Then f is a non-degenerate bilinear form on R^n. The matrix of f in the standard ordered basis is the $n \times n$ identity matrix:

$$f(X, Y) = X^tY.$$

This f is usually called the dot (or scalar) product. The reader is probably familiar with this bilinear form, at least in the case $n = 3$. Geometrically, the number $f(\alpha, \beta)$ is the product of the length of α, the length of β, and the cosine of the angle between α and β. In particular, $f(\alpha, \beta) = 0$ if and only if the vectors α and β are orthogonal (perpendicular).

Exercises

1. Which of the following functions f, defined on vectors $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in R^2, are bilinear forms?

(a) $f(\alpha, \beta) = 1$.
(b) $f(\alpha, \beta) = (x_1 - y_1)^2 + x_2y_2$.
(c) $f(\alpha, \beta) = (x_1 + y_1)^2 - (x_1 - y_1)^2$.
(d) $f(\alpha, \beta) = x_1y_2 - x_2y_1$.

2. Let f be the bilinear form on R^2 defined by

$$f((x_1, y_1), (x_2, y_2)) = x_1y_1 + x_2y_2.$$

Find the matrix of f in each of the following bases:

$$\{(1, 0), (0, 1)\}, \qquad \{(1, -1), (1, 1)\}, \qquad \{(1, 2), (3, 4)\}.$$

3. Let V be the space of all 2×3 matrices over R, and let f be the bilinear form on V defined by $f(X, Y) = \text{trace}\,(X^tAY)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the matrix of f in the ordered basis

$$\{E^{11}, E^{12}, E^{13}, E^{21}, E^{22}, E^{23}\}$$

where E^{ij} is the matrix whose only non-zero entry is a 1 in row i and column j.

4. Describe explicitly all bilinear forms f on R^3 with the property that $f(\alpha, \beta) = f(\beta, \alpha)$ for all α, β.

5. Describe the bilinear forms on R^3 which satisfy $f(\alpha, \beta) = -f(\beta, \alpha)$ for all α, β.

6. Let n be a positive integer, and let V be the space of all $n \times n$ matrices over the field of complex numbers. Show that the equation

$$f(A, B) = n\,\text{tr}\,(AB) - \text{tr}\,(A)\,\text{tr}\,(B)$$

defines a bilinear form f on V. Is it true that $f(A, B) = f(B, A)$ for all A, B?

7. Let f be the bilinear form defined in Exercise 6. Show that f is degenerate (not non-degenerate). Let V_1 be the subspace of V consisting of the matrices of trace 0, and let f_1 be the restriction of f to V_1. Show that f_1 is non-degenerate.

There are several comments we should make about the basis $\{\beta_1, \ldots, \beta_n\}$ of Theorem 5 and the associated subspaces V^+, V^-, and $V^\perp$ First, note that $V^\perp$ is exactly the subspace of vectors which are 'orthogonal' to all of V. We noted above that $V^\perp$ is contained in this subspace; but,

$$\dim V^\perp = \dim V - (\dim V^+ + \dim V^-) = \dim V - \operatorname{rank} f$$

so every vector α such that $f(\alpha, \beta) = 0$ for all β must be in $V^\perp$. Thus, the subspace $V^\perp$ is unique. The subspaces V^+ and V^- are not unique; however, their dimensions are unique. The proof of Theorem 5 shows us that dim V^+ is the largest possible dimension of any subspace on which f is positive definite. Similarly, dim V^- is the largest dimension of any subspace on which f is negative definite. Of course

$$\dim V^+ + \dim V^- = \operatorname{rank} f.$$

The number

$$\dim V^+ - \dim V^-$$

is often called the **signature** of f. It is introduced because the dimensions of V^+ and V^- are easily determined from the rank of f and the signature of f.

