3. Find the quotient and remainder in the division algorithm with divisor 17 and dividend

   a) 100
   b) 289
   c) -44
   d) -100.

4. What can you conclude if a and b are nonzero integers such that $a \mid b$ and $b \mid a$?

5. Show that if a, b, c, and d are integers with a and c nonzero such that $a \mid b$ and $c \mid d$, then $ac \mid bd$.

6. Are there integers a, b, and c such that $a \mid bc$, but $a \nmid b$ and $a \nmid c$?

7. Show that if a, b, and $c \neq 0$ are integers, then $a \mid b$ if and only if $ac \mid bc$.

8. Show that if a and b are positive integers and $a \mid b$, then $a \leqslant b$.

9. Give another proof of the division algorithm by using the well-ordering property. (Hint: When dividing a by b, take as the remainder the least positive integer in the set of integers $a-qb$.)

10. Show that if a and b are odd positive integers, then there are integers s and t such that $a = bs + t$, where t is odd and $|t| < b$.

11. When the integer a is divided by the interger b where $b > 0$, the division algorithm gives a quotient of q and a remainder of r. Show that if $b \nmid a$, when $-a$ is divided by b, the division algorithm gives a quotient of $-(q+1)$ and a remainder of $b - r$, while if $b \mid a$, the quotient is $-q$ and the remainder is zero.

12. Show that if a, b, and c are integers with $b > 0$ and $c > 0$, such that when a is divided by b the quotient is q and the remainder is r, and when q is divided by c the quotient is t and the remainder is s, then when a is divided by bc, the quotient is t and the remainder is $bs + r$.

13. a) Extend the division algorithm by allowing negative divisors. In particular, show that whenever a and $b \neq 0$ are integers, there are integers q and r such that $a = bq + r$, where $0 \leqslant r < |b|$.

    b) Find the remainder when 17 is divided by -7.

14. Show that if a and b are positive integers, then there are integers q, r and $e = \pm 1$ such that $a = bq + er$ where $-b/2 < er \leqslant b/2$.

15. Show that if a and b are real numbers, then $[a+b] \geqslant [a] + [b]$.

16. Show that if a and b are positive real numbers, then $[ab] \geqslant [a][b]$. What is the corresponding inequality when both a and b are negative? When one is negative and the other positive?

6. Let a be a positive integer. What is the greatest common divisor of a and $a + 2$?

7. Show that if a and b are integers, not both 0, and c is a nonzero integer, then $(ca, cb) = |c|(a, b)$.

8. Show that if a and b are integers with $(a, b) = 1$, then $(a + b, a - b) = 1$ or 2.

9. What is $(a^2 + b^2, a + b)$, where a and b are relatively prime integers that are not both 0?

10. Show that if a and b are both even integers that are not both 0, then $(a, b) = 2(a/2, b/2)$.

11. Show that if a is an even integer and b is an odd integer, then $(a, b) = (a/2, b)$.

12. Show that if a, b, and c are integers such that $(a, b) = 1$ and $c \mid (a + b)$, then $(c, a) = (c, b) = 1$.

13. Show that if a, b, and c are mutually relatively prime nonzero integers, then $(a, bc) = (a, b)(a, c)$.

☞ 14. a) Show that if a, b, and c are integers with $(a, b) = (a, c) = 1$, then $(a, bc) = 1$.
b) Use mathematical induction to show that if $a_1, a_2, \ldots, a_n$ are integers, and b is another integer such that $(a_1, b) = (a_2, b) = \cdots = (a_n, b) = 1$, then $(a_1 a_2 \cdots a_n, b) = 1$.

15. Find a set of three integers that are mutually relatively prime, but any two of which are not relatively prime. Do not use examples from the text.

16. Find four integers that are mutually relatively prime such that any three of these integers are not mutually relatively prime.

