Topology and its Applications 283 (2020) 107335

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Non-weak cut, shore and non-cut points in Whitney levels

Idalia-Guadalupe Bautista-Callejas, Mauricio Chacón-Tirado, Raúl Escobedo *

Benemérita Universidad Autónoma de Puebla, Facultad de Ciencias Físico-Matemáticas, Av. San Claudio y 18 sur, Col. San Manuel, Edificio FM1, Ciudad Universitaria, 72570, Puebla, Puebla, Mexico

A R T I C L E I N F O

Article history:
Received 15 May 2020
Received in revised form 20 July 2020
Accepted 21 July 2020
Available online 24 July 2020

MSC:
primary 54B20
secondary 54F15

Keywords:
Continuum
Cut point
Hyperspace
Weak cut point
Shore point
Whitney property

A B S T R A C T

We show that the property of having only non-weak cut points is a Whitney property, and that having only shore points is not a Whitney property. We also note that the property of having only non-cut (non-weak cut, shore) points is not a Whitney reversible property. This complements a result due to E. Matsuhashi.

1. Introduction

A *continuum* is a nonempty compact, connected, metric space. For a continuum X, the space of all subcontinua of X with the Hausdorff metric is denoted by $C(X)$, and it is called the *hyperspace of subcontinua of* X, [16, p. 1]. A *Whitney map* for $C(X)$ is a continuous function from $C(X)$ into the real line such that, for each point x in X, $\mu(\{x\}) = 0$, and for every two subcontinua A and B of X with A a proper subset of B, it holds that $\mu(A) < \mu(B)$, [16, (0.50) p. 24]. A *Whitney level* for $C(X)$ is a set of the form $\mu^{-1}(t)$, for some Whitney map μ, where $0 < t < \mu(X)$ [11, p. 159]. A topological property P is called a *Whitney property* provided that if a continuum X has property P, so does every Whitney level for $C(X)$; property P is called a *Whitney reversible property* provided that whenever X is a continuum such that every Whitney level for $C(X)$ has property P, then X has property P; property P is called a *strong Whitney reversible*

* Corresponding author.
E-mail addresses: dali_445@hotmail.com (I.-G. Bautista-Callejas), mauricio.chacon@correo.buap.mx (M. Chacón-Tirado), escobedo@fcfm.buap.mx (R. Escobedo).

https://doi.org/10.1016/j.topol.2020.107335
0166-8641/

property provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property P for some Whitney map μ for $C(X)$ and all $0 < t < \mu(X)$, then X has property P; property P is called a *sequential strong Whitney reversible property* provided that whenever X is a continuum such that there is a Whitney map μ for $C(X)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ converging to 0 such that $\mu^{-1}(t_n)$ has property P for each n, then X has property P (see [11, Definition 27.1 p. 232]). Many authors have studied these properties, in Chapter VIII of [11] there is a complete discussion of what was known in 1999.

A point p of a continuum X is called a *non-cut point* of X provided that $X - \{p\}$ is connected; otherwise, the point p is called a *cut point* of X. The point p is called a *non-weak cut point* of X provided that every two points of $X - \{p\}$ belong to a continuum contained in $X - \{p\}$; otherwise, the point p is called a *weak cut point* of X. The point p is a *shore point* of X provided that for each positive number ε there exists a subcontinuum A of X contained in $X - \{p\}$ and the distance in the Hausdorff metric between A and X is less than ε, [14]; otherwise the point p is called a *non-shore point* of X. We say that X is a continuum *having only non-cut* (*non-weak cut*, *shore*) *points* provided that each point of X is a non-cut (non-weak cut, shore) point of X. It is easy to see that a non-weak cut point is a shore point, and a shore point is a non-cut point; and that in locally connected continua these concepts coincide, [8, Remark 3.4]. Although the notions of cut and weak cut point are not new, [1], [15], recently several authors have been interested in them in connection with the notion of shore point and problems in continua and hyperspaces, [2], [3], [4], [5], [7], [8], [10], [12], [13], [14], [18], [19].