Perhaps we should make one final comment about the relation of symmetric bilinear forms on real vector spaces to inner products. Suppose V is a finite-dimensional real vector space and that V_1, V_2, V_3 are subspaces of V such that

$$V = V_1 \oplus V_2 \oplus V_3.$$

Suppose that f_1 is an inner product on V_1, and f_2 is an inner product on V_2. We can then define a symmetric bilinear form f on V as follows: If α, β are vectors in V, then we can write

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{and} \quad \beta = \beta_1 + \beta_2 + \beta_3$$

with α_j and β_j in V_j. Let

$$f(\alpha, \beta) = f_1(\alpha_1, \beta_1) - f_2(\alpha_2, \beta_2).$$

The subspace $V^\perp$ for f will be V_3, V_1 is a suitable V^+ for f, and V_2 is a suitable V^-. One part of the statement of Theorem 5 is that every symmetric bilinear form on V arises in this way. The additional content of the theorem is that an inner product is represented in some ordered basis by the identity matrix.

Exercises

1. The following expressions define quadratic forms q on R^2. Find the symmetric bilinear form f corresponding to each q.

(a) ax_1^2.
(b) bx_1x_2.
(c) cx_2^2.
(d) $2x_1^2 - \frac{1}{3}x_1x_2$.
(e) $x_1^2 + 9x_2^2$.
(f) $3x_1x_2 - x_2^2$.
(g) $4x_1^2 + 6x_1x_2 - 3x_2^2$.

2. Find the matrix, in the standard ordered basis, and the rank of each of the bilinear forms determined in Exercise 1. Indicate which forms are non-degenerate.

3. Let $q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ be the quadratic form associated with a symmetric bilinear form f on R^2. Show that f is non-degenerate if and only if $b^2 - 4ac \neq 0$.

4. Let V be a finite-dimensional vector space over a subfield F of the complex numbers, and let S be the set of all symmetric bilinear forms on V.

(a) Show that S is a subspace of $L(V, V, F)$.
(b) Find dim S.

Let Q be the set of all quadratic forms on V.

(c) Show that Q is a subspace of the space of all functions from V into F.
(d) Describe explicitly an isomorphism T of Q onto S, without reference to a basis.
(e) Let U be a linear operator on V and q an element of Q. Show that the equation $(U^\dagger q)(\alpha) = q(U\alpha)$ defines a quadratic form $U^\dagger q$ on V.
(f) If U is a linear operator on V, show that the function $U^\dagger$ defined in part (e) is a linear operator on Q. Show that $U^\dagger$ is invertible if and only if U is invertible.

5. Let q be the quadratic form on R^2 given by

$$q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2, \qquad a \neq 0.$$

Find an invertible linear operator U on R^2 such that

$$(U^\dagger q)(x_1, x_2) = ax_1^2 + \left(c - \frac{b^2}{a}\right)x_2^2.$$

(*Hint:* To find U^{-1} (and hence U), complete the square. For the definition of $U^\dagger$, see part (e) of Exercise 4.)

6. Let q be the quadratic form on R^2 given by

$$q(x_1, x_2) = 2bx_1x_2.$$

Find an invertible linear operator U on R^2 such that

$$(U^\dagger q)(x_1, x_2) = 2bx_1^2 - 2bx_2^2.$$

7. Let q be the quadratic form on R^3 given by

$$q(x_1, x_2, x_3) = x_1x_2 + 2x_1x_3 + x_3^2.$$

Find an invertible linear operator U on R^3 such that

$$(U^\dagger q)(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3^2.$$

(*Hint:* Express U as a product of operators similar to those used in Exercises 5 and 6.)

Show that: (a) $S^\perp$ and $S^\top$ are subspaces of V; (b) $S_1 \subseteq S_2$ implies $S_2^\perp \subseteq S_1^\perp$ and $S_2^\top \subseteq S_1^\top$; and (c) $\{0\}^\perp = \{0\}^\top = V$.

13.23. Prove: If f is a bilinear form on V, then rank $f = \dim V - \dim V^\perp = \dim V - \dim V^\top$ and hence $\dim V^\perp = \dim V^\top$.

13.24. Let f be a bilinear form on V. For each $u \in V$, let $\hat{u} : V \to K$ and $\tilde{u} : V \to K$ be defined by $\hat{u}(x) = f(x, u)$ and $\tilde{u}(x) = f(u, x)$. Prove:

(a) $\hat{u}$ and $\tilde{u}$ are each linear, i.e. $\hat{u}, \tilde{u} \in V^*$;

(b) $u \mapsto \hat{u}$ and $u \mapsto \tilde{u}$ are each linear mappings from V into V^*;

(c) rank f = rank $(u \mapsto \hat{u})$ = rank $(u \mapsto \tilde{u})$.

