14. Use the least-remainder algorithm to find $(384, 226)$.

15. Show that the least-remainder algorithm always produces the greatest common divisor of two integers.

** 16. Show that the least-remainder algorithm is always at least as fast as the Euclidean algorithm. (*Hint:* First show that if a and b are positive integers with $2b < a$, then the least-remainder algorithm can find (a, b) with no more steps than it uses to find $(a, a - b)$.)

* 17. Find a sequence of integers $v_0, v_1, v_2, \ldots$, such that the least-remainder algorithm takes exactly n divisions to find (v_{n+1}, v_{n+2}).

* 18. Show that the number of divisions needed to find the greatest common divisor of two positive integers using the least-remainder algorithm is less than $8/3$ times the number of digits in the smaller of the two numbers, plus $4/3$.

* 19. Let m and n be positive integers and let a be an integer greater than 1. Show that $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$.

* 20. Show that if m and n are positive integers, then $(f_m, f_n) = f_{(m,n)}$.

The next two exercises deal with the *game of Euclid*. Two players begin with a pair of positive integers and take turns making moves of the following type. A player can move from the pair of positive integers $\{x, y\}$ with $x \geq y$, to any of the pairs $\{x - ty, y\}$, where t is a positive integer and $x - ty \geq 0$. A *winning move* consists of moving to a pair with one element equal to 0.

21. Show that every sequence of moves starting with the pair $\{a, b\}$ must eventually end with the pair $\{0, (a, b)\}$.

* 22. Show that in a game beginning with the pair $\{a, b\}$, the first player may play a winning strategy if $a = b$ or if $a > b(1 + \sqrt{5})/2$; otherwise, the second player may play a winning strategy. (*Hint:* First show that if $y < x \leq y(1 + \sqrt{5})/2$, then there is a unique move from $\{x, y\}$ that goes to a pair $\{z, y\}$ with $y > z(1 + \sqrt{5})/2$.)

* 23. Show that the number of bit operations needed to use the Euclidean algorithm to find the greatest common divisor of two positive integers a and b with $a > b$ is $O((\log_2 a)^2)$. (*Hint:* First show that the complexity of division of the positive integer q by the positive integer d is $O(\log d \log q)$.)

* 24. Let a and b be positive integers and let r_j and q_j, $j = 1, 2, \ldots, n$ be the remainders and quotients of the steps of the Euclidean algorithm as defined in this section.
a) Find the value of $\sum_{j=1}^{n} r_j q_j$.
b) Find the value of $\sum_{j=1}^{n} r_j^2 q_j$.

25. Suppose that a and b are positive integers with $a \geq b$. Let q_i and r_i be the quotients and remainders in the steps of the Euclidean algorithm for $i = 1, 2, \ldots, n$, where r_n is the last nonzero remainder. Let $Q_i = \begin{pmatrix} q_i & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \prod_{i=0}^{n} Q_i$. Show that $\begin{pmatrix} a \\ b \end{pmatrix} = Q \begin{pmatrix} r_n \\ 0 \end{pmatrix}$.