The notation for a diagonal matrix does not include the size of the matrix. This will cause no confusion in what follows. The size of a diagonal matrix will always be clear from the context in which it appears.

EXERCISES

1. Show the ring $M_{n\times n}(\mathbb{Z})$ has infinitely many two-sided, proper ideals. Contrast this with the situation in Exercise 1 of Chapter 1.
2. Let $T = M_{n\times n}(R)$. Set

   $$S = \{E_{ij} \mid 1 \le i, j \le n\} \cup \{0\}$$

   Show S is closed under multiplication in T.
3. Suppose $R = R_1 \oplus \cdots \oplus R_s$ is a direct sum of (commutative) rings $R_1, \ldots, R_s$. Show

   $$M_{n\times n}(R) \cong M_{n\times n}(R_1) \oplus \cdots \oplus M_{n\times n}(R_s)$$

4. Let T be a ring. The center, $C(T)$, of T is defined as follows:

   $$C(T) = \{x \in T \mid xy = yx \text{ for all } y \in T\}$$

   Compute $C(M_{n\times n}(R))$.
5. Let T be a ring. Let $M_{n\times n}(T)$ denote the set of all $n \times n$ matrices with entries from T. Define addition and multiplication in the usual ways: If $A,B \in M_{n\times n}(T)$, then $[A + B]_{ij} = [A]_{ij} + [B]_{ij}$ and $[AB]_{ij} = \sum_{k=1}^{n}[A]_{ik}[B]_{kj}$ for all $i,j = 1, \ldots n$. Show that $M_{n\times n}(T)$ is an associative ring with identity. Is $M_{n\times n}(T)$ a T-algebra ?
6. Show $M_{n\times n}(M_{m\times m}(R)) \cong M_{nm\times nm}(R)$ by using block addition and multiplication.
7. Let X be an indeterminate over the commutative ring R. Let $(M_{n\times n}(R))[X]$ denote the set of all polynomials in X with coefficients from $M_{n\times n}(R)$. Show that $(M_{n\times n}(R))[X]$ is a ring with the usual definitions of addition and multiplication of polynomials. Is $(M_{n\times n}(R))[X]$ commutative?
8. Using the notation in Exercise 7, show $(M_{n\times n}(R))[X] \cong M_{n\times n}(R[X])$.
9. Let $\mathrm{Tr} : M_{n\times n}(R) \mapsto R$ denote the trace mapping. Thus, if $A \in M_{n\times n}(R)$, then $\mathrm{Tr}(A) = \sum_{i=1}^{n}[A]_{ii}$. Verify the following statements:
   (a) $\mathrm{Tr} \in \mathrm{Hom}_R(M_{n\times n}(R), R)$.
   (b) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in M_{n\times n}(R)$.
   (c) $\mathrm{Tr}(A^t) = \mathrm{Tr}(A)$ for all $A \in M_{n\times n}(R)$.
   (d) The R-submodule $\mathrm{Ker}(\mathrm{Tr})$ is generated as an R-module by the following set of matrices:

   $$\{E_{ij} \mid 1 \le i \ne j \le n\} \cup \{E_{11} - E_{ii} \mid i \ne 1\}$$

10. Prove the assertions in 2.13 and 2.14.