for evaluating f. Were it true that $kh' = h'k$, however, then we could evaluate $f(aa')$. Consider the commutator $h'kh'^{-1}k^{-1}$. Now $(h'kh'^{-1})k^{-1} \in K$ (for $h'kh'^{-1} \in K$ because K is normal), and, similarly, $h'(kh'^{-1}k^{-1}) \in H$ (because H is normal); therefore, $h'kh'^{-1}k^{-1} \in H \cap K = 1$ and h' and k commute. The reader can now check that f is a homomorphism and a bijection; that is, f is an isomorphism. ∎

We pause to give an example showing that all the hypotheses in Theorem 2.29 are necessary. Let $G = S_3$, $H = \langle(1\ 2\ 3)\rangle$, and $K = \langle(1\ 2)\rangle$. It is easy to see that $HK = G$ and $H \cap K = 1$; moreover, $H \lhd G$ but K is not a normal subgroup. The direct product $H \times K \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ is abelian, and so the nonabelian group $G = S_3$ is not isomorphic to $H \times K$.

Theorem 2.30. *If $A \lhd H$ and $B \lhd K$, then $A \times B \lhd H \times K$ and*

$$(H \times K)/(A \times B) \cong (H/A) \times (K/B).$$

Proof. The homomorphism $\varphi: H \times K \to (H/A) \times (K/B)$, defined by $\varphi(h, k) = (Ah, Bk)$, is surjective and $\ker \varphi = A \times B$. The first isomorphism theorem now gives the result. ∎

It follows, in particular, that if $N \lhd H$, then $N \times 1 \lhd H \times K$.

Corollary 2.31. *If $G = H \times K$, then $G/(H \times 1) \cong K$.*

There are two versions of the direct product $H \times K$: the ***external*** version, whose elements are ordered pairs and which contains isomorphic copies of H and K (namely, $H \times 1$ and $1 \times K$); the ***internal*** version which does contain H and K as normal subgroups and in which $HK = G$ and $H \cap K = 1$. By Theorem 2.29, the two versions are isomorphic. In the future, we shall not distinguish between external and internal; in almost all cases, however, our point of view is internal. For example, we shall write Corollary 2.31 as $(H \times K)/H \cong K$.

Exercises

• 2.66. Prove that $\mathbf{V} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

• 2.67. Show that it is possible for a group G to contain three distinct normal subgroups H, K, and L such that $G = H \times L = K \times L$; that is, $HL = G = KL$ and $H \cap L = 1 = K \cap L$. (*Hint*: Try $G = \mathbf{V}$).

2.68. Prove that an abelian group G of order p^2, where p is a prime, is either cyclic or isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. (We shall see in Corollary 4.5 that every group of order p^2 must be abelian).

• 2.69. Let G be a group with normal subgroups H and K. Prove that $HK = G$ and $H \cap K = 1$ if and only if each $a \in G$ has a unique expression of the form $a = hk$, where $h \in H$ and $k \in K$.

for vector-spaces, rings, and modules. Theorems that describe such an algebraic object in terms of direct products of more describable algebraic objects of the same kind (for example, the case of abelian groups above) are important theorems in general. Through such theorems we can reduce the study of a fairly complex algebraic situation to a much simpler one.

Problems

• 1. If A and B are groups, prove that $A \times B$ is isomorphic to $B \times A$.

2. If G_1, G_2, G_3 are groups, prove that $(G_1 \times G_2) \times G_3$ is isomorphic to $G_1 \times G_2 \times G_3$. Care to generalize?

• 3. If $T = G_1 \times G_2 \times \cdots \times G_n$ prove that for each $i = 1, 2, \ldots, n$ there is a homomorphism ϕ_i of T onto G_i. Find the kernel of ϕ_i.

• 4. Let G be a group and let $T = G \times G$.
   (a) Show that $D = \{(g, g) \in G \times G \mid g \in G\}$ is a group isomorphic to G.
   (b) Prove that D is normal in T if and only if G is abelian.

5. Let G be a finite abelian group. Prove that G is isomorphic to the direct product of its Sylow subgroups.

• 6. Let A, B be cyclic groups of order m and n, respectively. Prove that $A \times B$ is cyclic if and only if m and n are relatively prime.

7. Use the result of Problem 6 to prove the Chinese Remainder Theorem; namely, if m and n are relatively prime integers and u, v any two integers, then we can find an integer x such that $x \equiv u \bmod m$ and $x \equiv v \bmod n$.

8. Give an example of a group G and normal subgroups $N_1, \ldots, N_n$ such that $G = N_1N_2\cdots N_n$ and $N_i \cap N_j = (e)$ for $i \neq j$ and yet G is *not* the internal direct product of $N_1, \ldots, N_n$.

• 9. Prove that G is the internal direct product of the normal subgroups $N_1, \ldots, N_n$ if and only if
   1. $G = N_1 \cdots N_n$.
   2. $N_i \cap (N_1N_2\cdots N_{i-1}N_{i+1}\cdots N_n) = (e)$ for $i = 1, \ldots, n$.

• 10. Let G be a group, $K_1, \ldots, K_n$ normal subgroups of G. Suppose that $K_1 \cap K_2 \cap \cdots \cap K_n = (e)$. Let $V_i = G/K_i$. Prove that there is an isomorphism of G into $V_1 \times V_2 \times \cdots \times V_n$.

* 11. Let G be a finite abelian group such that it contains a subgroup $H_0 \neq (e)$ which lies in *every* subgroup $H \neq (e)$. Prove that G must be cyclic. What can you say about $o(G)$?

12. Let G be a finite abelian group. Using Problem 11 show that G is isomorphic to a subgroup of a direct product of a finite number of finite cyclic groups.