Exercises

Determine whether each set in Exercises 1–5 is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. If it is a function, find its domain and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function.

1. $\{(1, a), (2, a), (3, c), (4, b)\}$

2. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$

3. $\{(1, c), (2, d), (3, a), (4, b)\}$

4. $\{(1, d), (2, d), (4, a)\}$

5. $\{(1, b), (2, b), (3, b), (4, b)\}$

Draw the graphs of the functions in Exercises 6–9. The domain of each function is the set of real numbers. The codomain of each function is also the set of real numbers.

6. $f(x) = \lceil x \rceil - \lfloor x \rfloor$

7. $f(x) = x - \lfloor x \rfloor$

8. $f(x) = \lceil x^2 \rceil$

9. $f(x) = \lfloor x^2 - x \rfloor$

Determine whether each function in Exercises 10–15 is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all integers. The codomain of each function is also the set of all integers.

10. $f(n) = n + 1$

11. $f(n) = n^2 - 1$

12. $f(n) = \lceil n/2 \rceil$

13. $f(n) = |n|$

14. $f(n) = 2n$

15. $f(n) = n^3$

Determine whether each function in Exercises 16–21 is one-to-one, onto, or both. Prove your answers. The domain of each function is $\mathbf{Z} \times \mathbf{Z}$. *The codomain of each function is* $\mathbf{Z}$.

16. $f(m, n) = m - n$

17. $f(m, n) = m$

18. $f(m, n) = mn$

19. $f(m, n) = m^2 + n^2$

20. $f(m, n) = n^2 + 1$

21. $f(m, n) = m + n + 2$

22. Prove that the function f from $\mathbf{Z}^+ \times \mathbf{Z}^+$ to $\mathbf{Z}^+$ defined by $f(m, n) = 2^m 3^n$ is one-to-one but not onto.

Determine whether each function in Exercises 23–28 is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all real numbers. The codomain of each function is also the set of all real numbers.

23. $f(x) = 6x - 9$

24. $f(x) = 3x^2 - 3x + 1$

25. $f(x) = \sin x$

26. $f(x) = 2x^3 - 4$

27. $f(x) = 3^x - 2$

28. $f(x) = \dfrac{x}{1 + x^2}$

29. Give an example of a function different from those presented in the text that is one-to-one but not onto, and prove that your function has the required properties.

30. Give an example of a function different from those presented in the text that is onto but not one-to-one, and prove that your function has the required properties.

31. Give an example of a function different from those presented in the text that is neither one-to-one nor onto, and prove that your function has the required properties.

Each function in Exercises 32–37 is one-to-one on the specified domain X. By letting Y = range of f, we obtain a bijection from X to Y. Find each inverse function.

32. $f(x) = 4x + 2$, X = set of real numbers

33. $f(x) = 3^x$, X = set of real numbers

34. $f(x) = 3 \log_2 x$, X = set of positive real numbers

35. $f(x) = 3 + \dfrac{1}{x}$, X = set of nonzero real numbers

36. $f(x) = 4x^3 - 5$, X = set of real numbers

37. $f(x) = 6 + 2^{7x-1}$, X = set of real numbers

38. Given

$$g = \{(1, b), (2, c), (3, a)\},$$

a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and

$$f = \{(a, x), (b, x), (c, z), (d, w)\},$$

a function from Y to $Z = \{w, x, y, z\}$, write $f \circ g$ as a set of ordered pairs and draw the arrow diagram of $f \circ g$.

39. Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = 2n + 1, \qquad g(n) = 3n - 1.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

40. Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = n^2, \qquad g(n) = 2^n.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

41. Let f and g be functions from the nonnegative real numbers to the nonnegative real numbers defined by the equations

$$f(x) = \lfloor 2x \rfloor, \qquad g(x) = x^2.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

In Exercises 42–47, decompose the function into simpler functions as in Example 3.1.44.

42. $f(x) = \log_2(x^2 + 2)$

43. $f(x) = \dfrac{1}{2x^2}$

44. $f(x) = \sin 2x$

45. $f(x) = 2 \sin x$

46. $f(x) = (3 + \sin x)^4$

47. $f(x) = \dfrac{1}{(\cos 6x)^3}$

48. Given

$$f = \{(x, x^2) \mid x \in X\},$$

a function from $X = \{-5, -4, \ldots, 4, 5\}$ to the set of integers, write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one or onto?

49. How many functions are there from $\{1, 2\}$ to $\{a, b\}$? Which are one-to-one? Which are onto?

50. Given

$$f = \{(a, b), (b, a), (c, b)\},$$

a function from $X = \{a, b, c\}$ to X:

(a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.

