and so we have (2).

(3) By Proposition 4.9 we have

$$\begin{aligned} dim(im(\alpha)) &= dim(V) - dim(ker(\alpha)), \\ dim(im(\beta)) &= dim(W) - dim(ker(\beta)), \\ dim(im(\beta\alpha)) &= dim(V) - dim(ker(\beta\alpha)), \end{aligned}$$

and so (3) is true if and only if $dim(V) - dim(ker(\beta\alpha)) \geq dim(V) - dim(ker(\alpha)) + dim(W) - dim(ker(\beta)) - dim(W)$, which follows directly from (2). □

(4.11) PROPOSITION. *Let V, W, and Y be vector spaces over a field F. Let $\alpha: V \to W$ be a linear transformation and let $\beta: W \to Y$ be a monomorphism. For each $w \in W$ then $Inv(\alpha, w) = Inv(\beta\alpha, \beta(w))$.*

PROOF. If $v \in Inv(\alpha, w)$ then $\alpha(v) = w$ and so $\beta\alpha(v) = \beta(w)$, showing that $v \in Inv(\beta\alpha, \beta(w))$. Conversely, if $v \in Inv(\beta\alpha, \beta(w))$ then $\beta\alpha(v) = \beta(w)$ and, since β is monic, this implies that $\alpha(v) = w$ and so $v \in Inv(\alpha, w)$. □

Problems

1. Let $\alpha: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation over $\mathbb{R}$ satisfying $\alpha([1,0,1]) = [-1,3,4]$, $\alpha([1,-1,1]) = [0,1,0]$, and $\alpha([1,2,-1]) = [3,1,4]$. What is $\alpha([1,0,0])$?

2. Let $\alpha: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation over $\mathbb{R}$ satisfying $\alpha([1,1,0]) = [1,2,-1]$, $\alpha([1,0,-1]) = [0,1,1]$, and $\alpha([0,-1,1]) = [3,3,3]$. Find a vector $v \in \mathbb{R}^3$ satisfying $\alpha(v) = [1,0,0]$.

3. Does there exist a real number d such that the function $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $[a,b] \mapsto [a+b+d^2+1, a]$ is a linear transformation over $\mathbb{R}$?

4. Does there exist a real number d such that the function $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $[a,b] \mapsto [5da - db, 8d^2 - 8d - 6]$ is a linear transformation over $\mathbb{R}$?

5. Let V be a vector space over a field F and let W, W' be subspaces of F. Let Y be a vector space over F and assume that $\alpha: W \to Y$ and $\beta: W' \to Y$ are linear transformations satisfying $\alpha(v) = \beta(v)$ for all $v \in W \cap W'$. Find a linear transformation $\theta: W + W' \to Y$ the restriction of which to W equals α and the restriction of which to W' equals β.

6. Let F be a field and let α be the function from $\mathcal{M}_{k\times n}(F) \to F^{k+n}$ defined by $\alpha: A \mapsto [b_1, \ldots, b_k, c_1, \ldots, c_n]$ where

(i) For each $1 \leq i \leq k$, the scalar b_i is the sum of the entries in the ith row of A; and

(ii) For each $1 \leq j \leq n$, the scalar c_j is the sum of the entries in the jth column of A.

Is α a linear transformation?

7. Let $\alpha: \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $\alpha(a+b) = \alpha(a) + \alpha(b)$ for all $a, b \in \mathbb{R}$. Show that α is a linear transformation.

8. Let $F = \mathbb{Z}/(3)$ and let $g: F \to F$ be the function defined by

$$g: a \mapsto \begin{cases} 0, & \text{if } a = 0; \\ 2, & \text{if } a = 1; \\ 1, & \text{if } a = 2. \end{cases}$$

Let n be a positive integer and let $\alpha: F^n \to F^n$ be the function defined by

$$\alpha: [a_1, \dots, a_n] \mapsto [g(a_1), \dots, g(a_n)].$$

Is α a linear transformation?

