Problems

● 1. In the following determine whether the systems described are groups. If they are not, point out which of the group axioms fail to hold.

(a) G = set of all integers, $a \cdot b \equiv a - b$.

(b) G = set of all positive integers, $a \cdot b = ab$, the usual product of integers.

(c) $G = a_0, a_1, \ldots, a_6$ where

$$a_i \cdot a_j = a_{i+j} \qquad \text{if} \quad i + j < 7,$$

$$a_i \cdot a_j = a_{i+j-7} \quad \text{if} \quad i + j \geq 7$$

(for instance, $a_5 \cdot a_4 = a_{5+4-7} = a_2$ since $5 + 4 = 9 > 7$).

(d) G = set of all rational numbers with odd denominators, $a \cdot b \equiv a + b$, the usual addition of rational numbers.

● 2. Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n, $(a \cdot b)^n = a^n \cdot b^n$.

● 3. If G is a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$, show that G must be abelian.

● *4. If G is a group in which $(a \cdot b)^i = a^i \cdot b^i$ for three consecutive integers i for all $a, b \in G$, show that G is abelian.

5. Show that the conclusion of Problem 4 does not follow if we assume the relation $(a \cdot b)^i = a^i \cdot b^i$ for just two consecutive integers.

● 6. In S_3 give an example of two elements x, y such that $(x \cdot y)^2 \neq x^2 \cdot y^2$.

● 7. In S_3 show that there are four elements satisfying $x^2 = e$ and three elements satisfying $y^3 = e$.

● 8. If G is a finite group, show that there exists a positive integer N such that $a^N = e$ for all $a \in G$.

• 9. (a) If the group G has three elements, show it must be abelian.

(b) Do part (a) if G has four elements.

(c) Do part (a) if G has five elements.

● 10. Show that if every element of the group G is its own inverse, then G is abelian.

• 11. If G is a group of even order, prove it has an element $a \neq e$ satisfying $a^2 = e$.

● 12. Let G be a nonempty set closed under an associative product, which in addition satisfies:

(a) There exists an $e \in G$ such that $a \cdot e = a$ for all $a \in G$.

(b) Give $a \in G$, there exists an element $y(a) \in G$ such that $a \cdot y(a) = e$.

Prove that G must be a group under this product.