The preceding ideas can be generalized to relate the matrices $[T]_{C\leftarrow B}$ and $[T]_{C'\leftarrow B'}$ of a linear transformation $T: V \to W$, where $B$ and $B'$ are bases for $V$ and $C$ and $C'$ are bases for $W$. (See Exercise 44.)

We conclude this section by revisiting the Fundamental Theorem of Invertible Matrices and incorporating some results from this chapter.

**Theorem 6.30** **The Fundamental Theorem of Invertible Matrices: Version 4**

Let $A$ be an $n\times n$ matrix and let $T: V \to W$ be a linear transformation whose matrix $[T]_{C\leftarrow B}$ with respect to bases $B$ and $C$ of $V$ and $W$, respectively, is $A$. The following statements are equivalent:

a. $A$ is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of $A$ is $I_n$.
e. $A$ is a product of elementary matrices.
f. $\operatorname{rank}(A) = n$
g. $\operatorname{nullity}(A) = 0$
h. The column vectors of $A$ are linearly independent.
i. The column vectors of $A$ span $\mathbb{R}^n$.
j. The column vectors of $A$ form a basis for $\mathbb{R}^n$.
k. The row vectors of $A$ are linearly independent.
l. The row vectors of $A$ span $\mathbb{R}^n$.
m. The row vectors of $A$ form a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of $A$.
p. $T$ is invertible.
q. $T$ is one-to-one.
r. $T$ is onto.
s. $\ker(T) = \{\mathbf{0}\}$
t. $\operatorname{range}(T) = W$

**Proof** The equivalence (q) ⇔ (s) is Theorem 6.20, and (r) ⇔ (t) is the definition of onto. Since $A$ is $n\times n$, we must have $\dim V = \dim W = n$. From Theorems 6.21 and 6.24, we get (p) ⇔ (q) ⇔ (r). Finally, we connect the last five statements to the others by Theorem 6.28, which implies that (a) ⇔ (p).

## Exercises 6.6

*In Exercises 1–12, find the matrix $[T]_{C\leftarrow B}$ of the linear transformation $T: V \to W$ with respect to the bases $B$ and $C$ of $V$ and $W$, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.*

**1.** $T: \mathscr{P}_1 \to \mathscr{P}_1$ defined by $T(a + bx) = b - ax$, $\mathcal{B} = \mathcal{C} = \{1, x\}$, $\mathbf{v} = p(x) = 4 + 2x$

**2.** $T: \mathscr{P}_1 \to \mathscr{P}_1$ defined by $T(a + bx) = b - ax$, $\mathcal{B} = \{1 + x, 1 - x\}$, $\mathcal{C} = \{1, x\}$, $\mathbf{v} = p(x) = 4 + 2x$

**3.** $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x + 2)$, $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{1, x + 2, (x + 2)^2\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

4. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x+2)$, $\mathcal{B} = \{1, x+2, (x+2)^2\}$, $\mathcal{C} = \{1, x, x^2\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

5. $T: \mathscr{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$, $\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

6. $T: \mathscr{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$, $\mathcal{B} = \{x^2, x, 1\}$, $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\mathbf{v} = p(x) = a + bx + cx^2$

7. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+2b \\ -a \\ b \end{bmatrix}, \quad \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\},$$

$$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

8. Repeat Exercise 7 with $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

9. $T: M_{22} \to M_{22}$ defined by $T(A) = A^T$, $\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

10. Repeat Exercise 9 with $\mathcal{B} = \{E_{22}, E_{21}, E_{12}, E_{11}\}$ and $\mathcal{C} = \{E_{12}, E_{21}, E_{22}, E_{11}\}$.

