Proof. As we already know, there is a homomorphism θ of $\bar{G}$ onto $\bar{G}/\bar{N}$ defined by $\theta(\bar{g}) = \bar{N}\bar{g}$. We define the mapping $\psi: G \to \bar{G}/\bar{N}$ by $\psi(g) = \bar{N}\phi(g)$ for all $g \in G$. To begin with, ψ is onto, for if $\bar{g} \in \bar{G}$, $\bar{g} = \phi(g)$ for some $g \in G$, since ϕ is onto, so the typical element $\bar{N}\bar{g}$ in $\bar{G}/\bar{N}$ can be represented as $\bar{N}\phi(g) = \psi(g)$.

If $a, b \in G$, $\psi(ab) = \bar{N}\phi(ab)$ by the definition of the mapping ψ. However, since ϕ is a homomorphism, $\phi(ab) = \phi(a)\phi(b)$. Thus $\psi(ab) = \bar{N}\phi(a)\phi(b) = \bar{N}\phi(a)\bar{N}\phi(b) = \psi(a)\psi(b)$. So far we have shown that ψ is a homomorphism of G onto $\bar{G}/\bar{N}$. What is the kernel, T, of ψ? Firstly, if $n \in N$, $\phi(n) \in \bar{N}$, so that $\psi(n) = \bar{N}\phi(n) = \bar{N}$, the identity element of $\bar{G}/\bar{N}$, proving that $N \subset T$. On the other hand, if $t \in T$, $\psi(t) =$ identity element of $\bar{G}/\bar{N} = \bar{N}$; but $\psi(t) = \bar{N}\phi(t)$. Comparing these two evaluations of $\psi(t)$, we arrive at $\bar{N} = \bar{N}\phi(t)$, which forces $\phi(t) \in \bar{N}$; but this places t in N by definition of N. That is, $T \subset N$. The kernel of ψ has been proved to be equal to N. But then ψ is a homomorphism of G onto $\bar{G}/\bar{N}$ with kernel N. By Theorem 2.7.1 $G/N \approx \bar{G}/\bar{N}$, which is the first part of the theorem. The last statement in the theorem is immediate from the observation (following as a consequence of Theorem 2.7.1) that $\bar{G} \approx G/K$, $\bar{N} \approx N/K$, $\bar{G}/\bar{N} \approx (G/K)/(N/K)$.

Problems

- 1. In the following, verify if the mappings defined are homomorphisms, and in those cases in which they are homomorphisms, determine the kernel.
  - (a) G is the group of nonzero real numbers under multiplication, $\bar{G} = G$, $\phi(x) = x^2$ all $x \in G$.
  - (b) G, $\bar{G}$ as in (a), $\phi(x) = 2^x$.
  - (c) G is the group of real numbers under addition, $\bar{G} = G$, $\phi(x) = x + 1$ all $x \in G$.
  - (d) G, $\bar{G}$ as in (c), $\phi(x) = 13x$ for $x \in G$.
  - (e) G is any abelian group, $\bar{G} = G$, $\phi(x) = x^5$ all $x \in G$.
- 2. Let G be any group, g a fixed element in G. Define $\phi: G \to G$ by $\phi(x) = gxg^{-1}$. Prove that ϕ is an isomorphism of G onto G.
- 3. Let G be a finite abelian group of order $o(G)$ and suppose the integer n is relatively prime to $o(G)$. Prove that every $g \in G$ can be written as $g = x^n$ with $x \in G$. (*Hint:* Consider the mapping $\phi: G \to G$ defined by $\phi(y) = y^n$, and prove this mapping is an isomorphism of G onto G.)

4. (a) Given any group G and a subset U, let $\hat{U}$ be the smallest subgroup of G which contains U. Prove there is such a subgroup $\hat{U}$ in G. ($\hat{U}$ is called the *subgroup generated by* U.)