16. If N is a normal subgroup in the finite group such that $i_G(N)$ and $o(N)$ are relatively prime, show that any element $x \in G$ satisfying $x^{o(N)} = e$ must be in N.

17. Let G be defined as all formal symbols $x^i y^j$, $i = 0, 1$, $j = 0, 1, 2, \ldots, n - 1$ where we assume

$$x^i y^j = x^{i'} y^{j'} \text{ if and only if } i = i', j = j'$$

$$x^2 = y^n = e, \quad n > 2$$

$$xy = y^{-1}x.$$

(a) Find the form of the product $(x^i y^j)(x^k y^l)$ as $x^\alpha y^\beta$.
(b) Using this, prove that G is a non-abelian group of order $2n$.
(c) If n is odd, prove that the center of G is (e), while if n is even the center of G is larger than (e).

This group is known as a *dihedral* group. A geometric realization of this is obtained as follows: let y be a rotation of the Euclidean plane about the origin through an angle of $2\pi/n$, and x the reflection about the vertical axis. G is the group of motions of the plane generated by y and x.

18. Let G be a group in which, for some integer $n > 1$, $(ab)^n = a^n b^n$ for all $a, b \in G$. Show that
(a) $G^{(n)} = \{x^n \mid x \in G\}$ is a normal subgroup of G.
(b) $G^{(n-1)} = \{x^{n-1} \mid x \in G\}$ is a normal subgroup of G.

19. Let G be as in Problem 18. Show
(a) $a^{n-1} b^n = b^n a^{n-1}$ for all $a, b \in G$.
(b) $(aba^{-1}b^{-1})^{n(n-1)} = e$ for all $a, b \in G$.

20. Let G be a group such that $(ab)^p = a^p b^p$ for all $a, b \in G$, where p is a prime number. Let $S = \{x \in G \mid x^{p^m} = e$ for some m depending on $x\}$. Prove
(a) S is a normal subgroup of G.
(b) If $\bar{G} = G/S$ and if $\bar{x} \in \bar{G}$ is such that $\bar{x}^p = \bar{e}$ then $\bar{x} = \bar{e}$.

#21. Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad \neq 0$, under matrix multiplication. Let $N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}$. Prove that
(a) N is a normal subgroup of G.
(b) G/N is abelian.

2.7 Homomorphisms

The ideas and results in this section are closely interwoven with those of the preceding one. If there is one central idea which is common to all aspects of modern algebra it is the notion of homomorphism. By this one means