3.7 Exercises

1. For each of the following linear diophantine equations, either find all solutions, or show that there are no integral solutions.
   a) $2x + 5y = 11$
   b) $17x + 13y = 100$
   c) $21x + 14y = 147$
   d) $60x + 18y = 97$
   e) $1402x + 1969y = 1$

2. For each of the following linear diophantine equations, either find all solutions, or show that there are no integral solutions.
   a) $3x + 4y = 7$
   b) $12x + 18y = 50$
   c) $30x + 47y = -11$
   d) $25x + 95y = 970$
   e) $102x + 1001y = 1$

3. A Japanese businessman returning home from a trip to North America exchanges his U.S. and Canadian dollars for yen. If he receives 15,286 yen, and received 122 yen for each U.S. and 112 yen for each Canadian dollar, how many of each type of currency did he exchange?

4. A student returning from Europe changes his euros and Swiss francs into U.S. money. If she receives \$46.26, and received \$1.11 for each euro and 83¢ for each Swiss franc, how much of each type of currency did she exchange?

5. A professor returning home from conferences in Paris and London changes his euros and pounds into U.S. money. If he receives \$117.98, and received \$1.11 for each euro and \$1.69 for each pound, how much of each type of currency did he exchange?

6. The Indian astronomer and mathematician Mahavira, who lived in the ninth century, posed this puzzle: A band of 23 weary travelers entered a lush forest where they found 63 piles each containing the same number of plantains and a remaining pile containing seven plantains. They divided the plantains equally. How many plantains were in each of the 63 piles? Solve this puzzle.

7. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost him 25¢ each and oranges cost him 18¢ each, how many of each type of fruit did he order?

8. A shopper spends a total of \$5.49 for oranges, which cost 18¢ each, and grapefruit, which cost 33¢ each. What is the minimum number of pieces of fruit the shopper could have bought?

9. A postal clerk has only 14- and 21-cent stamps to sell. What combinations of these may be used to mail a package requiring postage of exactly each of the following amounts?

   a) \$3.50　　b) \$4.00　　c) \$7.77

10. At a clambake, the total cost of a lobster dinner is \$11 and of a chicken dinner is \$8. What can you conclude if the total bill is each of the following amounts?

    a) \$777　　b) \$96　　c) \$69

Hence, $1 = 5 - 2 \cdot 2 = 5 - (7 - 5 \cdot 1) \cdot 2 = 5 \cdot 3 - 2 \cdot 7 = (12 - 7 \cdot 1) \cdot 3 - 2 \cdot 7 = 12 \cdot 3 - 5 \cdot 7$. Therefore, a particular solution to the linear diophantine equation is $x_0 = -20$ and $y_0 = 12$. Hence, all solutions of the linear congruences are given by $x \equiv -20 \equiv 4 \pmod{12}$. ◀

Later we will want to know which integers are their own inverses modulo p, where p is prime. The following theorem tells us which integers have this property.

Theorem 4.11. Let p be prime. The positive integer a is its own inverse modulo p if and only if $a \equiv 1 \pmod{p}$ or $a \equiv -1 \pmod{p}$.

Proof. If $a \equiv 1 \pmod{p}$ or $a \equiv -1 \pmod{p}$, then $a^2 \equiv 1 \pmod{p}$, so that a is its own inverse modulo p.

Conversely, if a is its own inverse modulo p, then $a^2 = a \cdot a \equiv 1 \pmod{p}$. Hence, $p \mid (a^2 - 1)$. Since $a^2 - 1 = (a-1)(a+1)$, either $p \mid (a-1)$ or $p \mid (a+1)$. Therefore, either $a \equiv 1 \pmod{p}$ or $a \equiv -1 \pmod{p}$. ■

4.2 Exercises

1. Find all solutions of each of the following linear congruences.

   a) $2x \equiv 5 \pmod 7$
   b) $3x \equiv 6 \pmod 9$
   c) $19x \equiv 30 \pmod{40}$
   d) $9x \equiv 5 \pmod{25}$
   e) $103x \equiv 444 \pmod{999}$
   f) $980x \equiv 1500 \pmod{1600}$

2. Find all solutions of each of the following linear congruences.

   a) $3x \equiv 2 \pmod 7$
   b) $6x \equiv 3 \pmod 9$
   c) $17x \equiv 14 \pmod{21}$
   d) $15x \equiv 9 \pmod{25}$
   e) $128x \equiv 833 \pmod{1001}$
   f) $987x \equiv 610 \pmod{1597}$

3. Find all solutions to the congruence $6{,}789{,}783x \equiv 2{,}474{,}010 \pmod{28{,}927{,}591}$.

4. Suppose that p is prime and that a and b are positive integers with $(p, a) = 1$. The following method can be used to solve the linear congruence $ax \equiv b \pmod{p}$.

   a) Show that if the integer x is a solution of $ax \equiv b \pmod{p}$, then x is also a solution of the linear congruence

   $$a_1 x \equiv -b[m/a] \pmod{p},$$

   where a_1 is the least positive residue of p modulo a. Note that this congruence is of the same type as the original congruence, with a positive integer smaller than a as the coefficient of x.

   b) When the procedure of part (a) is iterated, one obtains a sequence of linear congruences with coefficients of x equal to $a_0 = a > a_1 > a_2 > \cdots$. Show that there is a positive integer n with $a_n = 1$, so that at the nth stage, one obtains a linear congruence $x \equiv B \pmod{p}$.

   c) Use the method described in part (b) to solve the linear congruence $6x \equiv 7 \pmod{23}$.