11. $T: M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

12. $T: M_{22} \to M_{22}$ defined by $T(A) = A - A^T$, $\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, $\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

13. Consider the subspace $W$ of $\mathscr{D}$, given by $W = \text{span}(\sin x, \cos x)$.
   (a) Show that the differential operator $D$ maps $W$ into itself.
   (b) Find the matrix of $D$ with respect to $\mathcal{B} = \{\sin x, \cos x\}$.
   (c) Compute the derivative of $f(x) = 3\sin x - 5\cos x$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

14. Consider the subspace $W$ of $\mathscr{D}$, given by $W = \text{span}(e^{2x}, e^{-2x})$.
   (a) Show that the differential operator $D$ maps $W$ into itself.
   (b) Find the matrix of $D$ with respect to $\mathcal{B} = \{e^{2x}, e^{-2x}\}$.
   (c) Compute the derivative of $f(x) = e^{2x} - 3e^{-2x}$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

15. Consider the subspace $W$ of $\mathscr{D}$, given by $W = \text{span}(e^{2x}, e^{2x}\cos x, e^{2x}\sin x)$.
   (a) Find the matrix of $D$ with respect to $\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.
   (b) Compute the derivative of $f(x) = 3e^{2x} - e^{2x}\cos x + 2e^{2x}\sin x$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

16. Consider the subspace $W$ of $\mathscr{D}$, given by $W = \text{span}(\cos x, \sin x, x\cos x, x\sin x)$.
   (a) Find the matrix of $D$ with respect to $\mathcal{B} = \{\cos x, \sin x, x\cos x, x\sin x\}$.
   (b) Compute the derivative of $f(x) = \cos x + 2x\cos x$ indirectly, using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

*In Exercises 17 and 18, $T: U \to V$ and $S: V \to W$ are linear transformations and $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are bases for $U$, $V$, and $W$, respectively. Compute $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}}$ in two ways: (a) by finding $S \circ T$ directly and then computing its matrix and (b) by finding the matrices of $S$ and $T$ separately and using Theorem 6.27.*

17. $T: \mathscr{P}_1 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$, $S: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - 2b \\ 2a - b \end{bmatrix}$, $\mathcal{B} = \{1, x\}$, $\mathcal{C} = \mathcal{D} = \{\mathbf{e}_1, \mathbf{e}_2\}$

18. $T: \mathscr{P}_1 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x+1)$, $S: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $S(p(x)) = p(x+1)$, $\mathcal{B} = \{1, x\}$, $\mathcal{C} = \mathcal{D} = \{1, x, x^2\}$

*In Exercises 19–26, determine whether the linear transformation $T$ is invertible by considering its matrix with respect to the standard bases. If $T$ is invertible, use Theorem 6.28 and the method of Example 6.82 to find $T^{-1}$.*

19. $T$ in Exercise 1

20. $T$ in Exercise 5

21. $T$ in Exercise 3

22. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p'(x)$

23. $T: \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(p(x)) = p(x) + p'(x)$

**24.** $T: M_{22} \to M_{22}$ defined by $T(A) = AB$, where

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

**25.** $T$ in Exercise 11 **26.** $T$ in Exercise 12

*In Exercises 27–30, use the method of Example 6.83 to evaluate the given integral.*

**27.** $\int (\sin x - 3 \cos x)\,dx$ (See Exercise 13.)

**28.** $\int 5e^{-2x}\,dx$ (See Exercise 14.)

**29.** $\int (e^{2x} \cos x - 2e^{2x} \sin x)\,dx$ (See Exercise 15.)

**30.** $\int (x \cos x + x \sin x)\,dx$ (See Exercise 16.)

*In Exercises 31–36, a linear transformation $T: V \to V$ is given. If possible, find a basis $\mathcal{C}$ for $V$ such that the matrix $[T]_{\mathcal{C}}$ of $T$ with respect to $\mathcal{C}$ is diagonal.*

**31.** $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4b \\ a + 5b \end{bmatrix}$

**32.** $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$

**33.** $T: \mathcal{P}_1 \to \mathcal{P}_1$ defined by $T(a + bx) = (4a + 2b) + (a + 3b)x$

**34.** $T: \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(p(x)) = p(x - 1)$

**35.** $T: \mathcal{P}_1 \to \mathcal{P}_1$ defined by $T(p(x)) = p(x) + xp'(x)$

**36.** $T: \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(p(x)) = p(3x + 2)$

**37.** Let $\ell$ be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Use the method of Example 6.85 to find the standard matrix of a reflection in $\ell$.