9. Let F be a field and let V be the subspace of $F[X]$ composed of all polynomials of degree at most 2. Let $\alpha: V \to F[X]$ be a linear transformation satisfying $\alpha(1_F) = X$, $\alpha(1_F + X) = X^3 + X^5$, and $\alpha(1_F + X + X^2) = 1_F - X^2 + X^4$. Calculate $\alpha(p)$ for an arbitrary element $p(X)$ of V.

10. Does there exist a linear transformation $\alpha: \mathbb{Q}^4 \to \mathbb{Q}[X]$ satisfying the conditions $\alpha([1, 3, 0, -1]) = 2$, $\alpha([-1, 1, 1, 1]) = X$, and $\alpha([-1, 5, 2, 1]) = X + 1$?

11. Let $F = \mathbb{Z}/(3)$ and let $\alpha: F^4 \to F^4$ be the function defined by $[a, b, c, d] \mapsto [a^3, b^3, d, c]$. Is α a linear transformation of vector spaces over F?

12. Let F be a field and let V be the set of infinite series $[a_1, a_2, \dots]$ of elements of F. This is a vector space over F. Given such a series and given $n \geq 1$, let $s_n = \sum_{i=1}^{n} a_i$ be its nth partial sum. Is the function $\alpha: V \to V$ defined by $[a_1, a_2, \dots] \mapsto [s_1, s_2, \dots]$ a linear transformation?

13. Let V be a vector space over $\mathbb{Q}$ having a countably-infinite basis $B = \{v_1, v_2, v_3, \dots\}$. We then know that each $v \in V$ can be uniquely represented in the form $v = \sum_{i=1}^{\infty} a_i(v)v_i$, where only finitely-many of the scalars $a_i(v)$ are nonzero. Define the function $\alpha: V \to \mathbb{Q}$ by $\alpha: v \mapsto \sum_{i=1}^{\infty} i^2 a_i(v)$. Is α a linear transformation?

14. Let V be the subspace of $\mathbb{R}^{\mathbb{R}}$ composed of all functions which are everywhere differentiable. For each $f \in V$ define the function $Df: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $Df: (a, b) \mapsto f'(a)b$. (This function is called the *differential* of f.) Is the function $D: V \to \mathbb{R}^{\mathbb{R} \times \mathbb{R}}$ defined by $f \mapsto Df$ a linear transformation?

15. Let W be the subspace of $\mathbb{R}^{\mathbb{R}}$ consisting of all twice-differentiable functions and let $\alpha: W \to \mathbb{R}^{\mathbb{R}}$ be the linear transformation defined by $\alpha: f \mapsto f''$. If $g \in \mathbb{R}^{\mathbb{R}}$ is the function defined by $g: x \mapsto x + 1$, find $Inv(\alpha, g)$.

16. Let $\alpha: \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation defined by

$$[a, b, c, d, e] \mapsto [b + c - 2d + e, a + 2b + 3c - 4d + e, 2a + 2c - 2e].$$

Find $ker(\alpha)$.

17. Let $F = \mathbb{Z}/(3)$ and let α be the linear transformation from the space F^3 to itself defined by $\alpha: [a, b, c] \mapsto [a + b, 2b + c, 0]$. Find $ker(\alpha)$.

18. Let $\alpha: \mathcal{M}_{3\times 3}(\mathbb{R}) \to \mathbb{R}$ be the function defined by

$$\alpha: [a_{ij}] \mapsto \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}.$$

Show that α is a linear transformation and find its kernel.

19. Let $F = \mathbb{Z}/(2)$ and let n be a positive integer. Let W be the set of all vectors $[a_1, \ldots, a_n]$ in F^n having an even number of nonzero entries. Show that W is a subspace of F^n by finding a linear transformation $F^n \to Y$ having kernel equal to W.