4. Find all the solutions of each of the following systems of linear congruences.

a) $x \equiv 4 \pmod{11}$
$x \equiv 3 \pmod{17}$

b) $x \equiv 1 \pmod{2}$
$x \equiv 2 \pmod{3}$
$x \equiv 3 \pmod{5}$

c) $x \equiv 0 \pmod{2}$
$x \equiv 0 \pmod{3}$
$x \equiv 1 \pmod{5}$
$x \equiv 6 \pmod{7}$

d) $x \equiv 2 \pmod{11}$
$x \equiv 3 \pmod{12}$
$x \equiv 4 \pmod{13}$
$x \equiv 5 \pmod{17}$
$x \equiv 6 \pmod{19}$

5. Find all the solutions to the system of linear congruences $x \equiv 1 \pmod{2}$, $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$, $x \equiv 4 \pmod{7}$, and $x \equiv 5 \pmod{11}$.

6. Find all the solutions to the system of linear congruences $x \equiv 1 \pmod{999}$, $x \equiv 2 \pmod{1001}$, $x \equiv 3 \pmod{1003}$, $x \equiv 4 \pmod{1004}$, and $x \equiv 5 \pmod{1007}$.

7. A troop of 17 monkeys store their bananas in 11 piles of equal size, each containing more than 1 banana, with a twelfth pile of 6 left over. When they divide the bananas into 17 equal groups, none remain. What is the smallest number of bananas they can have?

8. As an odometer check, a special counter measures the miles a car travels modulo 7. Explain how this counter can be used to determine whether the car has been driven 49,335; 149,335; or 249,335 miles when the odometer reads 49,335 and works modulo 100,000.

9. Chinese generals counted troops remaining after a battle by lining them up in rows of different lengths, counting the number left over each time, and calculating the total from these remainders. If a general had 1200 troops at the start of a battle and if there were 3 left over when they lined up 5 at a time, 3 left over when they lined up 6 at a time, 1 left over when they lined up 7 at a time, and none left over when they lined up 11 at a time, how many troops remained after the battle?

10. Find an integer that leaves a remainder of 9 when it is divided by either 10 or 11, but that is divisible by 13.

11. Find a multiple of 11 that leaves a remainder of 1 when divided by each of the integers 2, 3, 5, and 7.

12. Solve the following ancient Indian problem: If eggs are removed from a basket 2, 3, 4, 5, and 6 at a time, there remain, respectively, 1, 2, 3, 4, and 5 eggs. But if the eggs are removed 7 at a time, no eggs remain. What is the least number of eggs that could have been in the basket?

13. Show that there are arbitrarily long strings of consecutive integers each divisible by a perfect square greater than 1. (*Hint:* Use the Chinese remainder theorem to show that there is a simultaneous solution to the system of congruences $x \equiv 0 \pmod{4}$, $x \equiv -1 \pmod{9}$, $x \equiv -2 \pmod{25}$, $\ldots$, $x \equiv -k+1 \pmod{p_k^2}$, where p_k is the kth prime.)

* 14. Show that if a, b, and c are integers such that $(a, b) = 1$, then there is an integer n such that $(an + b, c) = 1$.

In Exercises 15–18, we will consider systems of congruences where the moduli of the congruences are not necessarily relatively prime.

21. What is the smallest number of lobsters in a tank if 1 lobster is left over when they are removed 2, 3, 5, or 7 at a time, but no lobsters are left over when they are removed 11 at a time?

22. An ancient Chinese problem asks for the least number of gold coins a band of 17 pirates could have stolen. The problem states that when the pirates divided the coins into equal piles, 3 coins were left over. When they fought over who should get the extra coins, one of the pirates was slain. When the remaining pirates divided the coins into equal piles, 10 coins were left over. When the pirates fought again over who should get the extra coins, another pirate was slain. When they divided the coins in equal piles again, no coins were left over. What is the answer to this problem?

23. Solve the following problem originally posed by Ch'in Chiu-Shao (using different weight units). Three farmers equally divide a quantity of rice with a weight that is an integral number of pounds. The farmers each sell their rice, selling as much as possible, at three different markets where the markets use weights of 83 pounds, 110 pounds, and 135 pounds, and only buy rice in multiples of these weights. What is the least amount of rice the farmers could have divided if the farmers return home with 32 pounds, 70 pounds, and 30 pounds, respectively?

24. Using the Chinese remainder theorem, explain how to add and how to multiply 784 and 813 on a computer of word size 100.

A positive integer $x \neq 1$ with n base b digits is called an *automorph to the base b* if the last n base b digits of x^2 are the same as those of x.

* 25. Find the base 10 automorphs with four digits (with initial zeros allowed).

* 26. How many base b automorphs are there with n or fewer base b digits, if b has prime-power factorization $b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$?

According to the theory of *biorhythms*, there are three cycles in your life that start the day you are born. These are the *physical*, *emotional*, and *intellectual cycles*, of lengths 23, 28, and 33 days, respectively. Each cycle follows a sine curve with period equal to the length of that cycle, starting with value 0, climbing to value 1 one-quarter of the way through the cycle, dropping back to value 0 one-half of the way through the cycle, dropping further to value -1 three-quarters of the way through the cycle, and climbing back to value 0 at the end of the cycle.

Answer the following questions about biorhythms, measuring time in quarter days (so that the units will be integers).

27. For which days of your life will you be at a triple peak, where all of your three cycles are at maximum values?

28. For which days of your life will you be at a triple nadir, where all three of your cycles have minimum values?

29. When in your life will all three cycles be at a neutral position (value 0)?

A set of congruences to distinct moduli greater than 1 that has the property that every integer satisfies at least one of the congruences is called a *covering set of congruences.*

30. Show that the set of congruences $x \equiv 0 \pmod 2$, $x \equiv 0 \pmod 3$, $x \equiv 1 \pmod 4$, $x \equiv 1 \pmod 6$, and $x \equiv 11 \pmod{12}$ is a covering set of congruences.