By conditions (4) and (4'), we have $\mu_j\pi_i = \pi_i\mu_j$ for all $1 \leq i \leq k$ and all $1 \leq j \leq t$. Therefore $\theta_{ij} = \pi_i\mu_j$ is an idempotent endomorphism of V for all such i and j. Moreover, $\theta_{ij}\theta_{st} = \sigma_0$ if $(i,j) \neq (s,t)$ and

$$\sigma_1 = \left(\sum_{i=1}^{k} \pi_i\right)\left(\sum_{j=1}^{t} \mu_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{t} \theta_{ij}.$$

This suffices to show that α and β can be simultaneously represented by diagonal matrices with respect to some basis of V over F. □

Problems

1. Let α be the endomorphism of $\mathbb{R}^3$ defined by

$$\alpha\colon [a,b,c] \mapsto [a-b, a+2b+c, -2a+b-c].$$

Find the eigenvalues of α and the eigenspaces associated with them.

2. Let $F = \mathbb{Z}/(2)$ and let A be a nonempty set having a given subset B. Let α be the endomorphism of the vector space F^A over F given by $\alpha\colon f \mapsto \chi_B f$, where χ_B is the characteristic function on B. Find the eigenvectors of α.

3. Let α be an endomorphism of a a vector space V finitely generated over a field F.

(i) Show that every eigenvector of α is also an eigenvector of $p(\alpha)$ for any $p(X) \in F[X]$;
(ii) Show that the converse of (i) is false.

4. Find the eigenvalues of the following matrices and the eigenspaces associated with them.

(i) $\begin{bmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \in \mathcal{M}_{3\times3}(\mathbb{R});$

(ii) $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix} \in \mathcal{M}_{3\times3}(\mathbb{C});$

(iii) $\begin{bmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{bmatrix} \in \mathcal{M}_{3\times3}(\mathbb{C}).$

5. Let n be a positive integer, let F be a field and let $A \in \mathcal{M}_{n\times n}(F)$ be nonsingular. Given the eigenvalues of A, find the eigenvalues of A^{-1}.

6. Let n be a positive integer, let F be a field and let $A = [a_{ij}] \in \mathcal{M}_{n\times n}(F)$ be a matrix having eigenvalue c. If $b, d \in F$, show that $bc + d$ is an eigenvalue of the matrix $bA + dI$.

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$ If $t \in \mathbb{R}$ is a zero of the polynomial $bX^2 + (a-d)X - c \in \mathbb{R}[X]$, show that $[1,t]$ is an eigenvector of A associated with the eigenvalue $a + bt$.

8. Let $A \in \mathcal{M}_{2\times 2}(\mathbb{C})$ have two distinct nonzero eigenvalues. Show that there are precisely four matrices $B \in \mathcal{M}_{2\times 2}(\mathbb{C})$ satisfying $B^2 = A$.

9. Find a matrix $A = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 1 & a_{32} & a_{33} \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ having eigenvectors $[1,1,1]$, $[1,0,-1]$, and $[1,-1,0]$.

10. Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{C})$ and for each eigenvalue find the associated eigenspace.

11. Let c be a nonzero complex number and let m and n be positive integers. Let $A = [a_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{C})$ and let $B = [b_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{C})$ be the matrix defined by $b_{ij} = c^{m+i-j}a_{ij}$ for all $1 \leq i,j \leq n$. Show that if $d \in \mathbb{C}$ is an eigenvalue of A then $r^m d$ is an eigenvalue of B.

12. Show that every matrix in $\mathcal{S}_{2\times 2}(\mathbb{R})$ has a real eigenvalue.

13. Characterize magic matrices in terms of their eigenvalues.

14. Let $A \in \mathcal{M}_{2\times 2}(\mathbb{C})$ have distinct eigenvalues a and b. Show that

$$A^n = \frac{a^n}{a-b}(A - bI) + \frac{b^n}{b-a}(A - aI)$$

for all integers $n > 1$.

15. Let $A \in \mathcal{M}_{2\times 2}(\mathbb{C})$ have a unique eigenvalue c. Show that $A^n = c^{n-1}(cA - (n-1)cI)$ for all integers $n > 1$.

16. Let n be a positive integer. Show that every eigenvector of a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is also an eigenvector of $adj(A)$.