13.25. Show that congruence of matrices is an equivalence relation, i.e., (i) A is congruent to A; (ii) if A is congruent to B, then B is congruent to A; (iii) if A is congruent to B and B is congruent to C, then A is congruent to C.

SYMMETRIC BILINEAR FORMS, QUADRATIC FORMS

13.26. Find the symmetric matrices belonging to the following quadratic polynomials:

(a) $q(x, y, z) = 2x^2 - 8xy + y^2 - 16xz + 14yz + 5z^2$

(b) $q(x, y, z) = x^2 - xz + y^2$

(c) $q(x, y, z) = xy + y^2 + 4xz + z^2$

(d) $q(x, y, z) = xy + yz$

13.27. For each of the following matrices A, find a nonsingular matrix P such that P^TAP is diagonal:

(a) $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, (b) $A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -9 \\ 3 & -9 & 4 \end{pmatrix}$, (c) $A = \begin{pmatrix} 1 & 1 & -2 & -3 \\ 1 & 2 & -5 & -1 \\ -2 & -5 & 6 & 9 \\ -3 & -1 & 9 & 11 \end{pmatrix}$

In each case find the rank and signature.

13.28. Let $S(V)$ be the set of symmetric bilinear forms on V. Show that:

(i) $S(V)$ is a subspace of $B(V)$; (ii) if $\dim V = n$, then $\dim S(V) = \frac{1}{2}n(n+1)$.

13.29. Suppose A is a real symmetric positive definite matrix. Show that there exists a nonsingular matrix P such that $A = P^TP$.

13.30. Consider a real quadratic polynomial $q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$, where $a_{ij} = a_{ji}$.

(i) If $a_{11} \neq 0$, show that the substitution

$$x_1 = y_1 - \frac{1}{a_{11}}(a_{12}y_2 + \cdots + a_{1n}y_n),\ x_2 = y_2, \ldots, x_n = y_n$$

yields the equation $q(x_1, \ldots, x_n) = a_{11}y_1^2 + q'(y_2, \ldots, y_n)$, where q' is also a quadratic polynomial.

(ii) If $a_{11} = 0$ but, say, $a_{12} \neq 0$, show that the substitution

$$x_1 = y_1 + y_2,\ x_2 = y_1 - y_2,\ x_3 = y_3, \ldots, x_n = y_n$$

yields the equation $q(x_1, \ldots, x_n) = \sum b_{ij}y_iy_j$, where $b_{11} \neq 0$, i.e., reduces this case to case (i). This method of diagonalizing q is known as "completing the square."

Since $\lambda_1 \cdots \lambda_n = \det L = \det P \det A \det P = \det P^2 \det A$, we have

$$I = \frac{\pi^{\pi/2}}{\det A^{1/2}}.$$

EXERCISES

1. Determine the rank and signature of each of the quadratic forms of Exercise 1, Section 9.

2. Show that the quadratic form $Q(x, y) = ax^2 + bxy + cy^2 (a, b, c \text{ real})$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$.

3. Show that if A is a real symmetric positive definite matrix, then there exists a real non-singular matrix P such that $A = P^T P$.

4. Show that if A is a real non-singular matrix, then $A^T A$ is positive definite.

5. Show that if A is a real symmetric non-negative semi-definite matrix—that is, A represents a non-negative semi-definite quadratic form—then there exists a real matrix R such that $A = R^T R$.

6. Show that if A is real, then $A^T A$ is non-negative semi-definite.

7. Show that if A is real and $A^T A = 0$, then $A = 0$.

8. Show that if A is real symmetric and $A^2 = 0$, then $A = 0$.

9. If $A_1, \ldots, A_r$ are real symmetric matrices, show that

$$A_1^2 + \cdots + A_r^2 = 0$$

implies $A_1 = A_2 = \cdots = A_r = 0$.