(b) Define

$$f^n = f \circ f \circ \cdots \circ f$$

to be the n-fold composition of f with itself. Write f^9 and f^{623} as sets of ordered pairs.

51. Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by

$$f(x) = 4x \bmod 5.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

52. Let f be the function from $X = \{0, 1, 2, 3, 4, 5\}$ to X defined by

$$f(x) = 4x \bmod 6.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

53. Verify the ISBN check character for this book.

54. Universal product codes (UPC) are the familiar bar codes that identify products so that they can be automatically priced at the checkout counter. A UPC is a 12-digit code in which the first digit characterizes the type of product (0 identifies an ordinary grocery item, 2 is an item sold by weight, 3 is a medical item, 4 is a special item, 5 is a coupon, and 6 and 7 are items not sold in retail stores). The next five digits identify the manufacturer, the next five digits identify the product, and the last digit is a check digit. (All UPC codes have a check digit. It is always present on the bar code, but it may not appear in the printed version.) For example, the UPC for a package of 10 Ortega taco shells is 0-54400-00800-5. The first zero identifies this as an ordinary grocery item, the next five digits 54400 identify the manufacturer Nabisco Foods, and the next five digits 00800 identify the product as a package of 10 Ortega taco shells.

The check digit is computed as follows. First compute s, where s is 3 times the sum of every other number starting with the first plus the sum of every other number, except the check digit, starting with the second. The check digit is the number c, between 0 and 9 satisfying $(c+s) \bmod 10 = 0$. For the code on the package of taco shells, we would have

$$s = 3(0+4+0+0+8+0)+5+4+0+0+0 = 45.$$

Since $(5+45) \bmod 10 = 0$, the check digit is 5.

Find the check digit for the UPC whose first 11 digits are 3-41280-21414.

For each hash function in Exercises 55–58, show how the data would be inserted in the order given in initially empty cells. Use the collision resolution policy of Example 3.1.14.

55. $h(x) = x \bmod 11$; cells indexed 0 to 10; data: 53, 13, 281, 743, 377, 20, 10, 796

56. $h(x) = x \bmod 17$; cells indexed 0 to 16; data: 714, 631, 26, 373, 775, 906, 509, 2032, 42, 4, 136, 1028

57. $h(x) = x^2 \bmod 11$; cells and data as in Exercise 55

58. $h(x) = (x^2 + x) \bmod 17$; cells and data as in Exercise 56

59. Suppose that we store and retrieve data as described in Example 3.1.14. Will any problem arise if we delete data? Explain.

60. Suppose that we store data as described in Example 3.1.14 and that we never store more than 10 items. Will any problem arise when retrieving data if we stop searching when we encounter an empty cell? Explain.

61. Suppose that we store data as described in Example 3.1.14 and retrieve data as described in Exercise 60. Will any problem arise if we delete data? Explain.

Let g be a function from X to Y and let f be a function from Y to Z. For each statement in Exercises 62–69, if the statement is true, prove it; otherwise, give a counterexample.

62. If g is one-to-one, then $f \circ g$ is one-to-one.

63. If f is onto, then $f \circ g$ is onto.

64. If g is onto, then $f \circ g$ is onto.

65. If f and g are onto, then $f \circ g$ is onto.

66. If f and g are one-to-one and onto, then $f \circ g$ is one-to-one and onto.

67. If $f \circ g$ is one-to-one, then f is one-to-one.

68. If $f \circ g$ is one-to-one, then g is one-to-one.

69. If $f \circ g$ is onto, then f is onto.

If f is a function from X to Y and $A \subseteq X$ and $B \subseteq Y$, we define

$$f(A) = \{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

We call $f^{-1}(B)$ the inverse image of B under f.

70. Let

$$g = \{(1, a), (2, c), (3, c)\}$$

be a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$. Let $S = \{1\}$, $T = \{1, 3\}$, $U = \{a\}$, and $V = \{a, c\}$. Find $g(S)$, $g(T)$, $g^{-1}(U)$, and $g^{-1}(V)$.

★71. Let f be a function from X to Y. Prove that f is one-to-one if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

for all subsets A and B of X. [When S is a set, we define $f(S) = \{f(x) \mid x \in S\}$.]

★72. Let f be a function from X to Y. Prove that f is one-to-one if and only if whenever g is a one-to-one function from any set A to X, $f \circ g$ is one-to-one.

★73. Let f be a function from X to Y. Prove that f is onto Y if and only if whenever g is a function from Y onto any set Z, $g \circ f$ is onto Z.

74. Let f be a function from X onto Y. Let

$$\mathcal{S} = \{f^{-1}(\{y\}) \mid y \in Y\}.$$

Show that $\mathcal{S}$ is a partition of X.