3.7 Exercises

1. For each of the following linear diophantine equations, either find all solutions, or show that there are no integral solutions.
   a) $2x + 5y = 11$
   b) $17x + 13y = 100$
   c) $21x + 14y = 147$
   d) $60x + 18y = 97$
   e) $1402x + 1969y = 1$

2. For each of the following linear diophantine equations, either find all solutions, or show that there are no integral solutions.
   a) $3x + 4y = 7$
   b) $12x + 18y = 50$
   c) $30x + 47y = -11$
   d) $25x + 95y = 970$
   e) $102x + 1001y = 1$

3. A Japanese businessman returning home from a trip to North America exchanges his U.S. and Canadian dollars for yen. If he receives 15,286 yen, and received 122 yen for each U.S. and 112 yen for each Canadian dollar, how many of each type of currency did he exchange?

4. A student returning from Europe changes his euros and Swiss francs into U.S. money. If she receives \$46.26, and received \$1.11 for each euro and 83¢ for each Swiss franc, how much of each type of currency did she exchange?

5. A professor returning home from conferences in Paris and London changes his euros and pounds into U.S. money. If he receives \$117.98, and received \$1.11 for each euro and \$1.69 for each pound, how much of each type of currency did he exchange?

6. The Indian astronomer and mathematician Mahavira, who lived in the ninth century, posed this puzzle: A band of 23 weary travelers entered a lush forest where they found 63 piles each containing the same number of plantains and a remaining pile containing seven plantains. They divided the plantains equally. How many plantains were in each of the 63 piles? Solve this puzzle.

7. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost him 25¢ each and oranges cost him 18¢ each, how many of each type of fruit did he order?

8. A shopper spends a total of \$5.49 for oranges, which cost 18¢ each, and grapefruit, which cost 33¢ each. What is the minimum number of pieces of fruit the shopper could have bought?

9. A postal clerk has only 14- and 21-cent stamps to sell. What combinations of these may be used to mail a package requiring postage of exactly each of the following amounts?

   a) \$3.50 b) \$4.00 c) \$7.77

10. At a clambake, the total cost of a lobster dinner is \$11 and of a chicken dinner is \$8. What can you conclude if the total bill is each of the following amounts?

    a) \$777 b) \$96 c) \$69

* 11. Find all integer solutions of each of the following linear diophantine equations.
   a) $2x + 3y + 4z = 5$
   b) $7x + 21y + 35z = 8$
   c) $101x + 102y + 103z = 1$

* 12. Find all integer solutions of each of the following linear diophantine equations.
   a) $2x_1 + 5x_2 + 4x_3 + 3x_4 = 5$
   b) $12x_1 + 21x_2 + 9x_3 + 15x_4 = 9$
   c) $15x_1 + 6x_2 + 10x_3 + 21x_4 + 35x_5 = 1$

13. Which combinations of pennies, dimes, and quarters have a total value of 99¢?

14. How many ways can change be made for one dollar, using each of the following coins?
   a) dimes and quarters
   b) nickels, dimes, and quarters
   c) pennies, nickels, dimes, and quarters

In Exercises 15–17, we consider simultaneous linear diophantine equations. To solve these, first eliminate all but two variables and then solve the resulting equation in two variables.

15. Find all integer solutions of the following systems of linear diophantine equations.

a)
$$\begin{aligned} x + y + z &= 100 \\ x + 8y + 50z &= 156 \end{aligned}$$

b)
$$\begin{aligned} x + y + z &= 100 \\ x + 6y + 21z &= 121 \end{aligned}$$

c)
$$\begin{aligned} x + y + z + w &= 100 \\ x + 2y + 3z + 4w &= 300 \\ x + 4y + 9z + 16w &= 1000 \end{aligned}$$

16. A piggy bank contains 24 coins, all of which are nickels, dimes, or quarters. If the total value of the coins is two dollars, what combinations of coins are possible?

17. Nadir Airways offers three types of tickets on their Boston–New York flights. First-class tickets are \$140, second-class tickets are \$110, and standby tickets are \$78. If 69 passengers pay a total of \$6548 for their tickets on a particular flight, how many of each type of ticket were sold?

18. Is it possible to have 50 coins, all of which are pennies, dimes, or quarters, with a total worth \$3?

Let a and b be relatively prime positive integers, and let n be a positive integer. A solution (x, y) of the linear diophantine equation $ax + by = n$ is *nonnegative* when both x and y are nonnegative.

* 19. Show that whenever $n \geq (a-1)(b-1)$, there is a nonnegative solution of $ax + by = n$.

* 20. Show that if $n = ab - a - b$, then there are no nonnegative solutions of $ax + by = n$.

* 21. Show that there are exactly $(a-1)(b-1)/2$ nonnegative integers $n < ab - a - b$ such that the equation has a nonnegative solution.