the equation $\phi(n) = 8$ implies that no prime exceeding 9 divides n (otherwise $\phi(n) > p_j - 1 > 8$). Furthermore, 7 cannot divide n because if it did, $7 - 1 = 6$ would be a factor of $\phi(n)$. It follows that $n = 2^a 3^b 5^c$, where a, b, and c are nonnegative integers. We can also conclude that $b = 0$ or $b = 1$ and that $c = 0$ or $c = 1$; otherwise, 3 or 5 would divide $\phi(n) = 8$.

To find all solutions we need only consider four cases. When $b = c = 0$, we have $n = 2^a$, where $a \geq 1$. This implies that $\phi(n) = 2^{a-1}$, which means that $a = 4$ and $n = 16$. When $b = 0$ and $c = 1$, we have $n = 2^a \cdot 5$, where $a \geq 1$. This implies that $\phi(n) = 2^{a-1} \cdot 4$, so $a = 2$ and $n = 20$. When $b = 1$ and $c = 0$, we have $n = 2^a \cdot 3$, where $a \geq 1$. This implies that $\phi(n) = 2^{a-1} \cdot 2 = 2^a$, so $a = 3$ and $n = 24$. Finally, when $b = 1$ and $c = 1$, we have $n = 2^a \cdot 3 \cdot 5$. We need to consider the case where $a = 0$, as well as the case where $a \geq 1$. When $a = 0$, we have $n = 15$, which is a solution because $\phi(15) = 8$. When $a \geq 1$, we have $\phi(n) = 2^{a-1} \cdot 2 \cdot 4 = 2^{a+2}$. This means that $a = 1$ and $n = 30$. Putting everything together, we see that all the solutions to $\phi(n) = 8$ are $n = 15, 16, 20, 24$ and 30. ◄

7.1 Exercises

1. Determine whether each of the following arithmetic functions is completely multiplicative. Prove your answers.

a) $f(n) = 0$	d) $f(n) = \log n$	g) $f(n) = n + 1$
b) $f(n) = 2$	e) $f(n) = n^2$	h) $f(n) = n^n$
c) $f(n) = n/2$	f) $f(n) = n!$	i) $f(n) = \sqrt{n}$

2. Find the value of the Euler phi-function at each of the following integers.

a) 100	d) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
b) 256	e) 10!
c) 1001	f) 20!

3. Show that $\phi(5186) = \phi(5187) = \phi(5188)$.

4. Find all positive integers n such that $\phi(n)$ has each of the following values. Be sure to prove that you have found all solutions.

   a) 1 b) 2 c) 3 d) 4

5. Find all positive integers n such that $\phi(n) = 6$. Be sure to prove that you have found all solutions.

6. Find all positive integers n such that $\phi(n) = 12$. Be sure to prove that you have found all solutions.

7. Find all positive integers n such that $\phi(n) = 24$. Be sure to prove that you have found all solutions.

8. Show that there is no positive integer n such that $\phi(n) = 14$.

9. Can you find a rule involving the Euler phi-function for producing the terms of the sequence $1, 2, 2, 4, 4, 4, 6, 8, 6, \ldots$?

10. Can you find a rule involving the Euler phi-function for producing the terms of the sequence $2, 3, 0, 4, 0, 4, 0, 5, 0, \ldots$?

11. For which positive integers n does $\phi(3n) = 3\phi(n)$?

12. For which positive integers n is $\phi(n)$ divisible by 4?

13. For which positive integers n is $\phi(n)$ equal to $n/2$?

14. For which positive integers n does $\phi(n) \mid n$?

15. Show that if n is a positive integer, then

$$\phi(2n) = \begin{cases} \phi(n) & \text{if } n \text{ is odd;} \\ 2\phi(n) & \text{if } n \text{ is even.} \end{cases}$$

