COROLLARY 1 *If R is a unique factorization domain then so is $R[x_1, \ldots, x_n]$.*

A special case of Corollary 1 but of independent interest and importance is

COROLLARY 2 *If F is a field then $F[x_1, \ldots, x_n]$ is a unique factorization domain.*

Problems

● 1. Prove that $R[x]$ is a commutative ring with unit element whenever R is.

2. Prove that $R[x_1, \ldots, x_n] = R[x_{i_1}, \ldots, x_{i_n}]$, where $(i_1, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$.

● 3. If R is an integral domain, prove that for $f(x)$, $g(x)$ in $R[x]$, $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.

● 4. If R is an integral domain with unit element, prove that any unit in $R[x]$ must already be a unit in R.

● 5. Let R be a commutative ring with no nonzero *nilpotent* elements (that is, $a^n = 0$ implies $a = 0$). If $f(x) = a_0 + a_1x + \cdots + a_mx^m$ in $R[x]$ is a zero-divisor, prove that there is an element $b \neq 0$ in R such that $ba_0 = ba_1 = \cdots = ba_m = 0$.

*6. Do Problem 5 dropping the assumption that R has no nonzero nilpotent elements.

*7. If R is a commutative ring with unit element, prove that $a_0 + a_1x + \cdots + a_nx^n$ in $R[x]$ has an inverse in $R[x]$ (i.e., is a unit in $R[x]$) if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent elements in R.

8. Prove that when F is a field, $F[x_1, x_2]$ is not a principal ideal ring.

9. Prove, completely, Lemma 3.11.2 and its corollary.

● 10. (a) If R is a unique factorization domain, prove that every $f(x) \in R[x]$ can be written as $f(x) = af_1(x)$, where $a \in R$ and where $f_1(x)$ is primitive.
(b) Prove that the decomposition in part (a) is unique (up to associates).

● 11. If R is an integral domain, and if F is its field of quotients, prove that any element $f(x)$ in $F[x]$ can be written as $f(x) = (f_0(x)/a)$, where $f_0(x) \in R[x]$ and where $a \in R$.

12. Prove the converse part of Lemma 3.11.4.

13. Prove Corollary 2 to Theorem 3.11.1.

14. Prove that a principal ideal ring is a unique factorization domain.

15. If J is the ring of integers, prove that $J[x_1, \ldots, x_n]$ is a unique factorization domain.