17. Find the greatest common divisor of each of the following sets of integers.

a) 8, 10, 12	d) 6, 15, 21
b) 5, 25, 75	e) −7, 28, −35
c) 99, 9999, 0	f) 0, 0, 1001

18. Find three mutually relatively prime integers from among the integers 66, 105, 42, 70, and 165.

19. Show that if $a_1, a_2, \ldots, a_n$ are integers that are not all 0 and c is a positive integer, then $(ca_1, ca_2, \ldots, ca_n) = c(a_1, a_2 \ldots, a_n)$.

20. Show that the greatest common divisor of the integers $a_1, a_2, \ldots, a_n$, not all 0, is the least positive integer that is a linear combination of $a_1, a_2, \ldots, a_n$.

21. Show that if k is an integer, then the integers $6k - 1$, $6k + 1$, $6k + 2$, $6k + 3$, and $6k + 5$ are pairwise relatively prime.

22. Show that if k is a positive integer, then $3k + 2$ and $5k + 3$ are relatively prime.

23. Show that $8a + 3$ and $5a + 2$ are relatively prime for all integers a.

24. Show that if a and b are relatively prime integers, then $(a + 2b, 2a + b) = 1$ or 3.

25. Show that every positive integer greater than 6 is the sum of two relatively prime integers greater than 1.

14. Use the least-remainder algorithm to find $(384, 226)$.

15. Show that the least-remainder algorithm always produces the greatest common divisor of two integers.

** 16. Show that the least-remainder algorithm is always at least as fast as the Euclidean algorithm. (*Hint:* First show that if a and b are positive integers with $2b < a$, then the least-remainder algorithm can find (a, b) with no more steps than it uses to find $(a, a - b)$.)

* 17. Find a sequence of integers $v_0, v_1, v_2, \ldots$, such that the least-remainder algorithm takes exactly n divisions to find (v_{n+1}, v_{n+2}).

* 18. Show that the number of divisions needed to find the greatest common divisor of two positive integers using the least-remainder algorithm is less than 8/3 times the number of digits in the smaller of the two numbers, plus 4/3.

* 19. Let m and n be positive integers and let a be an integer greater than 1. Show that $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$.

* 20. Show that if m and n are positive integers, then $(f_m, f_n) = f_{(m,n)}$.

The next two exercises deal with the *game of Euclid*. Two players begin with a pair of positive integers and take turns making moves of the following type. A player can move from the pair of positive integers $\{x, y\}$ with $x \geq y$, to any of the pairs $\{x - ty, y\}$, where t is a positive integer and $x - ty \geq 0$. A *winning move* consists of moving to a pair with one element equal to 0.

21. Show that every sequence of moves starting with the pair $\{a, b\}$ must eventually end with the pair $\{0, (a, b)\}$.

* 22. Show that in a game beginning with the pair $\{a, b\}$, the first player may play a winning strategy if $a = b$ or if $a > b(1 + \sqrt{5})/2$; otherwise, the second player may play a winning strategy. (*Hint:* First show that if $y < x \leq y(1 + \sqrt{5})/2$, then there is a unique move from $\{x, y\}$ that goes to a pair $\{z, y\}$ with $y > z(1 + \sqrt{5})/2$.)

* 23. Show that the number of bit operations needed to use the Euclidean algorithm to find the greatest common divisor of two positive integers a and b with $a > b$ is $O((\log_2 a)^2)$. (*Hint:* First show that the complexity of division of the positive integer q by the positive integer d is $O(\log d \log q)$.)

* 24. Let a and b be positive integers and let r_j and q_j, $j = 1, 2, \ldots, n$ be the remainders and quotients of the steps of the Euclidean algorithm as defined in this section.
   a) Find the value of $\sum_{j=1}^{n} r_j q_j$.
   b) Find the value of $\sum_{j=1}^{n} r_j^2 q_j$.

25. Suppose that a and b are positive integers with $a \geq b$. Let q_i and r_i be the quotients and remainders in the steps of the Euclidean algorithm for $i = 1, 2, \ldots, n$, where r_n is the last nonzero remainder. Let $Q_i = \begin{pmatrix} q_i & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \prod_{i=0}^{n} Q_i$. Show that $\begin{pmatrix} a \\ b \end{pmatrix} = Q \begin{pmatrix} r_n \\ 0 \end{pmatrix}$.