16. Show that if n is a positive integer having k distinct odd prime divisors, then $\phi(n)$ is divisible by 2^k.

17. For which positive integers n is $\phi(n)$ a power of 2?

18. Show that if n is an odd integer, then $\phi(4n) = 2\phi(n)$.

19. Show that if $n = 2\phi(n)$, where n is a positive integer, then $n = 2^j$ for some positive integer j.

20. Let p be prime. Show that $p \nmid n$, where n is a positive integer, if and only if $\phi(np) = (p-1)\phi(n)$.

21. Show that if m and n are positive integers and $(m, n) = p$, where p is prime, then $\phi(mn) = p\phi(m)\phi(n)/(p-1)$.

22. Show that if m and k are positive integers, then $\phi(m^k) = m^{k-1}\phi(m)$.

23. Show that if a and b are positive integers, then

$$\phi(ab) = (a, b)\phi(a)\phi(b)/\phi((a, b)).$$

Conclude that $\phi(ab) > \phi(a)\phi(b)$ when $(a, b) > 1$.

24. Find the least positive integer n such that the following hold.

a) $\phi(n) \geq 100$
b) $\phi(n) \geq 1000$
c) $\phi(n) \geq 10{,}000$
d) $\phi(n) \geq 100{,}000$

25. Use the Euler phi-function to show that there are infinitely many primes. (*Hint:* Assume there are only a finite number of primes $p_1, \ldots, p_k$. Consider the value of the Euler phi-function at the product of these primes.)

26. Show that if the equation $\phi(n) = k$, where k is a positive integer, has exactly one solution n, then $36 \mid n$.

27. Show that the equation $\phi(n) = k$, where k is a positive integer, has finitely many solutions in integers n whenever k is a positive integer.

28. Show that if p is prime, $2^a p + 1$ is composite for $a = 1, 2, \ldots, r$ and p is not a Fermat prime, where r is a positive integer, then $\phi(n) = 2^r p$ has no solution.

* 29. Show that there are infinitely many positive integers k such that the equation $\phi(n) = k$ has exactly two solutions, where n is a positive integer. (*Hint:* Take $k = 2 \cdot 3^{6j+1}$, where $j = 1, 2, \ldots$.)

30. Show that if n is a positive integer with $n \neq 2$ and $n \neq 6$, then $\phi(n) \geq \sqrt{n}$.

* 31. Show that if n is a composite positive integer and $\phi(n) \mid n - 1$, then n is square-free and is the product of at least three distinct primes.

32. Show that if m and n are positive integers with $m \mid n$, then $\phi(m) \mid \phi(n)$.

* 33. Prove Theorem 7.5, using the principle of inclusion-exclusion (see Exercise 16 of Appendix B).

34. Show that a positive integer n is composite if and only if $\phi(n) \leq n - \sqrt{n}$.

35. Let n be a positive integer. Define the sequence of positive integers $n_1, n_2, n_3, \ldots$ recursively by $n_1 = \phi(n)$ and $n_{k+1} = \phi(n_k)$ for $k = 1, 2, 3, \ldots$. Show that there is a positive integer r such that $n_r = 1$.

A multiplicative function is called *strongly multiplicative* if and only if $f(p^k) = f(p)$ for every prime p and every positive integer k.

36. Show that $f(n) = \phi(n)/n$ is a strongly multiplicative function.

Two arithmetic functions f and g may be multiplied using the *Dirichlet product*, which is defined by

$$(f * g)(n) = \sum_{d \mid n} f(d)g(n/d).$$

37. Show that $f * g = g * f$.

38. Show that $(f * g) * h = f * (g * h)$.

We define the ι *function* by

$$\iota(n) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n > 1. \end{cases}$$

39. a) Show that ι is a multiplicative function.
    b) Show that $\iota * f = f * \iota = f$ for all arithmetic functions f.

