The matrix (7-2) is called the **companion matrix** of the monic polynomial p_α.

Theorem 2. *If U is a linear operator on the finite-dimensional space W, then U has a cyclic vector if and only if there is some ordered basis for W in which U is represented by the companion matrix of the minimal polynomial for U.*

Proof. We have just observed that if U has a cyclic vector, then there is such an ordered basis for W. Conversely, if we have some ordered basis $\{\alpha_1, \ldots, \alpha_k\}$ for W in which U is represented by the companion matrix of its minimal polynomial, it is obvious that α_1 is a cyclic vector for U. ▮

Corollary. *If A is the companion matrix of a monic polynomial p, then p is both the minimal and the characteristic polynomial of A.*

Proof. One way to see this is to let U be the linear operator on F^k which is represented by A in the standard ordered basis, and to apply Theorem 1 together with the Cayley-Hamilton theorem. Another method is to use Theorem 1 to see that p is the minimal polynomial for A and to verify by a direct calculation that p is the characteristic polynomial for A. ▮

One last comment—if T is any linear operator on the space V and α is any vector in V, then the operator U which T induces on the cyclic subspace $Z(\alpha; T)$ has a cyclic vector, namely, α. Thus $Z(\alpha; T)$ has an ordered basis in which U is represented by the companion matrix of p_α, the T-annihilator of α.

Exercises

• 1. Let T be a linear operator on F^2. Prove that any non-zero vector which is not a characteristic vector for T is a cyclic vector for T. Hence, prove that either T has a cyclic vector or T is a scalar multiple of the identity operator.

• 2. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Prove that T has no cyclic vector. What is the T-cyclic subspace generated by the vector $(1, -1, 3)$?

• 3. Let T be the linear operator on C^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 1 & i & 0 \\ -1 & 2 & -i \\ 0 & 1 & 1 \end{bmatrix}.$$