S = \alpha_1 \cup \ldots \cup \alpha_n$.

We may assume that $a_3 \in \alpha_n$. If a_3 is an end-point of α_n, we define the arcs $\beta_n = \alpha_n \cup a_3a_1 \cup a_1e_1$ and $\beta_{n+1} = a_2e_2$. Clearly, $X = \alpha_1 \cup \ldots \cup \alpha_{n-1} \cup \beta_n \cup \beta_{n+1}$.

If a_3 is not an end-point of α_n, then there exist two subarcs γ_1, γ_2 of α_n such that $\alpha_n = \gamma_1 \cup \gamma_2$ and $\gamma_1 \cap \gamma_2 = \{a_3\}$. In this case, we define $\beta_n = \gamma_1 \cup a_3a_1 \cup a_1e_1$ and $\beta_{n+1} = \gamma_2 \cup a_3a_2 \cup a_2e_2$. Clearly, $X = \alpha_1 \cup \ldots \cup \alpha_{n-1} \cup \beta_n \cup \beta_{n+1}$.

This completes the induction and the proof of the theorem. □

Given a tree X and an arc $\alpha \subset X$, let $r_\alpha : X \to \alpha$ be the first point retraction, that is, for each $p \in X$, $r_\alpha(p)$ is the unique point in α such that the intersection of α and the arc joining p to $r_\alpha(p)$ contains only the point $r_\alpha(p)$. Note that if $q \in \alpha$ is such that $(r_\alpha)^{-1}(q)$ is non-degenerate, then q is either a ramification point of X or an end-point of α.

Theorem 3.4. *Let X be a tree. Suppose that X is a union of at most n arcs. Then $wgd(X) \leq n$.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be arcs in X whose union is X. For each $i \in \{1, \ldots, n\}$ let $f_i : \alpha_i \to [0,1]$ be a homeomorphism, and let $g_i : X \to [0,1]$ be given by $g_i = f_i \circ r_{\alpha_i} : X \to [0,1]$. Define $\mu : C(X) \to [0, \infty)$ by

$$\mu(A) = length(g_1(A)) + \cdots + length(g_n(A)).$$

We check that μ is a Whitney mapping. Let $A, B \in C(X)$ be such that $A \subsetneq B$. Since $B \setminus A$ is infinite, there exists a point $b \in B \setminus A$ such that b is not a ramification point of X and b is not an end-point of any α_i. Take $j \in \{1, \ldots, n\}$ be such that $b \in \alpha_j$. By the choice of b, $\{b\} = (r_{\alpha_j})^{-1}(b)$. This implies that $g_j(b) \notin g_j(A)$. Since $g_j(A)$ and $g_j(B)$ are subintervals of $[0,1]$ we conclude that $length(g_j(A)) < length(g_j(B))$. Thus $\mu(A) < \mu(B)$. Therefore μ is a Whitney mapping and $wgd(X) \leq n$. □

Lemma 3.5. *Let X be a tree. Let $\mu : C(X) \to [0, \infty)$ be a Whitney mapping generated by the mappings $g_1, \ldots, g_m$. For each $i \in \{1, \ldots, m\}$, let p_i, q_i be points of X such that $g_i(p_i) = min(g_i(X))$ and $g_i(q_i) = max(g_i(X))$. Then $E(X) \subset \{p_1, \ldots, p_m, q_1, \ldots, q_m\}$.*

Proof. Given $i \in \{1, \ldots, m\}$, if $p_i = q_i$, then g_i is a constant mapping, so if we omit g_i, μ is still a Whitney mapping. So we may assume that $p_i \neq q_i$. Let α_i be the unique arc in X joining p_i to q_i.

We define a sequence of subtrees $T_1, \ldots, T_m$ of X such that for each $i \in \{1, \ldots, m\}$,

(i) if $i < m$, then $T_i \subset T_{i+1}$,
(ii) the set of end-points of T_i is contained in $\{p_1, \ldots, p_i\} \cup \{q_1, \ldots, q_i\}$,
(iii) $\alpha_i \subset T_i$.

