The general boundary formulation of quantum theory

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Contents

The formulation of a quantum theory of gravity leads to technical as well as conceptual difficulties

Motivations from Quantum Gravity

The formulation of a quantum theory of gravity leads to technical as well as conceptual difficulties

Quantization problems

- Incompatibility between the foundational principles of GR and QT:
  - Standard quantization prescriptions require a fixed, non-dynamical background metric
  - GR: spacetime is a physical and dynamical system + diff invariance
- Failure of standard quantization technique for QFT: non-renormalizable field theory

$\Rightarrow$ Background independent quantum field theory

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    - GR: spacetime is a physical and dynamical system + diff invariance
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Interpretational problems

- The problem of time. (frozen picture, no time)
- What is an observable in QG? (locality/non locality issue)

Problems (I)

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  - In QFT evolution requires a fixed background metric independent of the fields.
  - Probability is conserved in time.
- In General Relativity
  - Dynamical time.
  - The metric is dynamically coupled with matter fields.
  - No background time at all.

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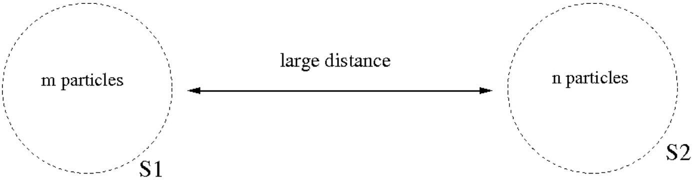

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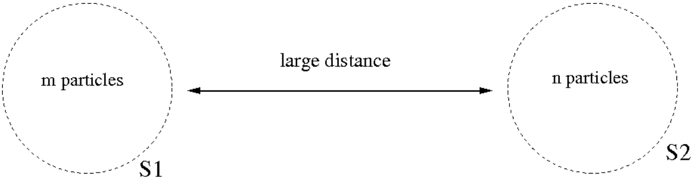

  We can separate the system from the rest of the universe.

- The notion of causal separation relies on the background metric. In a background independent context there is no way to isolate the system from the rest of the universe.

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Can we sufficiently extend the standard formulation of QT in order to render it compatible with the symmetries of GR?

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- description of physics in a manifestly local way

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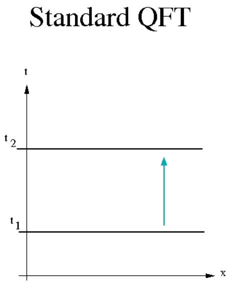
YES, using:

- The mathematical framework of topological quantum field theory.
- A generalization of the Born rule.

Standard Minkowski-based QFT

- States defined at instants of time, flat spacelike hypersurfaces
- Evolution in time implemented by a unitary operator
- Probability is conserved in time

This structure is dictated by canonical quantization prescriptions and the standard picture of dynamics.



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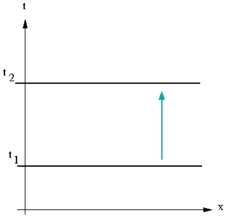
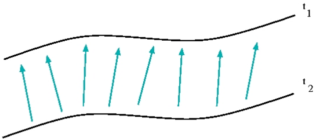


QFT on curved spacetime

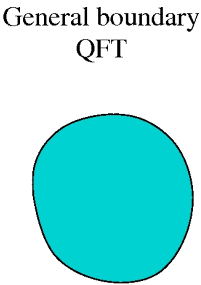
- States defined on spacelike Cauchy surfaces
- Evolution between Cauchy surfaces non-unitary in general

QFT in curved spacetime



General boundary formulation of QFT

- States and space states defined on general spacetime boundary hypersurfaces
- Evolution defined *inside* the region enclosed by the boundary
- Generalization of the notion of transition amplitude (for connected boundary there is no distinction between initial and final states)



There are aspects of the GBF that cannot be described within standard QT.