**38.** Let $W$ be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$. Use the method of Example 6.85 to find the standard matrix of an orthogonal projection onto $W$. Verify that your answer is correct by using it to compute the orthogonal projection of $\mathbf{v}$ onto $W$, where

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Compare your answer with Example 5.11. [*Hint:* Find an orthogonal decomposition of $\mathbb{R}^3$ as $\mathbb{R}^3 = W + W^{\perp}$ using an orthogonal basis for $W$. See Example 5.3.]

**39.** Let $T: V \to W$ be a linear transformation between finite-dimensional vector spaces and let $\mathcal{B}$ and $\mathcal{C}$ be bases for $V$ and $W$, respectively. Show that the matrix of $T$ with respect to $\mathcal{B}$ and $\mathcal{C}$ is unique. That is, if $A$ is a matrix such that $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for all $\mathbf{v}$ in $V$, then $A = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$. [*Hint:* Find values of $\mathbf{v}$ that will show this, one column at a time.]

*In Exercises 40–45, let $T: V \to W$ be a linear transformation between finite-dimensional vector spaces $V$ and $W$. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $V$ and $W$, respectively, and let $A = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$.*

**40.** Show that nullity($T$) = nullity($A$).

**41.** Show that rank($T$) = rank($A$).

**42.** If $V = W$ and $\mathcal{B} = \mathcal{C}$, show that $T$ is diagonalizable if and only if $A$ is diagonalizable.

**43.** Use the results of this section to give a matrix-based proof of the Rank Theorem (Theorem 6.19).

**44.** If $\mathcal{B}'$ and $\mathcal{C}'$ are also bases for $V$ and $W$, respectively, what is the relationship between $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ and $[T]_{\mathcal{C}'\leftarrow\mathcal{B}'}$? Prove your assertion.

**45.** If dim $V = n$ and dim $W = m$, prove that $\mathcal{L}(V, W) \cong M_{mn}$. (See the exercises for Section 6.4.) [*Hint:* Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $V$ and $W$, respectively. Show that the mapping $\varphi(T) = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$, for $T$ in $\mathcal{L}(V, W)$, defines a linear transformation $\varphi: \mathcal{L}(V, W) \to M_{mn}$ that is an isomorphism.]

**46.** If $V$ is a vector space, then the ***dual space*** of $V$ is the vector space $V^* = \mathcal{L}(V, \mathbb{R})$. Prove that if $V$ is finite-dimensional, then $V^* \cong V$.

# Exercises 6.3

*In Exercises 1–4:*
*(a) Find the coordinate vectors $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ of* **x** *with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, respectively.*
*(b) Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.*
*(c) Use your answer to part (b) to compute $[\mathbf{x}]_{\mathcal{C}}$, and compare your answer with the one found in part (a).*
*(d) Find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{B}$.*
*(e) Use your answers to parts (c) and (d) to compute $[\mathbf{x}]_{\mathcal{B}}$, and compare your answer with the one found in part (a).*

1. $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ in $\mathbb{R}^2$

2. $\mathbf{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \mathcal{C} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$ in $\mathbb{R}^2$

3. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^3$

4. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \mathcal{B} = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^3$

*In Exercises 5–8, follow the instructions for Exercises 1–4 using $p(x)$ instead of* **x**.

5. $p(x) = 2 - x, \mathcal{B} = \{1, x\}, \mathcal{C} = \{x, 1 + x\}$ in $\mathscr{P}_1$
6. $p(x) = 1 + 3x, \mathcal{B} = \{1 + x, 1 - x\}, \mathcal{C} = \{2x, 4\}$ in $\mathscr{P}_1$
7. $p(x) = 1 + x^2, \mathcal{B} = \{1 + x + x^2, x + x^2, x^2\},$
$\mathcal{C} = \{1, x, x^2\}$ in $\mathscr{P}_2$
8. $p(x) = 4 - 2x - x^2, \mathcal{B} = \{x, 1 + x^2, x + x^2\},$
$\mathcal{C} = \{1, 1 + x, x^2\}$ in $\mathscr{P}_2$

*In Exercises 9 and 10, follow the instructions for Exercises 1–4 using A instead of* **x**.

