let W be the group of real numbers 1 and -1 under multiplication. Define $\psi: S_n \to W$ by $\psi(s) = 1$ if s is an even permutation, $\psi(s) = -1$ if s is an odd permutation. By the rules 1, 2, 3 above ψ is a homomorphism onto W. The kernel of ψ is precisely A_n; being the kernel of a homomorphism A_n is a normal subgroup of S_n. By Theorem 2.7.1 $S_n/A_n \approx W$, so, since

$$2 = o(W) = o\left(\frac{S_n}{A_n}\right) = \frac{o(S_n)}{o(A_n)},$$

we see that $o(A_n) = \frac{1}{2}n!$. A_n is called the *alternating group* of degree n. We summarize our remarks in

LEMMA 2.10.3 *S_n has as a normal subgroup of index 2 the alternating group, A_n, consisting of all even permutations.*

At the end of the next section we shall return to S_n again.

Problems

* 1. Find the orbits and cycles of the following permutations:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$.

* 2. Write the permutations in Problem 1 as the product of disjoint cycles.

* 3. Express as the product of disjoint cycles:
(a) $(1, 2, 3)(4, 5)(1, 6, 7, 8, 9)(1, 5)$.
(b) $(1, 2)(1, 2, 3)(1, 2)$.

* 4. Prove that $(1, 2, \ldots, n)^{-1} = (n, n-1, n-2, \ldots, 2, 1)$.

* 5. Find the cycle structure of all the powers of $(1, 2, \ldots, 8)$.

6. (a) What is the order of an n-cycle?
(b) What is the order of the product of the disjoint cycles of lengths $m_1, m_2, \ldots, m_k$?
(c) How do you find the order of a given permutation?

* 7. Compute $a^{-1}ba$, where
(1) $a = (1, 3, 5)(1, 2)$, $b = (1, 5, 7, 9)$.
(2) $a = (5, 7, 9)$, $b = (1, 2, 3)$.

* 8. (a) Given the permutation $x = (1, 2)(3, 4)$, $y = (5, 6)(1, 3)$, find a permutation a such that $a^{-1}xa = y$.
(b) Prove that there is no a such that $a^{-1}(1, 2, 3)a = (1, 3)(5, 7, 8)$.
(c) Prove that there is no permutation a such that $a^{-1}(1, 2)a = (3, 4)(1, 5)$.

* 9. Determine for what m an m-cycle is an even permutation.

10. Determine which of the following are even permutations:
   (a) $(1, 2, 3)(1, 2)$.
   (b) $(1, 2, 3, 4, 5)(1, 2, 3)(4, 5)$.
   (c) $(1, 2)(1, 3)(1, 4)(2, 5)$.
11. Prove that the smallest subgroup of S_n containing $(1, 2)$ and $(1, 2, \ldots, n)$ is S_n. (In other words, these generate S_n.)
*12. Prove that for $n \geq 3$ the subgroup generated by the 3-cycles is A_n.
*13. Prove that if a normal subgroup of A_n contains even a single 3-cycle it must be all of A_n.
*14. Prove that A_5 has no normal subgroups $N \neq (e), A_5$.
15. Assuming the result of Problem 14, prove that any subgroup of A_5 has order at most 12.
16. Find all the normal subgroups in S_4.
*17. If $n \geq 5$ prove that A_n is the only nontrivial normal subgroup in S_n.

Cayley's theorem (Theorem 2.9.1) asserts that every group is isomorphic to a subgroup of $A(S)$ for some S. In particular, it says that every finite group can be realized as a group of permutations. Let us call the realization of the group as a group of permutations as given in the proof of Theorem 2.9.1 the *permutation representation* of G.

18. Find the permutation representation of a cyclic group of order n.
19. Let G be the group $\{e, a, b, ab\}$ of order 4, where $a^2 = b^2 = e$, $ab = ba$. Find the permutation representation of G.
20. Let G be the group S_3. Find the permutation representation of S_3. (*Note:* This gives an isomorphism of S_3 into S_6.)
21. Let G be the group $\{e, \theta, a, b, c, \theta a, \theta b, \theta c\}$, where $a^2 = b^2 = c^2 = \theta$, $\theta^2 = e$, $ab = \theta ba = c$, $bc = \theta cb = a$, $ca = \theta ac = b$.
   (a) Show that θ is in the center Z of G, and that $Z = \{e, \theta\}$.
   (b) Find the commutator subgroup of G.
   (c) Show that every subgroup of G is normal.
   (d) Find the permutation representation of G.
   (*Note:* G is often called the group of *quaternion units*; it, and algebraic systems constructed from it, will reappear in the book.)
22. Let G be the dihedral group of order $2n$ (see Problem 17, Section 2.6). Find the permutation representation of G.

Let us call the realization of a group G as a set of permutations given in Problem 1, Section 2.9 the *second permutation representation* of G.

23. Show that if G is an abelian group, then the permutation representation of G coincides with the second permutation representation of G (i.e., in the notation of the previous section, $\lambda_g = \tau_g$ for all $g \in G$.)