divides the characteristic polynomial of T.

(c) Let T be a linear operator on a finite-dimensional vector space V, and let x and y be elements of V. If W is the T-cyclic subspace generated by x, W′ is the T-cyclic subspace generated by y, and $W = W'$, then $x = y$.

(d) If T is a linear operator on a finite-dimensional vector space V, then for any $x \in V$ the T-cyclic subspace generated by x is the same as the T-cyclic subspace generated by $T(x)$.

(e) Let T be a linear operator on an n-dimensional vector space. Then there exists a polynomial $g(t)$ of degree n such that $g(T) = T_0$.

(f) Any polynomial of the form

$$(-1)^n(a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n)$$

is the characteristic polynomial of some linear operator.

(g) If T is a linear operator on a finite-dimensional vector space V, and if V is a direct sum of k T-invariant subspaces, then there is a basis β for V such that $[T]_\beta$ is a direct sum of k matrices.

2. For each of the following linear operators T, determine if the given subspace W is a T-invariant subspace of V.

(a) $V = P_3(R)$, $T(f) = f'$, and $W = P_2(R)$

(b) $V = P(R)$, $T(f)(x) = xf(x)$, and $W = P_2(R)$

(c) $V = R^3$, $T(a, b, c) = (a + b + c, a + b + c, a + b + c)$, and $W = \{(t, t, t): t \in R\}$

(d) $V = C([0, 1])$, $T(f)(t) = \left[\int_0^1 f(x)\,dx\right]t$, and $W = \{f \in V: f(t) = at + b \text{ for some } a \text{ and } b\}$

(e) $V = M_{2\times 2}(R)$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $W = \{A \in V: A^t = A\}$

3. Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

(a) $\{0\}$ and V

(b) N(T) and R(T)

(c) E_λ, for any eigenvalue λ of T

4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(T)$-invariant for any polynomial $g(t)$.

5. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

6. For each linear operator T on the vector space V find a basis for the T-cyclic subspace generated by the vector z.

(a) $V = R^4$, $T(a, b, c, d) = (a + b, b - c, a + c, a + d)$, and $z = e_1$

(b) $V = P_3(R)$, $T(f) = f''$, and $z = x^3$

(c) $V = M_{2\times 2}(R)$, $T(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(d) $V = M_{2\times 2}(R)$, $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$