(4) Decide which of these polynomials g in (3) are divisors of f. If $\deg(f) = n \geqslant 1$ and s is taken to be the greatest integer $\leqslant n/2$, then f is irreducible in $\mathbb{Q}[x]$ in case the method only yields constant polynomials as divisors. Otherwise, Kronecker's method yields a nontrivial factorization. By applying the method again to the factors and repeating the process, one eventually gets the canonical factorization of f. Use this procedure to find the canonical factorization of

$$f(x) = \tfrac{1}{3}x^6 - \tfrac{5}{3}x^5 + 2x^4 - x^3 + 5x^2 - \tfrac{17}{3}x - 1 \in \mathbb{Q}[x].$$

1.31. Construct the addition and multiplication table for $\mathbb{F}_2[x]/(x^3 + x^2 + x)$. Determine whether or not this ring is a field.

1.32. Let $[x+1]$ be the residue class of $x+1$ in $\mathbb{F}_2[x]/(x^4+1)$. Find the residue classes comprising the principal ideal $([x+1])$ in $\mathbb{F}_2[x]/(x^4+1)$.

1.33. Let F be a field and $a, b, g \in F[x]$ with $g \neq 0$. Prove that the congruence $af \equiv b \bmod g$ has a solution $f \in F[x]$ if and only if $\gcd(a, g)$ divides b.

1.34. Solve the congruence $(x^2+1)f(x) \equiv 1 \bmod (x^3+1)$ in $\mathbb{F}_3[x]$, if possible.

1.35. Solve $(x^4 + x^3 + x^2 + 1)f(x) \equiv (x^2+1) \bmod (x^3+1)$ in $\mathbb{F}_2[x]$, if possible.

1.36. Prove that $R[x]/(x^4 + x^3 + x + 1)$ cannot be a field, no matter what the commutative ring R with identity is.

1.37. Prove: given a field F, nonzero polynomials $f_1, \ldots, f_k \in F[x]$ that are pairwise relatively prime, and arbitrary polynomials $g_1, \ldots, g_k \in F[x]$, then the simultaneous congruences $h \equiv g_i \bmod f_i$, $i = 1, 2, \ldots, k$, have a unique solution $h \in F[x]$ modulo $f = f_1 \cdots f_k$. (Chinese Remainder Theorem for $F[x]$)

1.38. Evaluate $f(3)$ for $f(x) = x^{214} + 3x^{152} + 2x^{47} + 2 \in \mathbb{F}_5[x]$.

1.39. Let p be a prime and $a_0, \ldots, a_n$ integers with p not dividing a_n. Show that $a_0 + a_1 y + \cdots + a_n y^n \equiv 0 \bmod p$ has at most n different solutions y modulo p.

1.40. If $p > 2$ is a prime, show that there are exactly two elements $a \in \mathbb{F}_p$ such that $a^2 = 1$.

1.41. Show: if $f \in \mathbb{Z}[x]$ and $f(0) \equiv f(1) \equiv 1 \bmod 2$, then f has no roots in $\mathbb{Z}$.

1.42. Let p be a prime and $f \in \mathbb{Z}[x]$. Show: $f(a) \equiv 0 \bmod p$ holds for all $a \in \mathbb{Z}$ if and only if $f(x) = (x^p - x)g(x) + ph(x)$ with $g, h \in \mathbb{Z}[x]$.

1.43. Let p be a prime integer and c an element of the field F. Show that $x^p - c$ is irreducible over F if and only if $x^p - c$ has no root in F.

1.44. Show that for a polynomial $f \in F[x]$ of positive degree the following conditions are equivalent:

(a) f is irreducible over F;

(b) the principal ideal (f) of $F[x]$ is a maximal ideal;

(c) the principal ideal (f) of $F[x]$ is a prime ideal.