inversion formula is that we can turn this statement around. That is, if the summatory function F of an arithmetic function f is multiplicative, then so is f.

Theorem 7.17. Let f be an arithmetic function with summatory $F = \sum_{d|n} f(d)$. Then, if F is multiplicative, f is also multiplicative.

Proof. Suppose that m and n are relatively prime positive integers. We want to show that $f(mn) = f(m)f(n)$. To show this, first note that by Lemma 3.7, if d is a divisor of mn, then $d = d_1d_2$ where $d_1 \mid m$, $d_2 \mid n$, and $(d_1, d_2) = 1$. Using the Möbius inversion formula and the fact that μ and F are multiplicative, we see that

$$\begin{aligned} f(mn) &= \sum_{d|mn} \mu(d) F\left(\frac{mn}{d}\right) \\ &= \sum_{d_1|m,\, d_2|n} \mu(d_1d_2) F\left(\frac{mn}{d_1d_2}\right) \\ &= \sum_{d_1|m,\, d_2|n} \mu(d_1)\mu(d_2) F\left(\frac{m}{d_1}\right) F\left(\frac{n}{d_2}\right) \\ &= \sum_{d_1|m} \mu(d_1) F\left(\frac{m}{d_1}\right) \cdot \sum_{d_2|n} \mu(d_2) F\left(\frac{n}{d_2}\right) \\ &= f(m)f(n). \end{aligned}$$

■

7.4 Exercises

1. Find the following values of the Möbius function.

   a) $\mu(12)$ b) $\mu(15)$ c) $\mu(30)$ d) $\mu(50)$ e) $\mu(1001)$ f) $\mu(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13)$ g) $\mu(10!)$

2. Find the following values of the Möbius function.

   a) $\mu(33)$ b) $\mu(105)$ c) $\mu(110)$ d) $\mu(740)$ e) $\mu(999)$ f) $\mu(3 \cdot 7 \cdot 13 \cdot 19 \cdot 23)$ g) $\mu(10!/(5!)^2)$

3. Find the value of $\mu(n)$ for each integer n with $100 \le n \le 110$.
4. Find the value of $\mu(n)$ for each integer n with $1000 \le n \le 1010$.
5. Find all integers n, $1 \le n \le 100$ with $\mu(n) = 1$.
6. Find all composite integers n, $100 \le n \le 200$ with $\mu(n) = -1$.

The *Mertens function* $M(n)$ is defined by $M(n) = \sum_{i=1}^{n} \mu(i)$.

7. Find $M(n)$ for all positive integers not exceeding 10.
8. Find $M(n)$ for $n = 100$.

9. Show that $M(n)$ is the difference between the number of square-free positive integers not exceeding n with an even number of prime divisors and those with an odd number of prime divisors.

10. Show that if n is a positive integer, then $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

11. Prove or disprove that there are infinitely many positive integers n such that $\mu(n) + \mu(n+1) = 0$.

12. Prove or disprove that there are infinitely many positive integers n such that $\mu(n-1) + \mu(n) + \mu(n+1) = 0$.

13. For how many consecutive integers can the Möbius function $\mu(n)$ take a nonzero value?

14. For how many consecutive integers can the Möbius function $\mu(n)$ take the value 0?

15. Show that if n is a positive integer, then $\phi(n) = n \sum_{d|n} \mu(d)/d$. (*Hint:* Use the Möbius inversion formula.)

16. Use the Möbius inversion formula and the identity $n = \sum_{d|n} \phi(n/d)$, demonstrated in Section 7.1, to show the following.
    a) $\phi(p^t) = p^t - p^{t-1}$, whenever p is prime and t is a positive integer.
    b) $\phi(n)$ is multiplicative.

17. Suppose that f is a multiplicative function with $f(1) = 1$. Show that

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_k)),$$

where $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ is the prime-power factorization of n.

18. Use Exercise 17 to find a simple formula for $\sum_{d|n} d\mu(d)$ for all positive integers n.

19. Use Exercise 17 to find a simple formula for $\sum_{d|n} \mu(d)/d$ for all positive integers n.

20. Use Exercise 17 to find a simple formula for $\sum_{d|n} \mu(d)\tau(d)$ for all positive integers n.

21. Use Exercise 17 to find a simple formula for $\sum_{d|n} \mu(d)\sigma(d)$ for all positive integers n.

22. Let n be a positive integer. Show that

$$\prod_{d|n} \mu(d) = \begin{cases} -1 & \text{if } n \text{ is a prime;} \\ 0 & \text{if } n \text{ has a square factor;} \\ 1 & \text{if } n \text{ is square-free and composite.} \end{cases}$$

23. Show that

$$\sum_{d|n} \mu^2(d) = 2^{\omega(n)},$$

where $\omega(n)$ denotes the number of distinct prime factors of n.

24. Use Exercise 23 and the Möbius inversion formula to show that

$$\mu^2(n) = \sum_{d|n} \mu(d) 2^{\omega(n/d)}.$$

25. Show that $\sum_{d|n} \mu(d)\lambda(d) = 2^{\omega(n)}$ for all positive integers n, where $\omega(n)$ is the number of distinct prime factors of n. (See the preamble to Exercise 43 in Section 7.1 for a definition of $\lambda(n)$.)

26. Show that $\sum_{d|n} \lambda(n/d)2^{\omega(d)} = 1$ for all positive integers n.

Exercises 27–29 provide a proof of the Möbius inversion formula and Theorem 7.17 using the concepts of the Dirichlet product and the Dirichlet inverse, defined in the exercise set of Section 7.1.

27. Show that the Möbius function $\mu(n)$ is the Dirichlet inverse of the function $\nu(n) = 1$.

28. Use Exercise 38 in Section 7.1 and Exercise 27 to prove the Möbius inversion formula.

29. Prove Theorem 7.17 by noting that if $F = f \star \nu$, where $\nu = 1$ for all positive integers n, then $f = F \star \mu$.

The *Mangoldt function* Λ is defined for all positive integers n by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \text{ where } p \text{ is prime and } k \text{ is a positive integer;} \\ 0 & \text{otherwise.} \end{cases}$$

30. Show that $\sum_{d|n} \Lambda(d) = \log n$ whenever n is a positive integer.

31. Use the Möbius inversion formula and Exercise 30 to show that

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

7.4 Computational and Programming Exercises

Computations and Explorations

Using a computation program such as Maple or *Mathematica*, or programs you have written, carry out the following computations and explorations.

1. Find $\mu(n)$ for each of the following values of n.

   a) 421,602,180,943 b) 186,728,732,190 c) 737,842,183,177

2. Find $M(n)$, the value of the Mertens function at n, for each of the following integers. (See the preamble to Exercise 7 for the definition of $M(n)$.)

   a) 1000 b) 10,000 c) 100,000

3. A famous conjecture made in 1897 by F. Mertens, and disproved in 1985 by A. Odlyzko and H. te Riele (in [Odte85]), was that $|M(n)| < \sqrt{n}$ for all positive integers n, where $M(n)$ is the Mertens function. Show that this conjecture, called Mertens' conjecture, is true for all integers n for as large a range as you can. Do not expect to find a counterexample, because the smallest n for which the conjecture is false is fantastically large. What is known is that there is a counterexample less than $3.21 \cdot 10^{64}$. Before the conjecture was shown to be false, it had been checked by computer for all integers n up to 10^{10}. This shows that even a tremendous amount of evidence can be misleading, because the smallest counterexample to a conjecture can nevertheless be titanically large.