It is known that the property of having cut points is not a Whitney property, [11, Exercise 43.4]. In [13], answering Question 43.3 of [11], it was proved that the property of having cut points is not a Whitney reversible property; it follows that the property of having only non-cut points is not a Whitney property. In this paper we prove that the property of having only non-weak cut points is a Whitney property, and that the property of having only shore points is not a Whitney property. We also note that the property of having only non-cut (non-weak cut, shore) points is not a Whitney reversible property. We do not know whether having non-shore points is a Whitney reversible property.

2. Non-weak cut, shore and non-cut points in Whitney levels

For the next pair of lemmas we recall that an *arc* is a space homeomorphic to the closed interval $[0, 1]$, and an *order arc* is an arc in a hyperspace such that any two of its elements are comparable by inclusion of sets, [11, Definition 14.1 p. 110].

In Lemma 2.2 below, we prove that elements of Whitney levels are non-weak cut points if they contain non-weak cut points of the base continuum. In the proof of this lemma we use several times the following result due to S. B. Nadler, Jr.

Lemma 2.1. *[16, (4.8.1) p. 405]. Let X be a continuum, let μ be a Whitney map for $C(X)$ and let $0 < t < \mu(X)$. If A and B are distinct elements of $\mu^{-1}(t)$ such that $A \cap B$ is a nonempty set, then for each component K of $A \cap B$ there exists an arc $\mathcal{L}$ in the level $\mu^{-1}(t)$ having end points A and B and such that $K \subset L \subset A \cup B$ for each element L of the arc $\mathcal{L}$.*

Lemma 2.2. *If p is a non-weak cut point of a continuum X, then for each Whitney map μ for $C(X)$, for each $0 < t < \mu(X)$ and for each element A of $\mu^{-1}(t)$ containing p it happens that A is a non-weak cut point of $\mu^{-1}(t)$.*

Proof. Suppose p is a non-weak cut point of a continuum X. Let μ, t and A be as in the lemma, and let B and C be distinct elements of $\mu^{-1}(t) - \{A\}$. We will prove that there is a continuum $\mathcal{L}$ contained in $\mu^{-1}(t) - \{A\}$ which contains B and C. We analyze two cases:

(1) The intersection $B \cap C$ is not a subset of A: In this case we take a point x in $(B \cap C) - A$. By Lemma 2.1, there exists an arc $\mathcal{L}$ in $\mu^{-1}(t)$ containing B and C and such that each of its elements contains the point x. Since the point x is not in A, we have that A is not in $\mathcal{L}$. Thus, $\mathcal{L}$ is contained in $\mu^{-1}(t) - \{A\}$.

(2) The intersection $B \cap C$ is a subset of A: In this case we take points b in $B - A$ and c in $C - A$, and we notice that b, c and p are pairwise distinct points. Since p is a non-weak cut point of X, there exists a continuum D contained in $X - \{p\}$ which contains b and c. Next we analyze two subcases:

(2.1) $\mu(D) < t$: In this subcase we notice that $B \cup D$ is a continuum containing B as a proper subset. Thus, $\mu(D) < t < \mu(B \cup D)$. Hence, taking an order arc in $C(B \cup D)$ from D to $B \cup D$ [11, Theorem 14.6 p. 112], by continuity of μ we can find a subcontinuum E of X such that $D \subset E \subset B \cup D$ and $\mu(E) = t$. Similarly, we can find a subcontinuum F of X such that $D \subset F \subset C \cup D$ and $\mu(F) = t$. Since the point b is in $B \cap E$ and B and E are in the level $\mu^{-1}(t)$, by Lemma 2.1, there exists an arc $\mathcal{M}$ in $\mu^{-1}(t)$ containing B and E and such that each of its elements contains the point b. Since point b is not in A, we have that $\mathcal{M}$ is an arc in $\mu^{-1}(t) - \{A\}$. Similarly, there exists an arc $\mathcal{N}$ in $\mu^{-1}(t)$ contains both C and F and such that each of its elements contains the point c. Since the point c is not in A, we have that $\mathcal{N}$ is an arc in $\mu^{-1}(t) - \{A\}$. Moreover, since D is a subcontinuum of $E \cap F$ and E and F are in the level $\mu^{-1}(t)$, there exist an arc $\mathcal{P}$ in $\mu^{-1}(t)$ contains both E and F and such that each of its elements contains D, Lemma 2.1. Since D is not contained in A, we notice that A is not in $\mathcal{P}$. Let $\mathcal{L} = \mathcal{M} \cup \mathcal{P} \cup \mathcal{N}$. We have that $\mathcal{L}$ is a continuum contained in $\mu^{-1}(t) - \{A\}$ which contains B and C.