Let $T_1 = \alpha_1$. Suppose that $T_1, \ldots, T_k$ have been constructed and $k < m$.

If $T_k \cap \alpha_{k+1} \neq \emptyset$, then define $T_{k+1} = T_k \cup \alpha_{k+1}$. If $T_k \cap \alpha_{k+1} = \emptyset$, take the minimum subarc β in X joining a point in T_k to a point in α_{k+1}, and define $T_{k+1} = T_k \cup \beta \cup \alpha_{k+1}$.

Note that in both cases, each end-point of T_{k+1} is either an end-point of T_k or an end-point of α_{k+1}. Thus T_{k+1} satisfies (ii).

This completes the construction of the trees $T_1, \ldots, T_m$.

Since T_m contains all the arcs α_i, then T_m is a subcontinuum of X containing all the points p_i and q_i. This implies that for each $i \in \{1, \ldots, m\}$, $g_i(T_m) = g_i(X)$. Then $\mu(T_m) = \mu(X)$. Since μ is a Whitney mapping, we conclude that $T_m = X$. Therefore $E(X) \subset \{p_1, \ldots, p_m, q_1, \ldots, q_m\}$. □

Theorem 3.6. *Let X be a tree and $n \in \mathbb{N}$. Then the following are equivalent.*
(a) $wgd(X) \leq n$,
(b) X is the union of n arcs,
(c) $e(X) \leq 2n$.

Proof. By Theorem 3.4, (b) implies (a); by Theorem 3.3, (c) implies (b); and (a) implies (c) follows from Lemma 3.5. □

Corollary 3.7. *Let X be a tree and let $n \in \mathbb{N}$. Then the following are equivalent.*
(a) $wgd(X) = n$,
(b) $n = min\{m \in \mathbb{N} : X$ can be covered with m arcs$\}$,
(c) $e(X) \in \{2n-1, 2n\}$,
(d) $O(X) \in \{2n-1, 2n\}$,
(e) $dim[C(X)] \in \{2n-1, 2n\}$,
(f) $D^s(X) \in \{2n, 2n+1\}$.

Proof. In the case that $n \geq 2$, we have that $2n - 1 \geq 3$, so this result follows from Theorems 2.2 and 3.6. In the case $n = 1$, $2n - 1 = 1$ and $2n = 2$. Since there is not a tree X for which $e(X) = 1$ or $O(X) = 1$ or

$dim[C(X)] = 1$ or $D^s(X) = 2$, in this case, the conditions (c)-(f) become $2 = e(X) = O(X) = dim[C(X)]$ and $D^s(X) = 3$. While the conditions (a) and (b) become $wgd(X) = 1$ and the tree X can be covered by an arc. Thus by Theorems 2.2 and 3.6, the conditions are also equivalent for $n = 1$. Note that (topologically) the only continuum satisfying any of these conditions is the interval $[0, 1]$. □

Since, for each $n \in \mathbb{N}$, there exists a finite number of continua X such that $D^s(X) = n$ (see page 158 of [6]), by Corollary 3.7, we have the following corollary.

Corollary 3.8. *For each $n \in \mathbb{N}$, there exists a finite number of trees X such that $wgd(X) = n$.*

Theorem 3.9. *Let X be a tree and let Z be a compactification of the ray $[0, 1)$ with remainder X. Then $wgd(Z) = wgd(X) + 1$.*

Proof. By Theorem 2.4, we only need to show that $wgd(X) \neq wgd(Z)$. Suppose to the contrary that $k = wgd(X) = wgd(Z)$. Let $f_1, \ldots, f_k : Z \to [0, 1]$ be mappings that generate the Whitney mapping $\nu : C(Z) \to [0, \infty)$.

