2. Simplify each of the following expressions, expressing your answer in the form of a Gaussian integer $a + bi$.

   a) $(-1+i)^3(1+i)^3$ b) $(3+2i)(3-i)^2$ c) $(2+i)^2(5-i)^3$

3. Determine whether the Gaussian integer α divides the Gaussian integer β if

   a) $\alpha = 2 - i$, $\beta = 5 + 5i$. c) $\alpha = 5$, $\beta = 2 + 3i$.
   b) $\alpha = 1 - i$, $\beta = 8$. d) $\alpha = 3 + 2i$, $\beta = 26$.

4. Determine whether the Gaussian integer α divides the Gaussian integer β, where

   a) $\alpha = 3$, $\beta = 4 + 7i$. c) $\alpha = 5 + 3i$, $\beta = 30 + 6i$.
   b) $\alpha = 2 + i$, $\beta = 15$. d) $\alpha = 11 + 4i$, $\beta = 274$.

5. Give a formula for all Gaussian integers divisible by $4 + 3i$ and display the set of all such Gaussian integers in the plane.

6. Give a formula for all Gaussian integers divisible by $4 - i$ and display the set of all such Gaussian integers in the plane.

7. Show that if α, β, and γ are Gaussian integers and $\alpha \mid \beta$ and $\beta \mid \gamma$, then $\alpha \mid \gamma$.

8. Show that if α, β, γ, μ, and ν are Gaussian integers and $\gamma \mid \alpha$ and $\gamma \mid \beta$, then $\gamma \mid (\mu\alpha + \nu\beta)$.

9. Show that if ϵ is a unit for the Gaussian integers, then $\epsilon^5 = \epsilon$.

10. Find all Gaussian integers $\alpha = a + bi$ such that $\overline{\alpha} = a - bi$, the conjugate of α, is an associate of α.

11. Show that the Gaussian integers α and β are associates if $\alpha \mid \beta$ and $\beta \mid \alpha$.

12. Show that if α and β are Gaussian integers and $\alpha \mid \beta$, then $N(\alpha) \mid N(\beta)$.

13. Suppose that $N(\alpha) \mid N(\beta)$, where α and β are Gaussian integers. Does it necessarily follow that $\alpha \mid \beta$? Supply either a proof or a counterexample.

14. Show that if α divides β, where α and β are Gaussian integers, then $\overline{\alpha}$ divides $\overline{\beta}$.

15. Show that if $\alpha = a + bi$ is a nonzero Gaussian integer, then α has exactly one associate $c + di$ (including α itself), where $c > 0$ and $d \geq 0$.

16. For each pair of values for α and β, find the quotient γ and the remainder ρ when α is divided by β computed following the construction in the proof of Theorem 14.6, and verify that $N(\rho) < N(\beta)$.

   a) $\alpha = 14 + 17i$, $\beta = 2 + 3i$ c) $\alpha = 33$, $\beta = 5 + i$
   b) $\alpha = 7 - 19i$, $\beta = 3 - 4i$

17. For each pair of values for α and β, find the quotient γ and the remainder ρ when α is divided by β computed following the construction in the proof of Theorem 14.6, and verify that $N(\rho) < N(\beta)$.

   a) $\alpha = 24 - 9i$, $\beta = 3 + 3i$ c) $\alpha = 87i$, $\beta = 11 - 2i$
   b) $\alpha = 18 + 15i$, $\beta = 3 + 4i$

18. For each pair of values for α and β in Exercise 16, find a pair of Gaussian integers γ and ρ such that $\alpha = \beta\gamma + \rho$ and $N(\rho) < N(\beta)$ different from that computed following the construction in Theorem 14.6.

19. For each pair of values for α and β in Exercise 17, find a pair of Gaussian integers γ and ρ such that $\alpha = \beta\gamma + \rho$ and $N(\rho) < N(\beta)$ different from that computed following the construction in Theorem 14.6.

20. Show that for every pair of Gaussian integers α and β with $\beta \neq 0$ and $\beta \nmid \alpha$, there are at least two different pairs of Gaussian integers γ and ρ such that $\alpha = \beta\gamma + \rho$ and $N(\rho) < N(\beta)$.

* 21. Determine all possible values for the number of pairs of Gaussian integers γ and ρ such that $\alpha = \beta\gamma + \rho$ and $N(\rho) < N(\beta)$ when α and β are Gaussian integers and $\beta \neq 0$. (*Hint:* Analyze this geometrically by looking at the position of α/β in the square containing it and with four lattice points as its corners.)

22. Show that if a number of the form $r + si$, where r and s are rational numbers, is an algebraic integer, then r and s are integers.

23. Show that $1 + i$ divides a Gaussian integer $a + ib$ if and only if a and b are both even or both odd.

24. Show that if π is a Gaussian prime, then $N(\pi) = 2$ or $N(\pi) \equiv 1 \pmod 4$.

25. Find all Gaussian primes of the form $\alpha^2 + 1$, where α is a Gaussian integer.

26. Show that if $a + bi$ is a Gaussian prime, then $b + ai$ is also a Gaussian prime.

27. Show that the rational prime 7 is also a Gaussian prime by adapting the argument given in Example 14.6 that shows 3 is a Gaussian prime.