40. The arithmetic function g is said to be the *inverse* of the arithmetic function f if $f * g = g * f = \iota$. Show that the arithmetic function f has an *inverse* if and only if $f(1) \neq 0$. Show that if f has an inverse it is unique. (*Hint:* When $f(1) \neq 0$, find the inverse f^{-1} of f by calculating $f^{-1}(n)$ recursively, using the fact that $\iota(n) = \sum_{d \mid n} f(d) f^{-1}(n/d)$.)

41. Show that if f and g are multiplicative functions, then the Dirichlet product $f * g$ is also multiplicative.

42. Show that if f and g are arithmetic functions, $F = f * g$, and h is the Dirichlet inverse of g, then $f = F * h$.

We define *Liouville's function* $\lambda(n)$, named after French mathematician *Joseph Liouville*, by $\lambda(1) = 1$, and for $n > 1$, $\lambda(n) = (-1)^{a_1 + a_2 + \cdots + a_m}$, where the prime-power factorization of n is $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$.

43. Find $\lambda(n)$ for each of the following values of n.

a) 12	c) 210	e) 1001	g) 20!
b) 20	d) 1000	f) 10!	

44. Show that $\lambda(n)$ is completely multiplicative.

45. Show that if n is a positive integer, then $\sum_{d|n} \lambda(d)$ equals 0 if n is not a perfect square, and equals 1 if n is a perfect square.

46. Show that if f and g are multiplicative functions, then fg is also multiplicative, where $(fg)(n) = f(n)g(n)$ for every positive integer n.

47. Show that if f and g are completely multiplicative functions, then fg is also completely multiplicative.

48. Show that if f is completely multiplicative, then $f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \cdots f(p_m)^{a_m}$, where the prime-power factorization of n is $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$.

A function f that satisfies the equation $f(mn) = f(m) + f(n)$ for all relatively prime positive integers m and n is called *additive*, and if the above equation holds for all positive integers m and n, f is called *completely additive.*

49. Show that the function $f(n) = \log n$ is completely additive.

The function $\omega(n)$ is the function that denotes the number of distinct prime factors of the positive integer n.

50. Find $\omega(n)$ for each of the following integers.

a) 1 b) 2 c) 20 d) 84 e) 128

JOSEPH LIOUVILLE (1809–1882), born in Saint-Omer, France, was the son of a captain in Napoleon's army. He studied mathematics at the Collège St. Louis in Paris, and in 1825 he enrolled in the École Polytechnique; after graduating, he entered the École des Ponts et Chaussées (School of Bridges and Roads). Health problems while working on engineering projects and his interest in theoretical topics convinced him to pursue an academic career. He left the École des Ponts et Chaussées in 1830, but during his time there he wrote papers on electrodynamics, the theory of heat, and partial differential equations.

Liouville's first academic appointment was as an assistant at the École Polytechnique in 1831. He had a teaching load of around 40 hours a week at several different institutions. Some of his less able students complained that he lectured at too high a level. In 1836, Liouville founded the *Journal de Mathématiques Pures et Appliquées*, which played an important role in French mathematics in the nineteenth century. In 1837, he was appointed to lecture at the Collège de France and the following year he was appointed Professor at the École Polytechnique. Besides his academic interests, Liouville was also involved in politics. He was elected to Constituting Assembly in 1848 as a moderate republican, but lost in the election of 1849, embittering him. Liouville was appointed to a chair at the Collège de France in 1851, and the chair of mechanics at the Faculté des Sciences in 1857. Around this time, his heavy teaching load began to take its toll. Liouville was a perfectionist and was unhappy when he could not devote sufficient time to his lectures.

Liouville's work covered many diverse areas of mathematics, including mathematical physics, astronomy, and many areas of pure mathematics. He was the first person to provide an explicit example of a transcendental number. He is also known today for what is now called Sturm-Liouville theory, used in the solution of integral equations, and he made important contributions to differential geometry. His total output exceeds 400 papers in the mathematical sciences, with nearly half of those in number theory alone.