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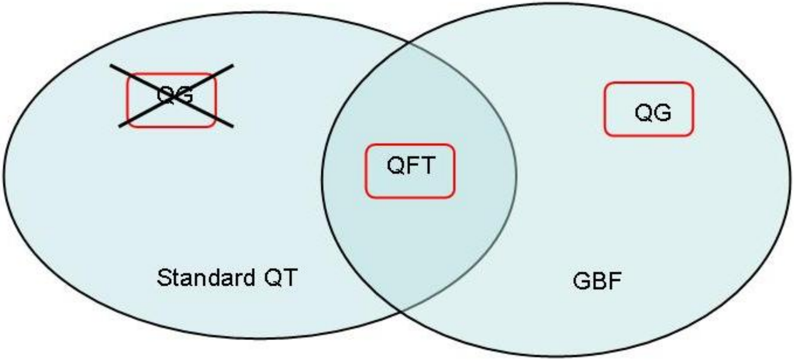
4. GBF and quantum gravity:
   - three dimensional quantum gravity is already formulated as a TQFT and fits "automatically" into the GBF;
   - the GBF is already used in spin foam approaches to quantum gravity (Rovelli's group in Marseille);
   - other approaches to quantum gravity can be adapted to the GBF (Group Field Theory).

Main statement:



Basic structures

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- To each oriented hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of states such that:
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  - **(Decomposition rule)** If $\Sigma = \Sigma_1 \cup \Sigma_2$ is the disjoint union of two disconnected hypersurfaces, then $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

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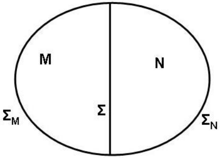
- To each spacetime region M with boundary ∂M associate a linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$, such that:
  - If $\partial M = \Sigma_1 \cup \Sigma_2$, the map ρ_M gives rise to an isomorphism of state spaces $\tilde{\rho}_M = \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$.
  - **(Gluing rule)** If M and N are adjacent regions, then $\rho_{M\cup N} = \rho_M \circ \rho_N$. The composition $\circ$ involves a sum over a complete basis on the boundary hypersurface Σ shared by M and N.

$$\tilde{\rho}_{M\cup N} : \mathcal{H}_{\Sigma_M} \to \mathcal{H}^*_{\Sigma_N}$$



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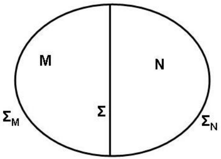


- Standard transition amplitudes of QFT can be recover from the GBF: $\rho_{[t_1,t_2]}(\psi_{t_1} \otimes \eta_{t_2}) = \langle \eta | U(t_1,t_2) | \psi \rangle$.

Generalized probability interpretation

In quantum theory, probabilities are generally conditional probabilities: probability to observe a specific state given that some other specific state was prepared. Then probability depends on two type of data: *preparation* and *observation*.

In the GBF, both type of data encoded through closed subspaces of the state space $\mathcal{H}_{\partial M}$:

- $\mathcal{S} \subset \mathcal{H}_{\partial M}$ representing *preparation*
- $\mathcal{A} \subset \mathcal{H}_{\partial M}$ representing *observation*

The probability that the system is described by $\mathcal{A}$ given that it is described by $\mathcal{S}$ is:

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_M \circ P_{\mathcal{S}} \circ P_{\mathcal{A}}|^2}{|\rho_M \circ P_{\mathcal{S}}|^2} \tag{1}$$

- $P_{\mathcal{S}}$ and $P_{\mathcal{A}}$ are the orthogonal projectors onto the subspaces.

Recovering of standard transition probability

The expression (1) of the generalized probability reduces to a standard transition probability for a standard transition amplitude.

