Proof   The proof is indirect. We will assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly *dependent* and show that this assumption leads to a contradiction.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent, then one of these vectors must be expressible as a linear combination of the previous ones. Let $\mathbf{v}_{k+1}$ be the first of the vectors $\mathbf{v}_i$ that can be so expressed. In other words, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, but there are scalars $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v}_{k+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{1}$$

Multiplying both sides of equation (1) by A from the left and using the fact that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for each i, we have

$$\lambda_{k+1}\mathbf{v}_{k+1} = A\mathbf{v}_{k+1} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k)$$

$$= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_kA\mathbf{v}_k \tag{2}$$

$$= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_k\lambda_k\mathbf{v}_k$$

Now we multiply both sides of equation (1) by λ_{k+1} to get

$$\lambda_{k+1}\mathbf{v}_{k+1} = c_1\lambda_{k+1}\mathbf{v}_1 + c_2\lambda_{k+1}\mathbf{v}_2 + \cdots + c_k\lambda_{k+1}\mathbf{v}_k \tag{3}$$

When we subtract equation (3) from equation (2), we obtain

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{k+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{k+1})\mathbf{v}_2 + \cdots + c_k(\lambda_k - \lambda_{k+1})\mathbf{v}_k$$

The linear independence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ implies that

$$c_1(\lambda_1 - \lambda_{k+1}) = c_2(\lambda_2 - \lambda_{k+1}) = \cdots = c_k(\lambda_k - \lambda_{k+1}) = 0$$

Since the eigenvalues λ_i are all distinct, the terms in parentheses $(\lambda_i - \lambda_{k+1})$, $i = 1, \ldots, k$, are all nonzero. Hence, $c_1 = c_2 = \cdots = c_k = 0$. This implies that

$$\mathbf{v}_{k+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$$

which is impossible, since the eigenvector $\mathbf{v}_{k+1}$ cannot be zero. Thus, we have a contradiction, which means that our assumption that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent is false. It follows that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ must be linearly independent.

Exercises 4.3

In Exercises 1–12, compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, and (d) the algebraic and geometric multiplicity of each eigenvalue.

1. $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -2 & 1 & 2 & -1 \end{bmatrix}$

12. $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

13. Prove Theorem 4.18(b).

14. Prove Theorem 4.18(c). [*Hint:* Combine the proofs of parts (a) and (b) and see the fourth Remark following Theorem 3.9 (p. 167).]

In Exercises 15 and 16, A is a 2×2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *corresponding to eigenvalues* $\lambda_1 = \frac{1}{2}$ *and* $\lambda_2 = 2$, *respectively, and* $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

15. Find $A^{10}\mathbf{x}$.

16. Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty$)?

In Exercises 17 and 18, A is a 3×3 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, *and* $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *corresponding to eigenvalues* $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = \frac{1}{3}$, *and* $\lambda_3 = 1$, *respectively, and* $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

17. Find $A^{20}\mathbf{x}$.

18. Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty$)?

19. (a) Show that, for any square matrix A, A^T and A have the same characteristic polynomial and hence the same eigenvalues.
    (b) Give an example of a 2×2 matrix A for which A^T and A have different eigenspaces.

20. Let A be a nilpotent matrix (that is, $A^m = O$ for some $m > 1$). Show that $\lambda = 0$ is the only eigenvalue of A.

21. Let A be an idempotent matrix (that is, $A^2 = A$). Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of A.

22. If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

23. (a) Find the eigenvalues and eigenspaces of
$$A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$$

(b) Using Theorem 4.18 and Exercise 22, find the eigenvalues and eigenspaces of $A^{-1}, A - 2I,$ and $A + 2I$.

24. Let A and B be $n \times n$ matrices with eigenvalues λ and μ, respectively.
    (a) Give an example to show that $\lambda + \mu$ need not be an eigenvalue of $A + B$.
    (b) Give an example to show that $\lambda\mu$ need not be an eigenvalue of AB.
    (c) Suppose λ and μ correspond to the *same* eigenvector $\mathbf{x}$. Show that, in this case, $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.

25. If A and B are two row equivalent matrices, do they necessarily have the same eigenvalues? Either prove that they do or give a counterexample.