For each $i \in \{1, \ldots, k\}$, let $m_i = min(f_i(X))$ and $M_i = max(f_i(X))$. Since $wgd(X) = k$, $m_i < M_i$. Take points $p_i \in X \cap f_i^{-1}(m_i)$ and $q_i \in X \cap f_i^{-1}(M_i)$. Let α_i be the only arc in X joining p_i and q_i.

Set $D = \{p_1, \ldots, p_k, q_1, \ldots, q_k\}$. By Lemma 3.5, $E(X) \subset D$. Set $G = X \cap (f_1^{-1}(\{m_1, M_1\}) \cup \ldots \cup f_k^{-1}(\{m_k, M_k\}))$.

We prove that X is not a subset of G. By Corollary 3.7, $e(X) \in \{2k-1, 2k\}$. In the case that $e(X) = 2k$, since $E(X) \subset D$, we have that $E(X) = D$. In this case, we are going to prove that $G = D$. By definition $D \subset G$. Take $u \in G$, by symmetry, we may assume that $u \in f_1^{-1}(m_1)$. Then we apply Lemma 3.5 to the set $D' = \{u, p_2, \ldots, p_k, q_1, \ldots, q_k\}$, to obtain that $E(X) \subset D'$. Since $e(X) = 2k$, we have that $E(X) = D'$, so $D = D'$, we conclude that $u = p_1 \in D$. We have shown that $G \subset D$ and $G = D$. Since G is finite, we conclude that $X \not\subset G$.

Now, we consider the case that $e(X) = 2k - 1$. Fix a point $z \in X \setminus E(X)$. In the case that $z \notin G$, we are done. Suppose then that $z \in G$. By symmetry we may assume that $f_1(z) = m_1$. Let α'_1 be the unique arc in X joining z to q_1. We apply Lemma 3.5 to the set $D' = \{z, p_2, \ldots, p_k, q_1, \ldots, q_k\}$, to obtain that $E(X) \subset D'$. Since $E(X)$ contains $2k - 1$ elements and $z \notin E(X)$, we obtain that $E(X) = D' \setminus \{z\}$. By the connectedness of α'_1, we can choose a point $w \in \alpha'_1 \setminus f_1^{-1}(\{m_1, M_1\})$. Then $w \notin \{z, q_1\}$. Note that $w \notin E(X)$. We claim that $w \notin G$. Suppose to the contrary that $w \in G$. By the choice of w, we have that $f_1(w) \notin \{m_1, M_1\}$. Then, by symmetry, we may assume that $f_2(w) = m_2$. Then we may apply Lemma 3.5 to obtain that $E(X)$ is contained in the set $D'' = \{z, w, p_3, \ldots, p_k, q_1, \ldots, q_k\}$. This is impossible since $E(X)$ has $2k-1$ elements and D'' contains at most $2k-2$ end-points of X. We have shown that $w \notin G$. This ends the proof that $X \not\subset G$. This contradicts Theorem 2.4 and ends the proof that $wgd(X) \neq wgd(Z)$. Therefore $wgd(Z) = wgd(X) + 1$. □

4. More examples

The proof of the following theorem has similar ideas as those used in Theorem 2.4, this result is useful to determine that for some important continua the Whitney generated degree is infinite.

Theorem 4.1. *Let X be a continuum such that there exists a one-to-one mapping $h : [0, \infty) \to X$ satisfying that $h([1, \infty))$ is dense in X. Then $wgd(X)$ is infinite.*

Proof. Suppose the contrary. Then there exists a Whitney mapping $\mu : C(X) \to [0, \infty)$ generated by a finite family of mappings $f_1, \ldots, f_n : X \to [0, 1]$. For each $i \in \{1, \ldots, n\}$, let $m_i = min(f_i(X))$ and