12 | Hermitian Forms

For the applications of forms to many problems, it turns out that a quadratic form obtained from a bilinear form over the complex numbers is not the most useful generalization of the concept of a quadratic form over the real numbers. As we see later, the property that a quadratic form over the real numbers be positive-definite is a very useful property. While x^2 is positive-definite for real x, it is not positive-definite for complex x. When dealing with complex numbers we need a function like $|x|^2 = \bar{x}x$, where $\bar{x}$ is the conjugate complex of x. $\bar{x}x$ is non-negative for all complex (and real) x, and it is zero only when $x = 0$. Thus $\bar{x}x$ is a form which has the property of being positive definite. In the spirit of these considerations, the following definition is appropriate.

Definition. Let F be the field of complex numbers, or a subfield of the complex numbers, and let V be a vector space over F. A scalar valued

We see that, even though we are dealing with complex numbers, this transformation multiplies the elements along the main diagonal of H' by positive real numbers.

Since $q(\alpha) = f(\alpha, \alpha) = \overline{f(\alpha, \alpha)}$, $q(\alpha)$ is always real. We can, in fact, apply without change the discussion we gave for the real quadratic forms. Let P denote the number of positive terms in the diagonal representation of q, and let N denote the number of negative terms in the main diagonal. The number $S = P - N$ is called the *signature* of the Hermitian quadratic form q. Again, $P + N = r$, the *rank* of q.

The proof that the signature of a Hermitian quadratic form is independent of the particular diagonalized representation is identical with the proof given for real quadratic forms.

A Hermitian quadratic form is called *non-negative semi-definite* if $S = r$. It is called *positive definite* if $S = n$. If f is a Hermitian form whose associated Hermitian quadratic form q is positive-definite (non-negative semi-definite), we say that the Hermitian form f is *positive-definite* (*non-negative semi-definite*).

A Hermitian matrix can be reduced to diagonal form by a method analogous to the method described in Section 10, as is shown by the proof of Theorem 12.1. A modification must be made because the associated Hermitian form is not bilinear, but complex bilinear.

Let α_1' be a vector for which $q(\alpha_1') \neq 0$. With this fixed α_1', $f(\alpha_1', \alpha)$ defines a linear functional ϕ_1' on V. If α_1' is represented by

$$(p_{11}, \ldots, p_{n1}) = P \text{ and } \alpha \text{ by } (x_1, \ldots, x_n) = X,$$

then

$$\begin{aligned} f(\alpha_1', \alpha) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{p_{i1}} h_{ij} x_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \overline{p_{i1}} h_{ij} \right) x_j. \end{aligned} \tag{12.6}$$

This means the linear functional ϕ_1' is represented by P^*H.

EXERCISES

1. Reduce the following Hermitian matrices to diagonal form.

(*a*) $\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

(*b*) $\begin{bmatrix} 1 & 1 - i \\ 1 + i & 1 \end{bmatrix}$

2. Let f be an arbitrary complex bilinear form. Define f^* by the rule, $f^*(\alpha, \beta) = \overline{f(\beta, \alpha)}$. Show that f^* is complex bilinear.

3. Show that if H is a positive definite Hermitian matrix—that is, H represents a positive definite Hermitian form—then there exists a non-singular matrix P such that $H = P^*P$.

4. Show that if A is a complex non-singular matrix, then A^*A is a positive definite Hermitian matrix.

5. Show that if H is a Hermitian non-negative semi-definite matrix—that is, H represents a non-negative semi-definite Hermitian quadratic form—then there exists a complex matrix R such that $H = R^*R$.

6. Show that if A is complex, then A^*A is Hermitian non-negative semi-definite.

7. Show that if A is complex and $A^*A = 0$, then $A = 0$.

8. Show that if A is hermitian and $A^2 = 0$, then $A = 0$.

9. If $A_1, \ldots, A_r$ are Hermitian matrices, show that $A_1^2 + \cdots + A_r^2 = 0$ implies $A_1 = \cdots = A_r = 0$.

10. Show by an example that, if A and B are Hermitian, it is not necessarily true that AB is Hermitian. What is true if A and B are Hermitian and $AB = BA$?