(2.2) $t \leq \mu(D)$: In this subcase we denote $\mathcal{D} = C(D) \cap \mu^{-1}(t)$, and we observe that $\mathcal{D} = \mu|_{C(D)}^{-1}(t)$. Thus, since $\mu|_{C(D)}$ is a Whitney map for $C(D)$, we have that $\mathcal{D}$ is a continuum in $\mu^{-1}(t)$, [16, Theorem 14.2 p. 400]. Moreover, $\cup\mathcal{D} = D$, [16, (1.213.1) p. 205]. This last equality implies that there exist elements H and K of $\mathcal{D}$ containing the points b and c respectively. Since, the point p is in A and it is not in D, we notice that A is not in $\mathcal{D}$. Thus, $\mathcal{D}$ is a continuum in $\mu^{-1}(t) - \{A\}$. On the other hand, since b is a point in $B \cap H$ and B and H are in $\mu^{-1}(t)$, by Lemma 2.1, there is an arc $\mathcal{B}$ in $\mu^{-1}(t)$ containing both B and H, and such that each of its elements contains the point b. Since the point b is not in A, we have that A is not in $\mathcal{B}$. So, $\mathcal{B}$ is an arc in $\mu^{-1}(t) - \{A\}$. Similarly, there is an arc $\mathcal{C}$ in $\mu^{-1}(t) - \{A\}$ containing both C and K, and such that each of its elements contains the point c. Finally, in this subcase we denote $\mathcal{L} = \mathcal{B} \cup \mathcal{D} \cup \mathcal{C}$. We have that $\mathcal{L}$ is a continuum contained in $\mu^{-1}(t) - \{A\}$ which contains B and C. □

Theorem 2.3. *The property of having only non-weak cut points is a Whitney property.*

Proof. Suppose X is a continuum having only non-weak cut points. Let μ be a Whitney map for $C(X)$, let $0 < t < \mu(X)$, and let A be an element of the level $\mu^{-1}(t)$. Take a point p in A. We have that p is a non-weak cut point of X. By Lemma 2.2, we have that A is a non-weak cut point of $\mu^{-1}(t)$. Thus, $\mu^{-1}(t)$ is a continuum having only non-weak cut points. □

We recall that the negation of a Whitney property is a sequential strong Whitney reversible property, [11, p. 233, (c)]. Thus, we have the following corollary.

Corollary 2.4. *The property of having weak cut points is a sequential strong Whitney reversible property.*

Remark 2.5. From the corollary above it follows that the property of having weak cut points is a strong Whitney reversible property, and so, it is also a Whitney reversible property, [11, p. 233, (a) and (b)]. Compare with [13, Theorem 2.2] and Question 2.9 below.

The proof of next theorem is given by the Example 2.7 below.

Theorem 2.6. *The property of having only shore points is not a Whitney property.*