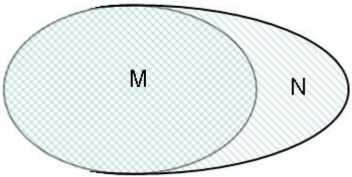
Spacetime region: $M = [t_1, t_2] \times \mathbb{R}^3$
Boundary: $\partial M = \Sigma_{t_1} \cup \overline{\Sigma}_{t_2}$
State space: $\mathcal{H}_{\partial M} = \mathcal{H}_{t_1} \otimes \mathcal{H}^*_{t_2}$

The preparation corresponds to the subspace $\mathcal{S} = \psi \otimes \mathcal{H}_{t_2} \subset \mathcal{H}_{\partial M}$.
The observation corresponds to the subspace $\mathcal{A} = \mathcal{H}_{t_1} \otimes \eta \subset \mathcal{H}_{\partial M}$.

Then formula (1) yields

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_M \circ P_{\mathcal{S}} \circ P_{\mathcal{A}}|^2}{|\rho_M \circ P_{\mathcal{S}}|^2} = \frac{|\rho_M(\psi \otimes \eta)|^2}{1} = |\langle \eta | U(t_1, t_2) | \psi \rangle|^2.$$

Probability conservation in time is generalized to probability conservation in spacetime. Consider a region M and an adjacent region N "deforming" M to $M' = M \cup N$:



- Suppose that the amplitude map $\rho_N : \mathcal{H}_{\partial N} \to \mathbb{C}$ associated with N induces a *unitary map* $\tilde{\rho} : \mathcal{H}_{\partial M} \to \mathcal{H}_{\partial M'}$.
- Let $\mathcal{S} \subset \mathcal{H}_{\partial M}$ and $\mathcal{A} \subset \mathcal{H}_{\partial M}$. Define the subspaces $\mathcal{S}' := \tilde{\rho}(\mathcal{S}) \subset \mathcal{H}_{\partial M'}$ and $\mathcal{A}' := \tilde{\rho}(\mathcal{A}) \subset \mathcal{H}_{\partial M'}$.
- Then, probability is conserved, $P(\mathcal{A}|\mathcal{S}) = P(\mathcal{A}'|\mathcal{S}')$, i.e. the probability for observing $\mathcal{A}$ given $\mathcal{S}$ on ∂M is the same as that for observing $\mathcal{A}'$ given $\mathcal{S}'$ on $\partial M'$.

Observables and expectation values

- As in AQFT, observables are associated to spacetime regions.
- An observable O in a region M is a linear map $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$, called observable map.

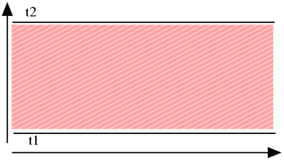
Observables can be composed in the same way as amplitudes.
Consider a region M and an adjacent region N:

- An observable map ρ_M^O in M gives rise in the region $M \cup N$ to an observable map $\rho_{M\cup N}^O : \mathcal{H}_{\partial M \cup N} \to \mathbb{C}$ given by $\rho_{M\cup N}^O = \rho_M^O \circ \rho_N$.
- An observable O in M can be composed with an observable P in N into a product observable $O \cdot P$ in $M \cup N$ represented by $\rho_{M\cup N}^{O\cdot P} = \rho_M^O \circ \rho_N^P$.

The expectation value of the observable O in a region M for the system being prepared in the subspace $\mathcal{S} \subset \mathcal{H}_{\partial M}$ is

$$\langle O \rangle_{\mathcal{S}} = \frac{\langle \rho_M \circ P_{\mathcal{S}}, \rho_M^O \circ P_{\mathcal{S}} \rangle}{|\rho_M \circ P_{\mathcal{S}}|^2},$$

(the inner product $\langle \cdot, \cdot \rangle$ is in the dual Hilbert space $\mathcal{H}_{\partial M}^*$)



- In the standard formulation of QT observables are associated to instants of time (equal-time hypersurfaces).
- To recover these we consider *empty regions*, i.e. the limiting case $t_2 = t_1 = t$.
- An observable O in this empty region is a linear map $\rho^O_{[t,t]} : \mathcal{H}_t \otimes \mathcal{H}_t^* \to \mathbb{C}$.
- A standard operator $\hat{O} : \mathcal{H}_t \to \mathcal{H}_t$ is recoverd as

$$\rho^O_{[t,t]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, \hat{O}\psi_1 \rangle \qquad \forall \psi_1, \psi_2 \in \mathcal{H}_t.$$

