$n = 4$. We can think of A and B as made up of blocks of size 2×2: $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. We now calculate matrices $P_1, \dots, P_6 \in \mathcal{M}_{2\times 2}(F)$ as above and use them to compute $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = AB$. To do this we need a total of 49 multiplications and $198 = 7 \cdot 18 + 4 \cdot 18$ additions, as opposed to 64 multiplications and 46 additions using the ordinary method. In general, if $n = 2^h$ then the number of multiplications needed to compute AB by the Strassen-Winograd method is $M(h) = 7^h$ and the number of additions is $A(h) = 6(7^h - 4^h)$. Thus $M(h) + A(h) < 7^{h+1}$ and we see that the number of arithmetic operations in F needed is on the order of n^c, where $c \leq log_2 7 = 2.807\dots$, as opposed to an order of n^3 by the ordinary method. If n is large, this can lead to a considerable savings in computational power. More sophisticated algorithms of this sort can reduce the number of arithmetic operations needed to an order of n^c, where $c < 2.4$.

Problems

1. Find the linear transformations from $\mathbb{Q}^4$ to $\mathbb{Q}^3$ represented by the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

with respect to each of the following pairs of bases:

(i) $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$ and $\{[1,0,0],[0,1,0],[0,0,1]\}$;
(ii) $\{[1,1,1,1],[0,1,1,1],[0,0,1,1],[0,0,0,1]\}$ and $\{[1,1,1],[0,1,1],[0,0,1]\}$;
(iii) $\{[1,0,0,1],[0,1,0,1],[0,0,1,1],[0,0,0,1]\}$ and $\{[2,0,0],[0,3,1],[0,0,1]\}$;
(iv) $\{[1,2,0,0],[0,1,2,0],[0,0,1,2],[0,0,0,1]\}$ and $\{[1,2,3],[2,1,2],[2,2,2]\}$.

2. Let V be the subspace of $\mathbb{R}[X]$ consisting of all those polynomials having degree at most 2 and let W be the subspace of $\mathbb{R}[X]$ consisting of all those polynomials having degree at most 3. Let $\alpha: V \to W$ be the linear transformation defined by $\alpha: a + bX + cX^2 \mapsto (a+b) + (b+c)X + (a+c)X^2 + (a+b+c)X^3$. Find $\Phi_{BD}(\alpha)$ when

(i) B and D are the canonical bases;
(ii) $B = \{1, X+1, X^2+X+1\}$ and $D = \{X^3 - X^2, X^2 - X, X - 1, 1\}$;
(iii) $B = \{X^2, X, 1\}$ and $D = \{X^3 - 3, X^2 - 3, X - 1, -1\}$.

3. Let $\alpha: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$\alpha: [a, b, c] \mapsto [a + b + c, b + c].$$

Find the matrices which represent α with respect to the following bases:

(i) The canonical bases;
(ii) $\{[-1,0,2],[0,1,1],[3,-1,0]\}$ and $\{[-1,1],[1,0]\}$;
(iii) $\{[3,-1,0],[0,1,1],[-1,0,2]\}$ and $\{[1,0],[-1,1]\}$;
(iv) $\{[0,1,1],[3,-1,0],[-1,0,2]\}$ and $\{[-1,1],[1,0]\}$.

4. Let α be the endomorphism of the vector space $\mathbb{R}^3$ defined by $[a,b,c] \mapsto [3a+2b, -a-c, a+3b]$. Find $\Phi_{BB}(\alpha)$ for each of the following bases B of $\mathbb{R}^3$ over $\mathbb{R}$:

(i) $\{[1,-1,0],[1,0,-1],[0,1,0]\}$;
(ii) $\{[-1,-1,-1],[2,0,3],[1,2,3]\}$;
(iii) $\{[1,1,2],[1,0,0],[0,-5,0]\}$.

5. Let α be the endomorphism of the vector space $\mathbb{R}^3$ over $\mathbb{R}$ represented with respect to some basis by the matrix $\begin{bmatrix} 0 & 2 & -1 \\ -2 & 5 & -2 \\ -4 & 8 & -3 \end{bmatrix}$. Is α idempotent?

6. Let V be the subspace of $\mathbb{R}[X]$ consisting of all polynomials of degree at most 2, and let $B = \{1, X, X^2\}$ and $D = \{1, 1+X, 3+4X+2X^2\}$ be bases for V. Let α be the endomorphism of V satisfying $\Phi_{BB}(\alpha) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Find $\Phi_{DD}(\alpha)$.