$M_i = max(f_i(X))$. We assume that $m_i < M_i$ (in the case that $m_i = M_i$, we can omit f_i and μ is still a Whitney mapping). Fix points $x_i, y_i \in X$ such that $f_i(x_i) = m_i$ and $f_i(y_i) = M_i$. Set $p = h(1)$ and let d be a metric for X. Let $\epsilon = \frac{1}{2}min(\{r > 0 : r = f_i(p) - m_i$ for some $i \in \{1, \ldots, n\}\} \cup \{r > 0 : r = M_i - f_i(p)$ for some $i \in \{1, \ldots, n\}\}$. Note that $\epsilon > 0$. Let $\delta > 0$ be such that if $x, y \in X$ and $d(x, y) < 2\delta$, then for each $i \in \{1, \ldots, n\}$, $|f_i(x) - f_i(y)| < \epsilon$.

By the density of $h([1, \infty))$ in X, for each $i \in \{1, \ldots, n\}$ we can fix numbers $s_i, t_i \in [1, \infty)$ such that $d(h(s_i), x_i) < \delta$ and $d(h(t_i), y_i) < \delta$. Let $R = max(\{s_1, \ldots, s_n\} \cup \{t_1, \ldots, t_n\})$ and $A = h([1, R])$. Choose a number $R_0 \in [0, 1)$ such that $diameter(h([R_0, 1])) < \delta$ and set $B = h([R_0, R])$. Then A and B are subcontinua of X such that $A \subsetneq B$.

We claim that $\mu(A) = \mu(B)$. In order to do this, let $i \in \{1, \ldots, n\}$. Since $d(x_i, h(s_i)) < \delta$, we have that $|m_i - f_i(h(s_i))| = |f_i(x_i) - f_i(h(s_i))| < \epsilon$. Thus $min(f_i(A)) < m_i + \epsilon$. Similarly, $M_i - \epsilon < max(f_i(A))$.

Given $s \in [R_0, 1]$ and $i \in \{1, \ldots, n\}$, $d(h(s), p) < \delta$, so $|f_i(h(s)) - f_i(p)| < \epsilon$.

In the case that $m_i < f_i(p) < M_i$, by the definition of ϵ, we have that $min(f_i(A)) < m_i + \epsilon < f_i(p) - \epsilon < f_i(h(s)) < f_i(p) + \epsilon < M_i - \epsilon < max(f_i(A))$. Thus $f_i(h(s)) \in f_i(A)$.

In the case that $m_i = f_i(p)$, we have that $min(f_i(A)) = m_i \leq f_i(h(s)) < f_i(p) + \epsilon < M_i - \epsilon < max(f_i(A))$. Thus, $f_i(p) \in f_i(A)$. Similarly, in the case that $f_i(p) = M_i$, we obtain that $f_i(p) \in f_i(A)$.

We have shown that for each $i \in \{1, \ldots, n\}$, $f_i([R_0, 1]) \subset f_i(A)$, and then $f_i(B) \subset f_i(A)$. This implies that $\mu(A) = \mu(B)$ and contradicts the fact that μ is a Whitney mapping. □

Since the Buckethandle continuum [6, 2.9] and the solenoids [6, 2.9] satisfy the hypothesis of Theorem 4.1, we obtain the following consequence.

Corollary 4.2. *Let X be either the Buckethandle continuum or a solenoid. Then $wgd(X)$ is infinite.*

5. Problems

Problem 5.1. Let P be the pseudo-arc [6, 1.23], what is the value of $wgd(P)$?

Problem 5.2. Is the unit interval $[0, 1]$ the unique continuum X for which $wdg(X) = 1$?

Problem 5.3. Is the unit interval $[0, 1]$ the unique hereditarily decomposable irreducible continuum X for which $wdg(X) = 1$?

It is easy to see that, with the usual metric, the diameter mapping is a Whitney mapping from $C([0, 1])$ onto $[0, 1]$. So, the following problem is related to the topic of this paper (Problem 1 of [2]).

Problem 5.4. Is the arc the only continuum X for which there exists a metric such that the diameter mapping from $C(X)$ into $[0, \infty)$ is a Whitney mapping?

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