Example 3.4. The seven consecutive integers beginning with $8! + 2 = 40{,}322$ are all composite. (However, these are much larger than the smallest seven consecutive composites, 90, 91, 92, 93, 94, 95, and 96.) ◀

Conjectures About Primes

Professional and amateur mathematicians alike find the prime numbers fascinating. It is not surprising that a tremendous variety of conjectures have been formulated concerning prime numbers. Some of these conjectures have been settled, but many still elude resolution. We will describe some of the best known of these conjectures here.

Looking at tables of primes led mathematicians in the first half of the nineteenth century to make conjectures that the distribution of primes satisfies some basic properties, such as this following conjecture.

Bertrand's Conjecture. In 1845, the French mathematician Joseph Bertrand conjectured that for every positive integer n with $n > 1$, there is a prime p such that $n < p < 2n$. Bertrand verified this conjecture for all n not exceeding 3,000,000, but he could not produce a proof. The first proof of this conjecture was found by Pafnuty Lvovich Chebyshev in 1852. Because this conjecture has been proved, it is often called *Bertrand's postulate*. (See Exercises 22–24 for an outline of a proof.)

Theorem 3.5 shows that the gap between consecutive primes is arbitrarily long. On the other hand, primes may often be close together. The only consecutive primes are 2 and 3, because 2 is the only even prime. However, many pairs of primes differ by two; these pairs of primes are called *twin primes*. Examples are the pairs 3, 5 and 7, 11 and 13, 101 and 103, and 4967 and 4969.

Evidence seems to indicate that there are infinitely many pairs of twin primes. There are 35 pairs of twin primes less than 10^3; 8169 pairs less than 10^6; 3,424,506 pairs less than 10^9; and 1,870,585,220 pairs less than 10^{12}. This leads to the following conjecture.

Twin Prime Conjecture. There are infinitely many pairs of primes p and $p + 2$.

JOSEPH LOUIS FRANÇOIS BERTRAND (1822–1900) was born in Paris. He studied at the École Polytechnique from 1839 until 1841 and at the École des Mines from 1841 to 1844. Instead of becoming a mining engineer, he decided to become a mathematician. Bertrand was appointed to a position at the École Polytechnique in 1856 and, in 1862, he also became professor at the Collège de France. In 1845, on the basis of extensive numerical evidence in tables of primes, Bertrand conjectured that there is at least one prime between n and $2n$ for every integer n with $n > 1$. This result was first proved by Chebyshev in 1852. Besides working in number theory, Bertrand worked on probability theory and differential geometry. He wrote several brief volumes on the theory of probability and on analyzing data from observations. His book *Calcul des probabilitiés*, written in 1888, contains a paradox on continuous probabilities now known as Bertrand's paradox. Bertrand was considered to be kind at heart, extremely clever, and full of spirit.

In 1966, Chinese mathematician J. R. Chen showed, using sophisticated sieve methods, that there are infinitely many primes p such that $p + 2$ has at most two prime factors. An active competition is under way to produce new largest pairs of twin primes. The current record for the largest pair of twin primes is $33{,}218{,}925 \cdot 2^{169{,}690} \pm 1$, a pair of primes with 51,090 digits each, discovered by Daniel Papp and Yves Gallot in 2002.

Viggo Brun showed that the sum $\sum_{\text{primes } p \text{ with } p+2 \text{ prime}} \frac{1}{p} = (1/3 + 1/5) + (1/5 + 1/7) + (1/11 + 1/13) + \cdots$ converges to a constant called *Brun's constant*, which is approximately equal to 1.9021605824. Surprisingly, the computation of Brun's constant has played a role in discovering flaws in Intel's original Pentium chip. In 1994, Thomas Nicely at Lynchburg College in Virginia computed Brun's constant in two different ways using different methods on a Pentium PC and came up with different answers. He traced the error back to a flaw in the Pentium chip and he alerted Intel to this problem. (See page 85 for more information about Nicely's discovery.)

We now discuss perhaps the most notorious conjecture about primes.

Goldbach's Conjecture. Every even positive integer greater than 2 can be written as the sum of two primes.

Example 3.5. The integers 10, 24, and 100 can be written as the sum of two primes in the following ways:

$$\begin{aligned} 10 &= 3 + 7 = 5 + 5, \\ 24 &= 5 + 19 = 7 + 17 = 11 + 13, \\ 100 &= 3 + 97 = 11 + 89 = 17 + 83 \\ &= 29 + 71 = 41 + 59 = 47 + 53. \end{aligned}$$

◄

This conjecture was stated by *Christian Goldbach* in a letter to Leonhard Euler in 1742. It has been verified for all even integers less than $4 \cdot 10^{14}$, with this limit increasing as computers become more powerful. Usually, there are many ways to write a particular even integer as the sum of primes, as Example 3.5 illustrates. However, a proof that there is always at least one way has not yet been found. The best result known to date is due to J. R. Chen, who showed (in 1966), using powerful sieve methods, that all sufficiently large integers are the sum of a prime and the product of at most two primes.