- Consider a state $\psi \in \mathcal{H}_t$ encoding a preparation. In the GBF we set

$$\mathcal{S} = \psi \otimes \mathcal{H}_t^*.$$

  We recover in this case the conventional expectation value of $\hat{O}$ with respect to ψ:

$$\langle O \rangle_{\mathcal{S}} = \frac{\langle \rho^{\mathcal{S}}_{[t,t]}, \rho^O_{[t,t]} \rangle}{|\rho^{\mathcal{S}}_{[t,t]}|^2} = \frac{\rho^O_{[t,t]}(\psi \otimes \psi^*)}{1} = \langle \psi, \hat{O}\psi \rangle.$$

GBF and QFT

Conjecture

Standard QFT can be formulated within the GBF

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2 quantization schemes have been studied, that transform a classical field theory into a general boundary quantum field theory:

- Schrödinger-Feynman quantization (non-rigorous but can treat interacting QFT perturbatively);
- holomorphic quantization (more rigorous but works only for linear theory, so far)

General Boundary Formulation of QFT: Schrödinger-Feynman quantization

- Schrödinger representation + Feynman path integral quantization
  The state space $\mathcal{H}_\Sigma$ for a hypersurface Σ is the space of functions on field configurations K_Σ on Σ.
- We write the inner product there as

$$\langle\psi_2|\psi_1\rangle = \int_{K_\Sigma} \mathcal{D}\varphi\, \psi_1(\varphi)\overline{\psi_2(\varphi)}.$$

- The amplitude for a region M and a state ψ in the state space $\mathcal{H}_{\partial M}$ associated to the boundary ∂M of M is

$$\rho_M(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi\, \psi(\varphi) \int_{K_M, \phi|_{\partial M}=\varphi} \mathcal{D}\phi\, e^{\mathrm{i}S_M(\phi)}.$$

  The inner integral, called *field propagator*, is over the space K_M of space-time field configurations ϕ in the interior of M which agree with φ on the boundary ∂M.
- A classical observable F in M is modeled as a function on K_M. The quantization of F is the linear map $\rho_M^F : \mathcal{H}_{\partial M} \to \mathbb{C}$ defined as

$$\rho_M^F(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi\, \psi(\varphi) \int_{K_M, \phi|_{\partial M}=\varphi} \mathcal{D}\phi\, F(\phi) e^{\mathrm{i}S_M(\phi)}.$$

General Boundary Formulation of QFT: Holomorphic quantization (I)

- Linear field theory: L_Σ is the vector space of solutions near the hypersurface Σ.
- For a region M, $L_{\tilde{M}}$ is the space of solutions in the interior of M; $L_{\tilde{M}} \subseteq L_{\partial M}$.
- L_Σ carries a non-degenerate symplectic structure ω_Σ and a complex structure $J_\Sigma : L_\Sigma \to L_\Sigma$ compatible with the symplectic structure:

$$J_\Sigma^2 = -\mathrm{id}_\Sigma \qquad \text{and} \qquad \omega_\Sigma(J_\Sigma(\cdot), J_\Sigma(\cdot)) = \omega_\Sigma(\cdot, \cdot).$$

- J_Σ and ω_Σ combine to a real inner product $g_\Sigma(\cdot, \cdot) = 2\omega_\Sigma(\cdot, J_\Sigma \cdot)$ and to a complex inner product $\{\cdot, \cdot\}_\Sigma = g_\Sigma(\cdot, \cdot) + 2\mathrm{i}\omega_\Sigma(\cdot, \cdot)$ which makes L_Σ into a complex Hilbert space.
- The Hilbert space $\mathcal{H}_\Sigma$ associated with Σ is the space of holomorphic functions on L_Σ with the inner product

$$\langle \psi, \psi' \rangle_\Sigma = \int_{L_\Sigma} \overline{\psi(\phi)} \psi'(\phi) \exp\left(-\frac{1}{2} g_\Sigma(\phi, \phi) \right) \mathrm{d}\mu(\phi),$$

where μ is a (fictitious) translation-invariant measure on L_Σ.