28. Show that every rational prime p of the form $4k + 3$ is also a Gaussian prime.

29. Suppose that α is a nonzero Gaussian integer which is neither a unit nor a prime. Show that a Gaussian integer β exists such that $\beta \mid \alpha$ and $1 < N(\beta) \leq \sqrt{N(\alpha)}$.

30. Explain how to adapt the sieve of Eratosthenes to find all the Gaussian primes with norm less than a specified limit.

31. Find all the Gaussian primes with norm less than 100.

32. Display all the Gaussian primes with norm less than 200 as lattice points in the plane.

We can define the notion of congruence for Gaussian integers. Suppose that α, β, and γ are Gaussian integers and that $\gamma \neq 0$. We say that α is *congruent* to β modulo γ and we write $\alpha \equiv \beta \pmod{\gamma}$ if $\gamma \mid (\alpha - \beta)$.

33. Suppose that μ is a nonzero Gaussian integer. Show that each of the following properties holds.
   a) If α is a Gaussian integer, then $\alpha \equiv \alpha \pmod{\mu}$.
   b) If $\alpha \equiv \beta \pmod{\mu}$, then $\beta \equiv \alpha \pmod{\mu}$.
   c) If $\alpha \equiv \beta \pmod{\mu}$ and $\beta \equiv \gamma \pmod{\mu}$, then $\alpha \equiv \gamma \pmod{\mu}$.

34. Suppose that $\alpha \equiv \beta \pmod{\mu}$ and $\gamma \equiv \delta \pmod{\mu}$, where $\alpha, \beta, \gamma, \delta$, and μ are Gaussian integers and $\mu \neq 0$. Show that each of these properties holds.
   a) $\alpha + \gamma \equiv \beta + \delta \pmod{\mu}$
   b) $\alpha - \gamma \equiv \beta - \delta \pmod{\mu}$
   c) $\alpha\gamma \equiv \beta\delta \pmod{\mu}$

35. Show that two Gaussian integers $\alpha = a_1 + ib_1$ and $\beta = a_2 + ib_2$ can multiplied using only three multiplications of rational integers, rather than the four in the equation shown

in the text, together with five additions and subtractions. (*Hint:* One way to do this uses the product $(a_1 + b_1)(a_2 + b_2)$. A second way uses the product $b_2(a_1 + b_1)$.)

36. When a and b are real numbers, let $\{a + bi\} = \{a\} + \{b\}i$, where $\{x\}$ is the closest integer to the real number x, rounding up in the case of a tie. Show that if z is a complex number, no Gaussian integer is closer to z than $\{z\}$ and $N(z - \{z\}) \leq 1/2$.

Let k be a nonnegative integer. The *Gaussian Fibonacci number* G_k is defined in terms of the Fibonacci numbers with $G_k = f_k + i f_{k+1}$. Exercises 37–39 involve Gaussian Fibonacci numbers.

37. a) List the terms of the Gaussian Fibonacci sequence for $k = 0, 1, 2, 3, 4, 5$. (Recall that $f_0 = 0$.)
b) Show that $G_k = G_{k-1} + G_{k-2}$ for $k = 2, 3, \ldots$.

38. Show that $N(G_k) = f_{2k+1}$ for all nonnegative integers k.

39. Show that $G_{n+2}G_{n+1} - G_{n+3}G_n = (-1)^n(2 + i)$, whenever n is a positive integer.

40. Show that every Gaussian integer can be written in the form $a_n(-1 + i)^n + a_{n-1}(-1 + i)^{n-1} + \cdots + a_1(-1 + i) + a_0$, where $a_j = 0$ or 1 for $j = 0, 1, \ldots, n - 1, n$.

41. Show that if α is a number of the form $r + si$, where r and s are rational numbers and α is a root of a monic quadratic polynomial with integer coefficients, then α is a Gaussian integer.

42. What can you conclude if $\pi = a + bi$ is a Gaussian prime and one of the Gaussian integers $(a + 1) + bi$, $(a - 1) + bi$, $a + (b + 1)i$, and $a + (b - 1)i$ is also a Gaussian prime?

43. Show that if $\pi_1 = a - 1 + bi$, $\pi_2 = a + 1 + bi$, $\pi_3 = a + (b - 1)i$, and $\pi_4 = a + (b + 1)i$ are all Gaussian primes and $|a| + |b| > 5$, then 5 divides both a and b and neither a nor b is zero.

44. Describe the block of Gaussian integers containing no Gaussian primes that can be constructed by first forming the product of all Gaussian integers $a + bi$ with a and b rational integers, $0 \leq a \leq m$, and $0 \leq b \leq n$.

45. Find all Gaussian integers α, β, and γ such that $\alpha\beta\gamma = \alpha + \beta + \gamma = 1$.

46. Show that if π is a Gaussian prime with $N(\pi) \neq 2$, then exactly one of the associates of π is congruent to either 1 or $3 + 2i$ modulo 4.

14.1 Computational and Programming Exercises

Computations and Explorations

Using a computation program such as Maple or *Mathematica*, or programs you have written, carry out the following computations and explorations.

1. Find all pairs of Gaussian integers γ and ρ such that $180 - 181i = (12 + 13i)\gamma + \rho$ and $N(\rho) < N(12 + 13i)$.

2. Use a version of the sieve of Eratosthenes to find all Gaussian primes with norm less than 1000.

3. Find as many different pairs of Gaussian primes that differ by 2 as you can.