Goldbach's conjecture asserts that infinitely many primes occur as pairs of consecutive odd numbers. However, consecutive primes may be far apart. A consequence of

JING RUN CHEN (1933–1996) was a student of the prominent Chinese number theorist Loo Keng Hua. Chen was almost entirely devoted to mathematical research. During the Cultural Revolution in China, he continued his research, working almost all day and night in a tiny room with no electric lights, no table or chairs, only a small bed and his books and papers. It was during this period that he made his most important discoveries concerning twin primes and Goldbach's conjecture. Although he was a mathematical prodigy, Chen was considered to be next to hopeless in other aspects of life. He died in 1996 after a long illness.

the prime number theorem is that as n grows, the average gap between the consecutive primes p_n and p_{n+1} is $\log n$. Number theorists have worked hard to prove results that show that the gaps between consecutive primes are much smaller than average for infinitely many primes. For example, it has been shown that $p_{n+1} - p_n < 0.2486 \log n$ for infinitely many positive integers n. Showing that for every positive real number ϵ, there are infinitely many positive integers n such that $(p_{n+1} - p_n)/\log n < \epsilon$ remains an elusive goal on the way toward the proof of Goldbach's conjecture.

There are many conjectures concerning the number of primes of various forms, such as the following conjecture.

The $n^2 + 1$ Conjecture. There are infinitely many primes of the form $n^2 + 1$, where n is a positive integer.

The smallest primes of the form $n^2 + 1$ are $5 = 2^2 + 1$, $17 = 4^2 + 1$, $37 = 6^2 + 1$, $101 = 10^2 + 1$, $197 = 14^2 + 1$, $257 = 16^2 + 1$, and $401 = 20^2 + 1$. The best result known

Pentium Chip Flaw

The story behind the Pentium chip flaw encountered by Thomas Nicely shows that answers produced by computers should not always be trusted. A surprising number of hardware and software problems arise that lead to incorrect computational results. This story also shows that companies risk serious problems when they hide errors in their products. In June 1994, testers at Intel discovered that Pentium chips did not always carry out computations correctly. However, Intel decided not to make public information about this problem. Instead, they concluded that because the error would not affect many users, it was unnecessary to alert the millions of owners of Pentium computers. The Pentium flaw involved an incorrect implementation of an algorithm for floating-point division. Although the probability is low that divisions of numbers affected by this error come up in a computation, such divisions arise in many computations in mathematics, science, and engineering, and even in spreadsheets running business applications.

Later in that same month, Nicely came up with two different results when he used a Pentium computer to compute Brun's constant in different ways. In October 1994, after checking all possible sources of computational error, Nicely contacted Intel customer support. They duplicated his computations and verified the existence of an error. Furthermore, they told him that this error had not been previously reported. After not hearing any additional information from Intel, Nicely sent e-mail to a few people telling them about this. These people forwarded the message to other interested parties, and within a few days, information about the bug was posted on an Internet newsgroup. By late November, this story was reported by CNN, the *New York Times*, and the Associated Press.

Surprised by the bad publicity, Intel offered to replace Pentium chips, but only for users running applications determined by Intel to be vulnerable to the Pentium division flaw. This offer did not mollify the Pentium user community. All the bad publicity drove Intel stock down several dollars a share and Intel became the object of many jokes, such as: "At Intel, quality is job 0.999999998." Finally, in December 1994, Intel decided to offer a replacement Pentium chip upon request. They set aside almost half a billion dollars to cover costs, and they hired hundreds of extra employees to handle customer requests. Nevertheless, this story does have a happy ending for Intel. Their corrected and improved version of the Pentium chip was extremely successful.

to date is that there are infinitely many integers n for which $n^2 + 1$ is either a prime or the product of two primes. This was shown by Henryk Iwaniec in 1973. Conjectures such as the $n^2 + 1$ conjecture may be easy to state, but are sometimes extremely difficult to resolve (see [Ri96] for more information).

3.2 Exercises

1. Find the smallest five consecutive composite integers.
2. Find one million consecutive composite integers.
3. Show that there are no "prime triplets," that is, primes p, $p+2$, and $p+4$, other than 3, 5 , and 7.
4. Find the smallest four sets of prime triplets of the form p, $p+2$, $p+6$.
5. Find the smallest four sets of prime triplets of the form p, $p+4$, $p+6$.
6. Find the smallest prime between n and $2n$ when n is

   a) 3. c) 19.
   b) 5. d) 31.

7. Find the smallest prime between n and $2n$ when n is

   a) 4. c) 23.
   b) 6. d) 47.

An unsettled conjecture asserts that for every positive integer n there is a prime between n^2 and $(n+1)^2$.

8. Find the smallest prime between n^2 and $(n+1)^2$ for all positive integers n with $n \leq 10$.
9. Find the smallest prime between n^2 and $(n+1)^2$ for all positive integers n with $11 \leq n \leq 20$.
10. Verify Goldbach's conjecture for each of the following values of n.

    a) 50 c) 102 e) 200
    b) 98 d) 144 f) 222

CHRISTIAN GOLDBACH (1690–1764) was born in Königsberg, Prussia (the city noted in mathematical circles for its famous bridge problem). He became professor of mathematics at the Imperial Academy of St. Petersburg in 1725. In 1728, Goldbach went to Moscow to tutor Tsarevich Peter II. In 1742, he entered the Russian Ministry of Foreign Affairs as a staff member. Goldbach is most noted for his correspondence with eminent mathematicians, in particular Leonhard Euler and Daniel Bernoulli. Besides his well-known conjectures that every even positive integer greater than 2 is the sum of two primes and that every odd positive integer greater than 5 is the sum of three primes, Goldbach made several notable contributions to analysis.