j	r_j	r_{j+1}	q_{j+1}	r_{j+2}	s_j	t_j
0	252	198	1	54	1	0
1	198	54	3	36	0	1
2	54	36	1	18	1	−1
3	36	18	2	0	−3	4
4					4	−5

The values of s_j and t_j, $j = 0, 1, 2, 3, 4$, are computed as follows:

$$\begin{aligned} s_0 &= 1, & t_0 &= 0,\\ s_1 &= 0, & t_1 &= 1,\\ s_2 &= s_0 - s_1q_1 = 1 - 0\cdot 1 = 1, & t_2 &= t_0 - t_1q_1 = 0 - 1\cdot 1 = -1,\\ s_3 &= s_1 - s_2q_2 = 0 - 1\cdot 3 = -3, & t_3 &= t_1 - t_2q_2 = 1 - (-1)3 = 4,\\ s_4 &= s_2 - s_3q_3 = 1 - (-3)\cdot 1 = 4, & t_4 &= t_2 - t_3q_3 = -1 - 4\cdot 1 = -5. \end{aligned}$$

Because $r_4 = 18 = (252, 198)$ and $r_4 = s_4a + t_4b$, we have

$$18 = (252, 198) = 4\cdot 252 - 5\cdot 198.$$ ◀

Note that the greatest common divisor of two integers may be expressed as a linear combination of these integers in an infinite number of ways. To see this, let $d = (a, b)$ and let $d = sa + tb$ be one way to write d as a linear combination of a and b, guaranteed to exist by the previous discussion. Then for all integers k,

$$d = (s + k(b/d))a + (t - k(a/d))b.$$

Example 3.15. With $a = 252$ and $b = 198$, we have $18 = (252, 198) = (4 + 11k)252 + (-5 - 14k)198$ for any integer k. ◀

3.4 Exercises

1. Use the Euclidean algorithm to find each of the following greatest common divisors.

   a) (45, 75) c) (666, 1414)
   b) (102, 222) d) (20785, 44350)

2. Use the Euclidean algorithm to find each of the following greatest common divisors.

   a) (51, 87) c) (981, 1234)
   b) (105, 300) d) (34709, 100313)

3. For each pair of integers in Exercise 1, express the greatest common divisor of the integers as a linear combination of these integers.

4. For each pair of integers in Exercise 2, express the greatest common divisor of the integers as a linear combination of these integers.

5. Find the greatest common divisor of each of the following sets of integers.

   a) 6, 10, 15 b) 70, 98, 105 c) 280, 330, 405, 490