17. Prove Theorem 3.2(a)–(d).

18. Prove Theorem 3.2(e)–(h).

19. Prove Theorem 3.3(c).

20. Prove Theorem 3.3(d).

21. Prove the half of Theorem 3.3(e) that was not proved in the text.

22. Prove that, for square matrices A and B, $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

In Exercises 23–25, if $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *find conditions on a, b, c, and d such that* $AB = BA$.

23. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ 24. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ 25. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

26. Find conditions on a, b, c, and d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

27. Find conditions on a, b, c, and d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with every 2×2 matrix.

28. Prove that if AB and BA are both defined, then AB and BA are both square matrices.

A square matrix is called ***upper triangular*** *if all of the entries below the main diagonal are zero. Thus, the form of an upper triangular matrix is*

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

where the entries marked $*$ *are arbitrary. A more formal definition of such a matrix* $A = [a_{ij}]$ *is that* $a_{ij} = 0$ *if* $i > j$.

● 29. Prove that the product of two upper triangular $n\times n$ matrices is upper triangular.

30. Prove Theorem 3.4(a)–(c).

31. Prove Theorem 3.4(e).

● 32. Using induction, prove that for all $n \geq 1$, $(A_1 + A_2 + \cdots + A_n)^T = A_1^T + A_2^T + \cdots + A_n^T$.

● 33. Using induction, prove that for all $n \geq 1$, $(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$.

34. Prove Theorem 3.5(b).

● 35. (a) Prove that if A and B are symmetric $n\times n$ matrices, then so is $A + B$.
(b) Prove that if A is a symmetric $n\times n$ matrix, then so is kA for any scalar k.

● 36. (a) Give an example to show that if A and B are symmetric $n\times n$ matrices, then AB need not be symmetric.
(b) Prove that if A and B are symmetric $n\times n$ matrices, then AB is symmetric if and only if $AB = BA$.

A square matrix is called ***skew-symmetric*** *if* $A^T = -A$.

● 37. Which of the following matrices are skew-symmetric?

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix}$

● 38. Give a componentwise definition of a skew-symmetric matrix.

● 39. Prove that the main diagonal of a skew-symmetric matrix must consist entirely of zeros.

● 40. Prove that if A and B are skew-symmetric $n\times n$ matrices, then so is $A + B$.

41. If A and B are skew-symmetric 2×2 matrices, under what conditions is AB skew-symmetric?

● 42. Prove that if A is an $n\times n$ matrix, then $A - A^T$ is skew-symmetric.

● 43. (a) Prove that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix. (*Hint:* Consider Theorem 3.5 and Exercise 42).
(b) Illustrate part (a) for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

The ***trace*** *of an* $n\times n$ *matrix* $A = [a_{ij}]$ *is the sum of the entries on its main diagonal and is denoted by* $\text{tr}(A)$. *That is,*

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

● 44. If A and B are $n\times n$ matrices, prove the following properties of the trace:
(a) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
(b) $\text{tr}(kA) = k\text{tr}(A)$, where k is a scalar

● 45. Prove that if A and B are $n\times n$ matrices, then $\text{tr}(AB) = \text{tr}(BA)$.

46. If A is any matrix, to what is $\text{tr}(AA^T)$ equal?

47. Show that there are no 2×2 matrices A and B such that $AB - BA = I_2$.