20. Set $F = \mathbb{Z}/(2)$. Let A and B be nonempty sets and let $\phi: A \to B$ be a given function. Define the function $\alpha: F^B \to F^A$ as follows: If $f \in F^B$ then

$$\alpha(f): a \mapsto \begin{cases} 1, & \text{if } f(\phi(a)) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Show that α is a linear transformation and find its kernel.

21. Let V be a vector space over a field F and let $\alpha: V^3 \to V$ be the function defined by $\alpha: [v, v', v''] \mapsto v + v' + v''$. Show that α is a linear transformation and find its kernel.

22. Let $F = \mathbb{Z}/(2)$ and let $\alpha: F^7 \to F^3$ be the linear transformation defined by

$$\alpha: [a_1, \ldots, a_7] \mapsto [a_4 + a_5 + a_6 + a_7, a_2 + a_3 + a_6 + a_7, a_1 + a_3 + a_5 + a_7].$$

Show that if $[0,0,0,0,0,0,0] \neq v \in ker(\alpha)$ then v has at least three components equal to 1.

23. Let V be a vector space over a field F having subspaces W and W'. Let $Y = \{[w, w'] \mid w \in W, w' \in W'\}$, which is a subspace of V^2. Let $\alpha: Y \to V$ be the linear transformation $\alpha: [w, w'] \mapsto w + w'$. Find the kernel of α and show that it is isomorphic to $W \cap W'$.

24. Let W be the subspace of $\mathbb{R}^{\mathbb{R}}$ consisting of all differentiable functions and let $\alpha: W \to \mathbb{R}^{\mathbb{R}}$ be the linear transformation defined by $\alpha(f): x \mapsto f'(x) + cos(x)f(x)$ for all $x \in \mathbb{R}$. Find $ker(\alpha)$.

25. Calculate the kernel of the linear transformation $\alpha: \mathbb{Q}[X] \to \mathbb{Q}[\sqrt{3}]$ (between vector spaces over $\mathbb{Q}$) defined by $\alpha: p(X) \mapsto p(\sqrt{3})$.

26. Let V and W be vector spaces over a field F and let $\alpha, \beta: V \to W$ be monomorphisms. Is $\alpha + \beta$ necessarily a monomorphism?

27. Let $F = \mathbb{Z}/(7)$. How many distinct monomorphisms are there from F^2 to F^4?

28. Let W be the subspace of $\mathbb{R}^6$ composed of all vectors $[a_1, a_2, a_3, a_4, a_5, a_6]$ satisfying $\sum_{i=1}^{6} a_i = 0$. Does there exist a monomorphism from W to $\mathbb{R}^4$?

29. Let n be a positive integer and let V be the subspace of $\mathbb{R}[X]$ consisting of all polynomials having degree less than or equal to n. Let $\alpha: V \rightarrow V$ be the linear transformation defined by $\alpha: p(X) \mapsto p(X+1) - p(X)$. Find $ker(\alpha)$ and $im(\alpha)$.

30. Let $\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by

$$[a, b, c] \mapsto [a + b + c, -a - c, b].$$

Find $ker(\alpha)$ and $im(\alpha)$.

31. Find a linear transformation $\alpha: \mathbb{Q}^3 \rightarrow \mathbb{Q}^4$ satisfying the condition that $im(\alpha) = \mathbb{Q}\{[\frac{2}{3}, -1, 3, 0], [2, 1, 1, -4]\}$.

32. Let V be a vector space over a field F and let $\mathbb{N}$ be the set of nonnegative integers. Let $W = \{f \in V^{\mathbb{N}} \mid f(i) = 0_V \text{ when } i \text{ is odd}\}$ and let Let $W' = \{f \in V^{\mathbb{N}} \mid f(i) = 0_V \text{ when } i \text{ is even}\}$. Find a linear transformation α from $V^{\mathbb{N}}$ to itself having kernel equal to W and image equal to W'.

33. Let V and W be vector spaces over a field F, where $dim(V) > 0$. Show that $W = \sum\{im(\alpha) \mid \alpha \in Hom(V, W)\}$.