9. $A = \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix}$, $\mathcal{B}$ = the standard basis,
$\mathcal{C} = \left\{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ in $M_{22}$

10. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$
$\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\},$
$\mathcal{C} = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ in $M_{22}$

*In Exercises 11 and 12, follow the instructions for Exercises 1–4 using $f(x)$ instead of* **x**.

11. $f(x) = 2 \sin x - 3 \cos x, \mathcal{B} = \{\sin x + \cos x, \cos x\},$
$\mathcal{C} = \{\sin x, \cos x\}$ in span$(\sin x, \cos x)$
12. $f(x) = \sin x, \mathcal{B} = \{\sin x + \cos x, \cos x\}, \mathcal{C} = \{\cos x - \sin x, \sin x + \cos x\}$ in span$(\sin x, \cos x)$
13. Rotate the $xy$-axes in the plane counterclockwise through an angle $\theta = 60°$ to obtain new $x'y'$-axes. Use the methods of this section to find (a) the $x'y'$-coordinates of the point whose $xy$-coordinates are $(3, 2)$ and (b) the $xy$-coordinates of the point whose $x'y'$-coordinates are $(4, -4)$.
14. Repeat Exercise 13 with $\theta = 135°$.
15. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathbb{R}^2$. If $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is
$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$
find $\mathcal{B}$.
16. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathscr{P}_2$. If $\mathcal{B} = \{x, 1 + x, 1 - x + x^2\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is
$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$
find $\mathcal{C}$.

*In calculus, you learn that a* ***Taylor polynomial of degree n about a*** *is a polynomial of the form*
$$p(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n$$
*where $a_n \neq 0$. In other words, it is a polynomial that has been expanded in terms of powers of $x - a$ instead of powers of $x$. Taylor polynomials are very useful for approximating functions that are "well behaved" near $x = a$.*

• *The set $\mathcal{B} = \{1, x - a, (x - a)^2, \ldots, (x - a)^n\}$ is a basis for $\mathcal{P}_n$ for any real number a. (Do you see a quick way to show this? Try using Theorem 6.7.) This fact allows us to use the techniques of this section to rewrite a polynomial as a Taylor polynomial about a given a.*

**17.** Express $p(x) = 1 + 2x - 5x^2$ as a Taylor polynomial about $a = 1$.

**18.** Express $p(x) = 1 + 2x - 5x^2$ as a Taylor polynomial about $a = -2$.

**19.** Express $p(x) = x^3$ as a Taylor polynomial about $a = -1$.

**20.** Express $p(x) = x^3$ as a Taylor polynomial about $a = \frac{1}{2}$.

• **21.** Let $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be bases for a finite-dimensional vector space $V$. Prove that

$$P_{\mathcal{D}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{B}}$$

• **22.** Let $V$ be an $n$-dimensional vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $P$ be an invertible $n \times n$ matrix and set

$$\mathbf{u}_i = p_{1i}\mathbf{v}_1 + \cdots + p_{ni}\mathbf{v}_n$$

for $i = 1, \ldots, n$. Prove that $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $V$ and show that $P = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

## 6.4 Linear Transformations

We encountered linear transformations in Section 3.6 in the context of matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section, we extend this concept to linear transformations between arbitrary vector spaces.

**Definition** A ***linear transformation*** from a vector space $V$ to a vector space $W$ is a mapping $T: V \to W$ such that, for all $\mathbf{u}$ and $\mathbf{v}$ in $V$ and for all scalars $c$,

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

It is straightforward to show that this definition is equivalent to the requirement that $T$ preserve all linear combinations. That is,

$T: V \to W$ is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$$

for all $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $V$ and scalars $c_1, \ldots, c_k$.

**Example 6.49**

Every matrix transformation is a linear transformation. That is, if $A$ is an $m \times n$ matrix, then the transformation $T_A: \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n$$

is a linear transformation. This is a restatement of Theorem 3.30.