7. Let V be the subspace of $\mathbb{R}^3$ with basis $\{[1,1,0],[0,1,1]\}$ and let W be the subspace of $\mathbb{R}^5$ with basis $\{[1,1,1,0,0],[0,0,1,1,0],[0,0,0,1,1]\}$. Let $\alpha\colon V \to \mathbb{R}^5$ be the linear transformation defined by

$$\alpha\colon [a,b,c] \mapsto [a+b+c, a+b+c, a-b+c, -2b, 0].$$

Show that $im(\alpha) \subseteq W$ and find the matrix which represents α with respect to the given bases.

8. Let W be the subspace of $\mathbb{R}^{\mathbb{R}}$ generated by the linearly-independent set of vectors $D = \{1, x, e^x, xe^x\}$ and let δ be the endomorphism of W defined by $\delta\colon f \mapsto f'$. Find the matrix represented δ with respect to D.

9. Let $D = \{1+i, 2+i\}$ be a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$ and let α be the endomorphism of $\mathbb{C}$ given by $\alpha\colon z \mapsto \bar{z}$. Find the matrix representing α with respect to D.

10. Let $F = \mathbb{Z}/(3)$. Let $\alpha\colon F^3 \to F^2$ be the linear transformation defined by $\alpha\colon [a,b,c] \mapsto [a-b, 2a-c]$ and let $\beta\colon F^2 \to F^4$ the linear transformation defined by $\beta\colon [a,b] \mapsto [b,a,2b,2a]$. Find the matrix which represents $\beta\alpha$ with respect to the canonical bases.

11. Calculate $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 4 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix}$ in $\mathcal{M}_{3\times 3}(\mathbb{Z}/(5))$.

12. If $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 4 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{3\times 4}(\mathbb{R})$, find all matrices $B \in \mathcal{M}_{4\times 3}(\mathbb{R})$ satisfying $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix} \in \mathcal{M}_{4\times3}(\mathbb{R})$, find all matrices $B \in \mathcal{M}_{3\times4}(\mathbb{R})$ satisfying $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

14. Find the set of all matrices $A \in \mathcal{M}_{4\times3}(\mathbb{R})$ satisfying

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

15. Let n be a positive integer. A matrix $[a_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called a *stochastic matrix* if and only if $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{i=1}^{n} a_{ij} = 1$ for all $1 \leq j \leq n$. Show that the set of all stochastic matrices is closed under matrix multiplication.

16. Do the elements $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$ of $\mathcal{M}_{4\times4}(\mathbb{Q})$ satisfy $AB = BA$?

17. Let F be a field, let n be a positive integer, and let $H \in \mathcal{M}_{n\times n}(F)$ be a given matrix. Let $\alpha: \mathcal{M}_{n\times n}(F) \to \mathcal{M}_{n\times n}(F)$ be the function defined by $\alpha: A \mapsto AH + HA$. Is α an endomorphism of the vector space $\mathcal{M}_{n\times n}(F)$?

18. Let α be the endomorphism of $\mathbb{R}^4$ represented with respect to the canonical bases by the matrix

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

If $v \in \mathbb{R}^4$ is a vector satisfying the condition that all components of $\alpha(v)$ are nonnegative. Show that all components of v are nonnegative.

19. Let V and W be vector spaces over a field F, which are not necessarily finitely generated over F. Pick bases $\{v_i \mid i \in \Omega\}$ and $\{w_j \mid j \in \Lambda\}$ for V and W respectively. Let $p: \Omega \times \Lambda \to F$ be function satisfying the condition that the set $\{j \in \Lambda \mid p(i,j) \neq 0_F\}$ is finite for all $i \in \Omega$ and let $\alpha_p: V \to W$ be the function defined as follows: if $v = \sum_{i\in\Gamma} a_i v_i$, where Γ is a finite subset of Ω and the a_i are scalars in F, then $\alpha_p(v) = \sum\{a_i p(i,j) w_j \mid i \in \Gamma; j \in \Lambda\}$. Show that α_p is a linear transformation and that all of the linear transformations from V to W are of this form.

20. Let k and n be positive integers and let $v \in \mathbb{R}^n$. If $A \in \mathcal{M}_{k\times n}(\mathbb{R})$, show that $Av^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ if and only if $A^T A v^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.