34. Let V, W, and Y be vector spaces finitely generated over a field F and let $\alpha: V \rightarrow W$ be a linear transformation. Show that the set of all linear transformations $\beta: W \rightarrow Y$ satisfying the condition that $im(\alpha) \subseteq ker(\beta)$ is a subspace of $Hom(W, Y)$ and calculate its dimension.

35. Let V be a vector space over $\mathbb{C}$ and let n be a positive integer. Does there exist a linear transformation $\alpha: V \rightarrow \mathbb{C}^n$ other than the 0-map satisfying the condition that $im(\alpha) \subseteq \mathbb{R}^n$?

36. Let V and W be vector spaces over a field F and let V' be a proper subspace of V. Decide which are the following sets are subspaces of $Hom(V, W)$:

(i) $\{\alpha \in Hom(V, W) \mid ker(\alpha) = V'\}$;
(ii) $\{\alpha \in Hom(V, W) \mid ker(\alpha) \subseteq V'\}$;
(iii) $\{\alpha \in Hom(V, W) \mid ker(\alpha) \supseteq V'\}$.

37. Let V and W be vector spaces over a field F and let $\alpha, \beta: V \rightarrow W$ be epimorphisms. Is $\alpha + \beta$ necessarily an epimorphism?

38. Let W be a subspace of a vector space V over a field F. If $v \in V$ then we will denote the set $\{v + w \mid w \in W\}$ by v/W. The collection of all subsets of V of this form will be denoted by V/W. Define operations of addition and scalar multiplication on V/W as follows:

(i) $v/W + v'/W = (v + v')/W$ for all $v, v' \in W$; and
(ii) $a(v/W) = (av)/W$ for all $v \in V$ and all $a \in F$.

Show that these operations are well-define and induce on V/W the structure of a vector space over F. Moreover, show that the function from V to V/W given by $\alpha: v \mapsto v/W$ is an epimorphism of vector spaces over F, the kernel of which is W.

39. Let k be a positive integer and let V be a vector space having finite dimension $2k$ over a field F. Show that there exists an isomorphism $\alpha: V \rightarrow V$ satisfying the condition $\alpha^2(v) = -v$ for all $v \in V$.

40. Let V and W be vector spaces over a field F and let $\alpha: V \to W$ be a linear transformation which satisfies the condition that $\alpha\beta\alpha$ is not the 0-function for any linear transformation $\beta: W \to V$ which is not the 0-function. Show that α is an isomorphism.

41. Let F be a field and let $\mathbb{N}$ be the set of nonnegative integers. Let $V = \{f \in F^{\mathbb{N}} \mid f(i) = 0_F \text{ when } i \text{ is even}\}$ and let $W = \{f \in F^{\mathbb{N}} \mid f(i) = 0_F \text{ when } i \text{ is odd}\}$. Show that $V \cong F^{\mathbb{N}} \cong W$.

42. Let F be a field and let $\alpha: F^3 \to F[X]$ be the linear transformation defined by $[a, b, c] \mapsto (a + b)X + (a + c)X^5$. Find the rank and nullity of α.

43. Let W be a subspace of a vector space V over a field F and let W' be a complement of W in V. If $\alpha \in Hom(W, W')$, show that $Y = \{w + \alpha(w) \mid w \in W\}$ is a subspace of V isomorphic to W.

44. Show that there is no vector space over any field which has precisely 15 elements.

45. Let B be a Hamel basis for $\mathbb{R}$ and let $1 \neq a \in \mathbb{R}$. Show that there exists an element $b \in B$ satisfying $ab \notin B$.

46. Let V and W be vector spaces over a field F, Let $\alpha, \beta \in Hom(V, W)$ satisfy the condition that for each $v \in V$ there exists a scalar $c_v \in F$ (depending on v) such that $\beta(v) = c_v\alpha(v)$. Show that there exists a scalar $c \in F$ such that $\beta = c\alpha$.