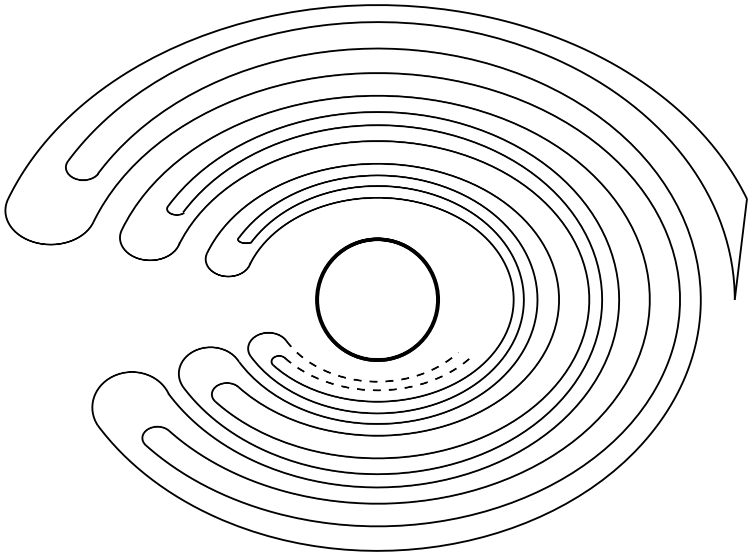

Fig. 1. A continuum having only shore points.

Example 2.7. We show a continuum having only shore points and such that some of its Whitney levels have non-shore points. Let Z be the closure in the Euclidean plane of the set $\{(x, \sin\frac{1}{x}) : x \in [-1, 1] - \{0\}\}$. Let Y be the continuum obtained from Z by identifying the points $(-1, \sin(-1))$ and $(1, \sin(1))$ to a point. The continuum Y is represented in (5) of Figure 20 in p. 63 of [11]. Now, let X be the continuum obtained from Y by identifying the points $(0, -1)$ and $(0, 1)$ to a point denoted by p. The continuum X is represented in Fig. 1. We notice that X is a compactification of the real line having a circle as its remainder, in such a way that sequences of return points in the spirals converge to point p in the circle. We observe that every point of X is a shore point, so X is a continuum having only shore points.

On the other hand, let S be the circle in X. If μ is a Whitney map for $C(X)$ and $0 < t < \mu(S)$, then the level $\mu^{-1}(t)$ is as the space illustrated in Fig. 2. Denote the circle of this level by $\mathcal{C}$. Notice that if A is an element of $\mathcal{C}$ such that A is the limit of a sequence of elements in $\mu^{-1}(t) - \mathcal{C}$, then A is a non-shore point of the level $\mu^{-1}(t)$.

Remark 2.8. Example 2.7 also proves that having non-shore points is neither a sequential strong Whitney reversible property nor a strong Whitney reversible property.

In [13, Theorem 2.2], the author shows a continuum Z having only non-cut points and such that every Whitney level has a cut point. Thus, each Whitney level of Z has a non-shore point. However, we notice that such continuum Z has many non-shore points. We do not know if there exists a continuum having only shore points and such that each of its Whitney levels has a non-shore point. So, we ask the following question.

Question 2.9. Is the property of having non-shore points a (strong) Whitney reversible property?

For next example we recall that a Hilbert cube is a space homeomorphic to a countable cartesian product of closed unit intervals, and a dendrite is a locally connected continuum which contains no simple closed curve [17, p. 4 and p. 165, respectively].

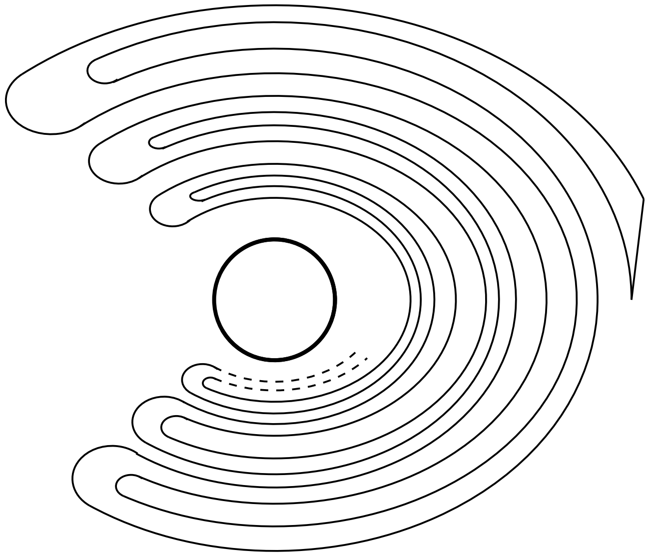

Fig. 2. A small Whitney level for the continuum in Fig. 1.

