Exercises

Exercise 4.1 Let $[P]$ be the transition matrix for a finite state Markov chain and let state i be recurrent. Prove that i is aperiodic if $P_{ii} > 0$.

Exercise 4.2 Show that every Markov chain with $\mathsf{M} < \infty$ states contains at least one recurrent set of states. Explaining each of the following statements is sufficient.

(a) If state i_1 is transient, then there is some other state i_2 such that $i_1 \to i_2$ and $i_2 \not\to i_1$.

(b) If the i_2 of (a) is also transient, there is a third state i_3 such that $i_2 \to i_3, i_3 \not\to i_2$; that state must satisfy $i_3 \neq i_2, i_3 \neq i_1$.

(c) Continue iteratively to repeat (b) for successive states, $i_1, i_2, \ldots$. That is, if $i_1, \ldots, i_k$ are generated as above and are all transient, generate i_{k+1} such that $i_k \to i_{k+1}$ and $i_{k+1} \not\to i_k$. Then $i_{k+1} \neq i_j$ for $1 \leq j \leq k$.

(d) Show that for some $k \leq \mathsf{M}$, k is not transient, i.e., it is recurrent, so a recurrent class exists.

Exercise 4.3 Consider a finite-state Markov chain in which some given state, say state 1, is accessible from every other state. Show that the chain has exactly one recurrent class $\mathcal{R}$ of states and state $1 \in \mathcal{R}$. (Note that the chain is then a unichain.)

Exercise 4.5 (Proof of Theorem 4.2.11) **(a)** Show that an ergodic Markov chain with M states must contain a cycle with $\tau < \mathsf{M}$ states. Hint: Use ergodicity to show that the smallest cycle cannot contain M states.

(b) Let ℓ be a fixed state on this cycle of length τ. Let $\mathcal{T}(m)$ be the set of states accessible from ℓ in m steps. Show that for each $m \geq 1$, $\mathcal{T}(m) \subseteq \mathcal{T}(m + \tau)$. Hint: For any given state $j \in \mathcal{T}(m)$, show how to construct a walk of $m + \tau$ steps from ℓ to j from the assumed walk of m steps.

***1.** Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

2. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed?

3. In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine **P** for this Markov chain.

***4.** Consider a process $\{X_n, n = 0, 1, \ldots\}$ which takes on the values 0, 1, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$
$$= \begin{cases} P_{ij}^{\mathrm{I}}, & \text{when } n \text{ is even} \\ P_{ij}^{\mathrm{II}}, & \text{when } n \text{ is odd} \end{cases}$$

where $\sum_{j=0}^{2} P_{ij}^{\mathrm{I}} = \sum_{j=0}^{2} P_{ij}^{\mathrm{II}} = 1$, $i = 0, 1, 2$. Is $\{X_n, n \geqslant 0\}$ a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

5. A Markov chain $\{X_n, n \geqslant 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $E[X_3]$.

6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\mathbf{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{Vmatrix}$$