17. Let n be a positive integer and let $\mathcal{A}$ be the set of all matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$ be the set of all matrices having the property that their eigenvectors generate all of $\mathbb{C}^n$. Is $\mathcal{A}$ closed under addition? Is it closed under multiplication?

18. Let n be a positive integer and let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. If $p(X) \in \mathbb{C}[X]$, calculate $|p(A)|$ using the eigenvalues of A.

19. Let n be a positive integer and let c be a nonzero real number. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be the matrix every entry of which equals c. Find the eigenvalues of A and, for each eigenvalue, find the associated eigenspace.

20. Let n be a positive integer and let $A \in \mathcal{M}_{n\times n}(\mathbb{Q})$. Find infinitely-many different matrices in $\mathcal{M}_{n\times n}(\mathbb{Q})$ having the same eigenvectors as A.

21. Find the spectra of the following matrices.

(i) $\begin{bmatrix} a & b & c \\ a-d & b+d & c \\ a-e & b & c+e \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$;

(ii) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{Z}/(2))$.

22. Find the characteristic polynomials of the following matrices.

(i) $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ 2 & -4 & -1 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$;

(ii) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$;

(iii) $\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \in \mathcal{M}_{5\times 5}(\mathbb{R})$;

(iv) $\begin{bmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{bmatrix} \in \mathcal{M}_{4\times 4}(\mathbb{R})$.

23. Let n be a positive integer and let $\alpha\colon \mathcal{M}_{n\times n}(\mathbb{C}) \to \mathbb{C}^n$ be the function defined by $\alpha\colon A \mapsto [b_0, \ldots, b_{n-1}]$, where $X^n + b_{n-1}X^{n-1} + \cdots + b_1X + b_0$ is the characteristic polynomial of A. Is α a linear transformation?

24. Let n be a positive integer, let F be a field, and let $A, B \in \mathcal{M}_{n\times n}(F)$. Show that AB and BA have the same characteristic polynomial.

25. Find six different matrices in $\mathcal{M}_{2\times 2}(\mathbb{R})$ which annihilate the polynomial $X^2 - 1$.

26. Are the matrices $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ in $\mathcal{M}_{2\times 2}(\mathbb{R})$ similar?

27. Find diagonal matrices in $\mathcal{M}_{3\times 3}(\mathbb{R})$ which are similar to each of the following matrices:

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;

(ii) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$;

(iii) $\begin{bmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{bmatrix}$;

(iv) $\begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 2 \\ -2 & -1 & 4 \end{bmatrix}$;

(v) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

28. Show that

$$\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

for all $a, b \in \mathbb{R}$.

29. Let $F = \mathbb{Z}/(5)$. Show that the matrices $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ in $\mathcal{M}_{3\times3}(F)$ are similar.

30. Show that the matrix $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \in \mathcal{M}_{2\times2}(\mathbb{C})$ is not similar to a diagonal matrix.

31. Let a be an element of a field F and let $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \in \mathcal{M}_{3\times3}(F)$. For any polynomial $p(X) \in F[X]$, show that

$$p(A) = \begin{bmatrix} p(a) & p'(a) & \frac{1}{2}p''(a) \\ 0 & p(a) & p'(a) \\ 0 & 0 & p(a) \end{bmatrix},$$

where $f'(X)$ denotes the formal derivative of a polynomial $f(X) \in F[X]$.

32. Let n be a positive integer and let V be the vector space over $\mathbb{R}$ consisting of all polynomial functions from $\mathbb{R}$ to itself having degree no more than n. Find the minimal polynomial of the differentiation endomorphism $\delta\colon f \mapsto f'$ of V.

33. Let n be a positive integer and let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a matrix of rank h. Show that the degree of the minimal polynomial of A is at most $h+1$.

34. Let n be a positive integer and let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that there exist nonsingular matrices B and C in $\mathcal{M}_{n\times n}(\mathbb{R})$ satisfying $A = B + C$.

35. Let α and β be the endomorphisms of $\mathbb{Q}^4$ represented respectively by the matrices $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$. Does there exist a basis of $\mathbb{Q}^4$ relative which both endomorphisms can be represented by diagonal matrices?

36. Let $a \neq -1$ be a real number and let $A = \begin{bmatrix} 1-a+a^2 & 1-a \\ a-a^2 & a \end{bmatrix} \in \mathcal{M}_{2\times2}(\mathbb{R})$. Calculate A^n for each $n > 1$.