Supplementary Problems

1. Let R be a commutative ring; an ideal P of R is said to be a *prime ideal* of R if $ab \in P$, $a, b \in R$ implies that $a \in P$ or $b \in P$. Prove that P is a prime ideal of R if and only if R/P is an integral domain.
2. Let R be a commutative ring with unit element; prove that every maximal ideal of R is a prime ideal.
3. Give an example of a ring in which some prime ideal is not a maximal ideal.
4. If R is a finite commutative ring (i.e., has only a finite number of elements) with unit element, prove that every prime ideal of R is a maximal ideal of R.
5. If F is a field, prove that $F[x]$ is isomorphic to $F[t]$.
6. Find all the automorphisms σ of $F[x]$ with the property that $\sigma(f) = f$ for every $f \in F$.
7. If R is a commutative ring, let $N = \{x \in R \mid x^n = 0 \text{ for some integer } n\}$. Prove
   (a) N is an ideal of R.
   (b) In $\bar{R} = R/N$ if $\bar{x}^m = 0$ for some m then $\bar{x} = 0$.
8. Let R be a commutative ring and suppose that A is an ideal of R. Let $N(A) = \{x \in R \mid x^n \in A \text{ for some } n\}$. Prove
   (a) $N(A)$ is an ideal of R which contains A.
   (b) $N(N(A)) = N(A)$.
   $N(A)$ is often called the *radical* of A.
9. If n is an integer, let J_n be the ring of integers mod n. Describe N (see Problem 7) for J_n in terms of n.
10. If A and B are ideals in a ring R such that $A \cap B = (0)$, prove that for every $a \in A$, $b \in B$, $ab = 0$.
11. If R is a ring, let $Z(R) = \{x \in R \mid xy = yx \text{ all } y \in R\}$. Prove that $Z(R)$ is a subring of R.
12. If R is a division ring, prove that $Z(R)$ is a field.
13. Find a polynomial of degree 3 irreducible over the ring of integers, J_3, mod 3. Use it to construct a field having 27 elements.
14. Construct a field having 625 elements.
15. If F is a field and $p(x) \in F[x]$, prove that in the ring

$$R = \frac{F[x]}{(p(x))},$$

   N (see Problem 7) is (0) if an only if $p(x)$ is not divisible by the square of any polynomial.