Example 2.10. Let D_3 be the standard universal dendrite of order 3, [6, p. 235]. By [9, Theorem (4.8) p. 691] every Whitney level in the hyperspace of D_3 is a Hilbert cube. Thus, each of these Whitney levels is a continuum having only non-cut points. On the other hand, we know that the dendrite D_3 has uncountably many cut points, [17, Theorem 10.8 p. 168]. By local connectedness, with this dendrite we observe the same fact for non-weak cut and shore points, [8, Remark 3.4]. So, this example proves the following result.

Theorem 2.11. *The properties of having only non-weak cut points, having only shore points and having only non-cut points are not Whitney reversible properties.*

Acknowledgements

The authors wish to thank Ana-Luisa Ramírez-Bautista for useful discussions on the topic of this paper. The authors also wish to thank the referee for their helpful suggestions.

References

[1] R.H. Bing, Some characterizations of arcs and simple closed curves, Am. J. Math. 70 (1948) 497–506.
[2] J. Bobok, R. Marciña, P. Pyrih, B. Vejnar, Union of shore sets in a dendroid, Topol. Appl. 161 (2014) 206–214.
[3] J. Bobok, P. Pyrih, B. Vejnar, Non-cut, shore and non-block points in continua, Glas. Mat. 51 (71) (2016) 237–253.
[4] J. Bobok, P. Pyrih, B. Vejnar, On blockers in continua, Topol. Appl. 202 (2016) 346–355.
[5] J. Camargo, D. Maya, L. Ortiz, The hyperspace of nonblockers of $\mathcal{F}_1(X)$, Topol. Appl. 251 (2019) 70–81.
[6] J.J. Charatonik, W.J. Charatonik, Dendrites, Soc. Mat. Mex., Aportaciones Mat., Ser. Comun. 22 (1998) 227–253.
[7] R. Escobedo, M.J. López, H. Villanueva, Nonblockers in hyperspaces, Topol. Appl. 159 (2012) 3614–3618.
[8] R. Escobedo, C. Estrada-Obregón, J. Sánchez-Martínez, On hyperspaces of non-cut sets of continua, Topol. Appl. 217 (2017) 97–106.
[9] J.T. Goodykoontz, S.B. Nadler Jr., Whitney levels in hyperspaces of certain Peano continua, Trans. Am. Math. Soc. (1982) 671–694.
[10] A. Illanes, Finite unions of shore sets, Rend. Circ. Mat. Palermo (2) 50 (2001) 483–498.
[11] A. Illanes, S.B. Nadler Jr., Hyperspaces, Fundamentals and Recent Advances, Monogr. Textbooks Pure App. Math., vol. 216, Marcel Dekker, Inc., New York, Basel, 1999.
[12] R. Leonel, Shore points of a continuum, Topol. Appl. 161 (2014) 433–441.
[13] E. Matsuhashi, Having cut-points is not a Whitney reversible property, Proc. Am. Math. Soc. 137 (2009) 3543–3545.
[14] L. Montejano-Peimbert, I. Puga-Espinosa, Shore points in dendroids and conical pointed hyperspaces, Topol. Appl. 46 (1992) 41–54.
[15] R.L. Moore, Concerning simple continuous curves, Trans. Am. Math. Soc. 21 (1920) 333–347.

[16] S.B. Nadler Jr., Hyperspaces of Sets, Monogr. Textbooks Pure Appl. Math., vol. 49, Marcel Dekker, Inc., New York, 1978.
[17] S.B. Nadler Jr., Continuum Theory. An Introduction, Monogr. Texbooks Pure Appl. Math., vol. 158, Marcel Dekker, Inc., New York, 1992.
[18] V. Neumann-Lara, I. Puga-Espinosa, Shore points and dendrites, Proc. Am. Math. Soc. 118 (3) (1993) 939–942.
[19] V.C. Nall, Centers and shore points of a dendroid, Topol. Appl. 154 (2007) 2167–2172.