Let $p(x)$ be the polynomial
$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

*The **companion matrix** of $p(x)$ is the $n \times n$ matrix*

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (4)$$

26. Find the companion matrix of $p(x) = x^2 - 7x + 12$ and then find the characteristic polynomial of $C(p)$.

27. Find the companion matrix of $p(x) = x^3 + 3x^2 - 4x + 12$ and then find the characteristic polynomial of $C(p)$.

28. (a) Show that the companion matrix $C(p)$ of $p(x) = x^2 + ax + b$ has characteristic polynomial $\lambda^2 + a\lambda + b$.
    (b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

29. (a) Show that the companion matrix $C(p)$ of $p(x) = x^3 + ax^2 + bx + c$ has characteristic polynomial $-(\lambda^3 + a\lambda^2 + b\lambda + c)$.
    (b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

Solving for A, we have $A = PDP^{-1}$, which makes it easy to find powers of A. We compute

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

and, generally, $A^n = PD^nP^{-1}$ for all $n \geq 1$. (You should verify this by induction. Observe that this fact will be true for *any* diagonalizable matrix, not just the one in this example.)

Since

$$D^n = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

we have

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2(-1)^n + 2^n}{3} & \dfrac{(-1)^{n+1} + 2^n}{3} \\ \dfrac{2(-1)^{n+1} + 2^{n+1}}{3} & \dfrac{(-1)^{n+2} + 2^{n+1}}{3} \end{bmatrix}$$

Since we were only asked for A^{10}, this is more than we needed. But now we can simply set $n = 10$ to find

$$A^{10} = \begin{bmatrix} \dfrac{2(-1)^{10} + 2^{10}}{3} & \dfrac{(-1)^{11} + 2^{10}}{3} \\ \dfrac{2(-1)^{11} + 2^{11}}{3} & \dfrac{(-1)^{12} + 2^{11}}{3} \end{bmatrix} = \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$$

Exercises 4.4

In Exercises 1–4, show that A and B are not similar matrices.

1. $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

In Exercises 5–7, a diagonalization of the matrix A is given in the form $P^{-1}AP = D$. List the eigenvalues of A and bases for the corresponding eigenspaces.

5. $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix} =$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{5}{8} & -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} =$

$\begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 8–15, determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

8. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ **9.** $A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **11.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ **13.** $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **15.** $A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

In Exercises 16–23, use the method of Example 4.29 to compute the indicated power of the matrix.

16. $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}^9$ **17.** $\begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^{10}$

18. $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}^{-6}$ **19.** $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}^k$

20. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}^8$ **21.** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2002}$

22. $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^{-5}$ **23.** $\begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}^k$

In Exercises 24–29, find all (real) values of k for which A is diagonalizable.

24. $A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$ **25.** $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

26. $A = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$ **27.** $A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **29.** $A = \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{bmatrix}$

30. Prove Theorem 4.21(c).

31. Prove Theorem 4.22(b).

32. Prove Theorem 4.22(c).

33. Prove Theorem 4.22(e).

34. If A and B are invertible matrices, show that AB and BA are similar.

35. Prove that if A and B are similar matrices, then $\text{tr}(A) = \text{tr}(B)$. (*Hint:* Find a way to use Exercise 45 from Section 3.2.)

+ *In general, it is difficult to show that two matrices are similar. However, if two similar matrices are diagonalizable, the task becomes easier. In Exercises 36–39, show that A and B are similar by showing that they are similar to the same diagonal matrix. Then find an invertible matrix P such that $P^{-1}AP = B$.*

36. $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

37. $A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 2 & -1 \\ 2 & 2 & -4 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 & 0 \\ 6 & 5 & 0 \\ 4 & 4 & -1 \end{bmatrix}$

40. Prove that if A is similar to B, then A^T is similar to B^T.

41. Prove that if A is diagonalizable, so is A^T.

42. Let A be an invertible matrix. Prove that if A is diagonalizable, so is A^{-1}.

43. Prove that if A is a diagonalizable matrix with only one eigenvalue λ, then A is of the form $A = \lambda I$. (Such a matrix is called a ***scalar matrix***.)

44. Let A and B be $n \times n$ matrices, each with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.

45. Let A and B be similar matrices. Prove that the algebraic multiplicities of the eigenvalues of A and B are the same.