General Boundary Formulation of QFT: Holomorphic quantization (II)

- The amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ associated with the spacetime region M for a state $\psi \in \mathcal{H}_{\partial M}$ is given by

$$\rho_M(\psi) = \int_{L_\Sigma} \psi(\phi) \exp\left(-\frac{1}{4} g_{\partial M}(\phi, \phi)\right) \mathrm{d}\mu_{\tilde{M}}(\phi),$$

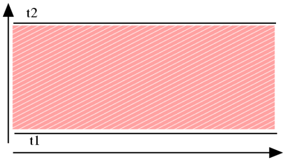
  where $\mu_{\tilde{M}}$ is a (fictitious) translation-invariant measure on $L_{\tilde{M}}$.
- The observable map associated to a classical observable F in a region M is

$$\rho_M^F(\psi) = \int_{L_\Sigma} \psi(\phi) F(\phi) \exp\left(-\frac{1}{4} g_{\partial M}(\phi, \phi)\right) \mathrm{d}\mu_{\tilde{M}}(\phi).$$

S-matrix in the standard setting

Usually, interacting QFT is described via the S-matrix:

Assume interaction is relevant only after the initial time t_1 and before the final time t_2. The S-matrix is the asymptotic limit of the amplitude between free states at early and at late time:

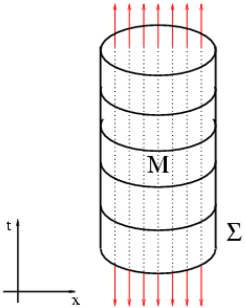


$$\langle\psi_2|\mathcal{S}|\psi_1\rangle = \lim_{\substack{t_1\to-\infty \\ t_2\to+\infty}} \langle\psi_2|U_{int}(t_1,t_2)|\psi_1\rangle$$

Spatially asymptotic S-matrix

Similarly, we can describe interacting QFT via a spatially asymptotic amplitude. Assume interaction is relevant only within a radius R from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$\mathcal{S}(\psi) = \lim_{R\to\infty} \rho_R(\psi)$$



Result

The S-matrices are equivalent when both are valid.

Some applications of the GBF to QFT

- Description of quantum states on timelike hypersurfaces. This permits the quantization of evanescent waves that are ”invisible” in traditional quantization prescriptions.
- Description of general interacting QFT in Minkowski spacetime.
- Description of new types of asymptotic amplitudes, generalizing the S-matrix framework.
- Description of a Euclidean theory in a bounded region of spacetime: local description of dynamics.
- Application to de Sitter spacetime
- New representation of the Feynman propagator in de Sitter spacetime and the S-matrix. Derivation of the Polyakov propagator.
- General structure of the vacuum state in a wide class of curved spaces.
- General structure of the complex structure used in the Schrödinger representation of QFT.
- Unitary evolution in curved spacetime.
- General structure of the S-matrix and the Feynman propagator in a wide class of curved spaces.
- Pairing between the holomorphic and the Schrödinger representation.

Conclusions

- The GBF provides a viable description of the dynamics of quantized fields
- It represents an extension of standard QFT, and offers a new perspective on QFT (geometrical aspects, holography)
- It is suitable to formulate background independent QFT.
- Results have been obtained.
- It may provide a new approach to the problem of quantum gravity (solves the problem of time and the problem of locality)

Outlook

- The GBF is still work in progress
- Application to more general spacetimes and regions: compact region, S-matrix in AdS,...
- Study of physical effects: Unruh, Hawking
- Develop suitable quantization prescriptions for interacting QFT
- Application to quantum gravity